Lower-Upper Symmetric-Gauss-Seidel Method for the Euler and Navier-Stokes Equations

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Abstract

A NEW multigrid relaxation scheme is developed for the steady-state solution of the Euler and Navier-Stokes equations. The lower-upper Symmetric-Gauss-Seidel method (LU-SGS) does not require flux splitting for approximate Newton iteration. The present method, which is vectorizable and unconditionally stable, needs only scalar diagonal inversions. Application to transonic flow shows that the new method is efficient and robust.

Contents

Recently, several implicit schemes have been developed successfully in conjunction with a multigrid method for steady-state solution of the unsteady Euler equations.[1,2] Although the alternating direction implicit scheme could be improved to achieve the expected efficiency of the multigrid method in two dimensions,[1] its inherent limitations in three dimensions suggest alternative approaches.[2] An alternative implicit scheme that is stable in any number of space dimensions is based on LU factorization. The LU implicit scheme proved to be robust in calculations over a wide range of Mach numbers.[3,4]

The Newton iteration method has been investigated to solve the steady Euler or Navier-Stokes equations. Because of the rapid growth of the operation count with the number of mesh cells, the system was solved indirectly. In this paper, an efficient multigrid relaxation scheme is developed for approximate Newton iteration. The new LU-SGS method permits scalar diagonal inversions, whereas the conventional line (or plane in three dimensions) Gauss-Seidel method requires block matrix inversions. The use of scalar diagonal inversions offers the potential for order-of-magnitude speedups when large systems of partial differential equations must be solved. It is desirable that the matrix be diagonally dominant to assure the convergence of a relaxation method. The new method achieves this without the flux splitting. Flux splittings substantially increase the computational work per cycle. Unlike the conventional Gauss-Seidel method, the present method in three dimensions does not need additional relaxation or factorization on a plane of sweep.

The Navier-Stokes equations represent gas flow in thermodynamic equilibrium. Let x, y, and t, be Cartesian coordinates and time, F and G convective flux vectors, and F_v and G_v the flux vectors for the viscous terms. Then, for a two-dimensional flow, these equations can be written as

$$\frac{\partial W}{\partial t}+\frac{\partial F}{\partial x}+\frac{\partial G}{\partial y}=\frac{\partial F_v}{\partial x}+\frac{\partial G_v}{\partial y} \tag{1}$$

where W is the vector of dependent variables.[5]

Using a finite-volume method for space discretization allows one to handle arbitrary geometries and helps to avoid problems with the metric singularities that are usually associated with finite-difference methods. Finite-volume methods do not require special treatment on a composite grid.[6] Using a central-difference scheme when calculating flows with discontinuities typically produces flowfield oscillations in the neighborhood of shock waves where the pressure gradients are severe. To suppress the tendency for spurious odd and even point oscillations and to prevent nonphysical overshoots near shock waves, we augment the semidiscrete finite-volume method by adaptive numerical dissipation, which gives upwind bias.[5] For more accurate capturing of oblique shock waves in hypersonic flows, a total variation diminishing (TVD) scheme can be used.[7]

Let A and B be the Jacobian matrices of the convective flux vectors, D_x and D_y central-difference operators that approximate ∂/∂_x and ∂/∂_y, and δW the correction. Then the linearized implicit scheme for the Euler equations can be written as

$$[I+\beta\Delta t(D_xA+D_yB)]\delta W+\Delta t(D_xF+D_yG)=0 \tag{2}$$

where I is the identity matrix. The unfactored implicit scheme, Eq. (2), produces a large block-banded matrix that can be inverted only by performing a great many computations. In addition, a large amount of storage is required. If $\beta = 1$, the scheme reduces to a Newton iteration in the limit $\Delta t \to \infty$:

$$(D_xA+D_yB)\delta W+(D_xF+D_yG)=0 \tag{3}$$

The LU-SGS method for approximate Newton iteration can be derived as[5]

$$\begin{aligned}&(D_x^-A^+ + D_y^+B^+ - A^- - B^-)(A^+ + B^+ - A^- - B^-)^{-1}\\ &\times(D_x^+A^- + D_y^+B^- + A^+ + B^+)\delta W\\ &= -(D_xF+D_yG)\end{aligned} \tag{4}$$

where D_x^- and D_y^- are backward-difference operators, and D_x^+ and D_y^+ are forward-difference operators; A^+, A^-, B^+, and B^- are constructed so that the eigenvalues of " + " matrices are nonnegative and those of " − " matrices are nonpositive:

$$A^+=\tfrac{1}{2}(A+r_AI),\qquad A^-=\tfrac{1}{2}(A-r_AI)$$

$$B^+=\tfrac{1}{2}(B+r_BI),\qquad B^-=\tfrac{1}{2}(B-r_BI) \tag{5}$$

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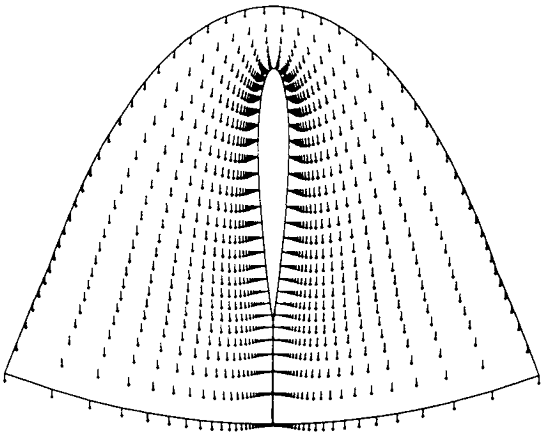

Fig. 1 Velocity vectors for viscous laminar flow.

