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Aircraft Configurations**

A. Jameson and T.J. Baker, Princeton Univ.,
Princeton, NJ

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MULTIGRID SOLUTION OF THE EULER EQUATIONS FOR AIRCRAFT CONFIGURATIONS

Antony Jameson and Timothy J. Baker
Department of Mechanical and Aerospace Engineering
Princeton University
Princeton, N.J. 08544

Abstract

A multigrid scheme for solving the Euler equations is presented. The method has been successfully applied to two-dimensional airfoil calculations on both O-type and C-type meshes. In three dimensions the scheme has proved equally effective and calculations of flows over wing/body combinations are possible with convergence achieved in less than 100 cycles.

1. Introduction

Multigrid has emerged as a particularly efficient technique for accelerating the convergence of numerical calculations. Although it was originally devised and applied to solve elliptic equations, considerable interest has been generated by the possibility of employing this technique for other types of partial differential equation^{1,2}. The application of multigrid to solve transonic flow problems using the potential equation has been an area of particularly active research^{3,4,5}. Here the equation is of mixed type and the presence of shockwaves leads to numerical difficulties not normally encountered with elliptic equations where the solutions are expected to possess a reasonable degree of smoothness. Further difficulties were caused by the wide variation in mesh cell aspect ratio which is typical of the meshes used for calculating the flow around aerodynamic shapes. This particular problem arose because the driving routine used in a multigrid scheme must provide heavy damping of all high frequency modes. In two and three dimensions the amplification factor depends on the mesh cell aspect ratio and it is by no means easy to devise schemes with good high frequency damping for a wide variation in the shape and size of mesh cells. For two-dimensional flow calculations using the potential equation this difficulty was successfully resolved by Jameson⁴ by the introduction of an alternating direction method. Numerous calculations by this technique have demonstrated that multigrid can indeed achieve very rapid convergence for transonic flow calculations.

The assumption of potential flow implies that the flow is irrotational. This is not strictly correct when shock waves are present and an exact description of transonic inviscid flow requires the solution of the Euler equations. This is a system of hyperbolic equations which in integral form can be written as

$$\frac{\partial}{\partial t} \iiint_{\Omega} W d\Omega + \iint_{\partial\Omega} \underline{F} \cdot d\underline{S} \quad (1)$$

for a fixed region Ω with boundary $\partial\Omega$. Here W represents the conserved quantity and $\underline{F}$ is the corresponding flux term. If we write p, ρ, u, v, w, E and H for the pressure, density, Cartesian velocity components, total energy and total enthalpy, then equation (1) represents mass conservation if we put

$$W = \rho, \quad \underline{F} = (\rho u, \rho v, \rho w).$$

For conservation of momentum in the x direction we have

$$W = u, \quad \underline{F} = (\rho u^2 + p, \rho uv, \rho uw)$$

in equation (1). Conservation of momentum in the y and z directions is similarly defined. Finally energy conservation is given by

$$W = \rho E, \quad \underline{F} = (\rho Hu, \rho Hv, \rho Hw).$$

While the available theorems in the theory of multigrid methods generally assume ellipticity, it seems that it ought to be possible to accelerate the evolution of a hyperbolic system to a steady state by using large time steps on coarse grids, so that disturbances are more rapidly expelled through the outer boundary. The interpolation of corrections back to the fine grid will introduce errors, however, which cannot be rapidly expelled from the fine grid, and ought to be locally damped, if a fast rate of convergence is to be attained. Thus it remains important that the driving scheme should have the property of rapidly damping out high frequency modes.

In a novel multigrid scheme proposed by Ni⁶, the flow variables are stored at mesh nodes, and the rates of change of mass, momentum and energy in each mesh cell are estimated from the flux integral (1). The corresponding change δW_0 in the solution at the cell center is then distributed unequally between the nodes at its four corners by the rule

$$\delta W_C = 1/4 (\delta W_0 \pm A\delta W_0 \pm B\delta W_0)$$

where δW_C is the correction at a corner, and A and B are the Jacobian matrices. The signs are varied in such a way that the accumulated correction at each node corresponds to the first two terms of a Taylor series in time, like a Lax Wendroff scheme. As it stands, this scheme does not damp odd and even point oscillations. Ni introduces artificial viscosity by adding a further correction proportional to the difference between the value at each corner and the average of the values at the four corners of the cell. Residuals on the coarse grid are formed by taking weighted averages of the corrections at neighboring nodes of the fine grid, and corrections are then assigned to the corners of coarse grid cells by the same distribution rule.

When several grid levels are used, the distribution rule is applied once on each grid down to the coarsest grid, and the corrections are then interpolated back to the fine grid.

Using 3 or 4 grid levels, Ni obtained a mean error reduction of about .95 per multigrid cycle, leading to a fairly well converged solution in 100-200 cycles. The accuracy of the results which can be obtained by the algorithm in its published form, however, appears to be limited by the action of the artificial viscosity, which introduces an error of first order.