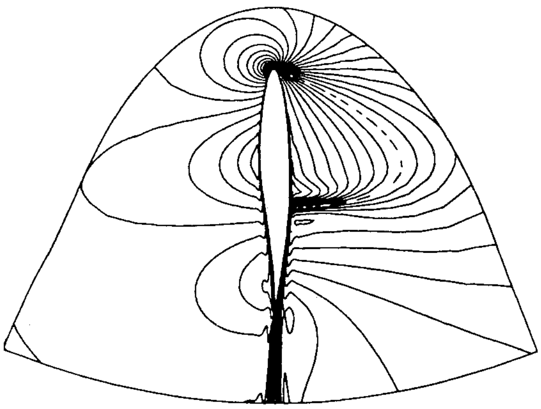

Fig. 2 Mach number contours for viscous turbulent flow.

where

$$r_A \geq \max(|\lambda_A|), \qquad r_B \geq \max(|\lambda_B|) \tag{6}$$

Here, r_A and r_B represent eigenvalues of Jacobian matrices. If we take " + " and " − " matrices as given in Eq. (5), a diagonal entry in any factor becomes

$$A^+ + B^+ - A^- - B^- = (r_A + r_B)\,I \tag{7}$$

Hence, the present numerical method eliminates the need for block diagonal inversions without using a diagonalization procedure. For the simplicity of programming, an approximate form of Eq. (4) may be written as

$$(D_x^- A^+ + D_y^- B^+ - A^- - B^-)(D_x^+ A^- + D_y^+ B^- + A^+ + B^+)\delta W = -(r_A + r_B)(D_x F + D_y G) \tag{8}$$

which can be inverted in two steps. Although Eq. (8) is not completely consistent with Eq. (4), extensive numerical experiments confirm that both forms work very well in practice. For the Navier-Stokes equations, F and G are replaced by $F - F_v$ and $G - G_v$ in Eq. (8)

$$(D_x^- A^+ + D_y^- B^+ - A^- - B^-)(D_x^+ A^- + D_y^+ B^- + A^+ + B^+)\delta W = -(\gamma_A + \gamma_B)[(D_x(F - Fv) + D_y(G - Gv) + \text{numerical dissipation}] \tag{9}$$

Since one-sided difference schemes are naturally dissipative, no implicit smoothing is required on the left side. Only numerical dissipation terms are explicitly added to the residual on the right side. The LU family of algorithms are vectorizable along $i + j =$ const lines on a vector computer.

The underlying idea of a multigrid method is to transfer some of the task of tracking the evolution of the system to a sequence of successively coarser meshes. This has two advantages. First, the computational effort per cycle is reduced on a coarser mesh. Second, the use of larger control volumes on the coarser grids tracks the evolution on a larger scale, with the consequence that global equilibrium can be more rapidly attained. The cells of the fine mesh can be amalgamated into larger cells, which form a coarser mesh. Then, in each coarse mesh cell the conservation laws are represented by summing the flux balances of its fine mesh cells; consequently, the evolution on the coarse mesh is driven by the disequilibrium of the fine-mesh equations. The multigrid method used here is the cell-centered method that was used for the implicit schemes.[2]

The first test case was for inviscid transonic flow past the NACA 0012 airfoil at 1.25 deg angle of attack. The freestream Mach number was 0.8. Nonreflecting boundary conditions were used to absorb the waves impinging on the farfield boundary. Five-level V-cycle multigrid calculations were performed on a 128×32 C-mesh without grid sequencing. Uniform flow was given as the initial condition. The convergence rate of the present method is about 30% faster than that of the LU implicit scheme. Moreover, the computational work per cycle for the new method is about 30% less than that for the LU implicit scheme, since the present method does not need block diagonal inversion. The overall CPU time is reduced by a factor of 2. The next case was for viscous laminar flow past the NACA 0012 airfoil at Mach 0.5, Reynolds number 5000, and 0 deg angle of attack. The adiabatic wall boundary condition was used at the body surface. Calculations were performed on a stretched 192×48 C-mesh. The enthalpy damping technique was not used for the viscous flow calculations. Figure 1 shows velocity vectors. The last case was for viscous turbulent flow past the RAE 2822 airfoil at Mach 0.73, Reynolds number 6.5×10^6, and 2.79 deg angle of attack. The Reynolds-averaged Navier-Stokes equations were solved using a Baldwin-Lomax turbulence model. Transition was fixed at 3% chord. Mach number contours are shown in Fig. 2 (the dashed line denotes the sonic line).

The LU-SGS method was combined with a flux-limited TVD scheme.[7,8] It was also applied to chemically reacting nonequilibrium flows in scramjet combustors.[9]

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