The multistage scheme previously developed by the authors⁷ forms the basis of a multigrid method recently introduced by Jameson⁸. This finite volume discretization has a number of attractive features. Central differencing is used for the spatial derivatives which leads to second order accuracy in regions where the flow is smoothly varying. The controlled addition of artificial dissipation permits the capture of sharply defined shocks. Away from shocks where the flow varies smoothly, a third order dissipation is applied to prevent the occurrence of odd-even point oscillations. In the neighborhood of a shockwave, however, a switch, sensitive to second differences in the pressure, turns on a first order dissipation that successfully inhibits preshock oscillations. Unlike a single stage scheme, the use of multiple stages permits a stable time stepping operation when central differencing is used to evaluate the flux term. Furthermore the construction of these multistage schemes ensures that the steady state solution is independent of the time step Δt , an advantage not shared by several other schemes.

In this paper it is shown that the multistage time stepping scheme can be adapted for use in conjunction with a multigrid technique to produce a rapidly convergent algorithm for calculating steady state solutions of the Euler equations. Several results are presented showing the performance of this algorithm in two and three dimensions and for flows over aircraft-like configurations.

2. Finite Volume Scheme and Time Stepping

The computational domain is divided into hexahedral cells. Assuming that the dependent variables are known at the center of each cell, a system of ordinary differential equations is obtained by applying equation (1) separately to each cell. These have the form

$$\frac{d}{dt} (V_{ijk} W_{ijk}) + Q(W)_{ijk} = 0 \quad (2)$$

Q_{ijk} represents the net flux out of the cell which is balanced by the rate of change of W in the cell whose volume is V . In order to suppress the tendency for odd and even point oscillations and to limit overshoots near shock waves, the scheme is augmented by a dissipative term so that equation (2) becomes

$$\frac{d}{dt} (V_{ijk} W_{ijk}) + Q_{ijk} - D_{ijk} = 0 \quad (3)$$

Here D_{ijk} is the dissipation which is constructed so that it is of third order in smooth regions of flow. For the density equation $D_{ijk}(\rho)$ has the form

$$d_{i+1/2,j,k} = d_{i-1/2,j,k} + d_{i,j+1/2,k} - d_{i,j-1/2,k} + d_{i,j,k+1/2} - d_{i,j,k-1/2} \quad (4)$$

where

$$d_{i+1/2,j,k} = \frac{V_{i+1/2,j,k}}{\Delta t} (\epsilon_{i+1/2,j,k}^{(2)} \Delta_x \rho_{i,j,k} - \epsilon_{i+1/2,j,k}^{(4)} \Delta_x^3 \rho_{i-1,j,k}) \quad (5)$$

and Δ_x is the forward difference operator defined by

$$\Delta_x \rho_{ijk} = \rho_{i+1,j,k} - \rho_{ijk}$$

The coefficient $\epsilon_{i+1/2,j,k}^{(2)}$ is made proportional to the normalized second difference of the pressure

$$v_{ijk} = \left| \frac{p_{i+1,j,k} - 2p_{ijk} + p_{i-1,j,k}}{p_{i+1,j,k} + 2p_{ijk} + p_{i-1,j,k}} \right| \quad (6)$$

In adjacent cells. This quantity is of second order except in regions containing a steep pressure gradient. The fourth differences provide background dissipation throughout the domain. In the neighborhood of a shock wave, however, v_{ijk} is of order one and the second differences become the dominant dissipative terms. The dissipative terms for the other equations are constructed from similar formulas with the exception of the energy equation where the differences are of ρH rather than ρE . The purpose of this is to allow a steady state solution for which H remains constant.

The cell volume V_{ijk} is independent of time so we can write (3) as

$$\frac{dW}{dt} + R(W) = 0 \quad (7)$$

where

$$R(W) = \frac{1}{V_{ijk}} (Q_{ijk} - D_{ijk}).$$

A class of four stage schemes to advance equations (7) in time can be written as

$$\begin{aligned} W^{(0)} &= W^n \\ W^{(1)} &= W^{(0)} - \alpha_1 \Delta t R^{(0)} \\ W^{(2)} &= W^{(0)} - \alpha_2 \Delta t R^{(1)} \\ W^{(3)} &= W^{(0)} - \alpha_3 \Delta t R^{(2)} \\ W^{(4)} &= W^{(0)} - \Delta t R^{(3)} \\ W^{n+1} &= W^{(4)} \end{aligned}$$

Here W^n and W^{n+1} are the values at the beginning and end of the n th time step, and in the $(q+1)$ st stage $R^{(q)}$ is evaluated as

$$\frac{1}{V_{ijk}} (Q_{ijk}(W^{(q)}) - D_{ijk}(W^{(0)})) \quad (8)$$

so that the dissipative terms are only evaluated once in each time step. The standard four stage scheme has the coefficients

$$\alpha_1 = 1/4, \alpha_2 = 1/3, \alpha_3 = 1/2. \quad (9)$$

3. Multigrid Method

The new multigrid scheme preserves the second order accuracy and effective shock capturing capability of the original multi-stage scheme. Auxiliary meshes are introduced by doubling the mesh spacing and the values of the flow variables are transferred to a coarser grid by the rule

$$w^{(0)}_{2h} = \sum w_h w_h / V_{2h}$$

where the subscripts denote values of the mesh spacing parameter. In three dimensions the sum is over the 8 cells on the fine grid composing each cell on the coarse grid. This rule conserves mass, momentum and energy. A forcing function is then defined as

$$P_{2h} = \sum R_h(w_h) - R_{2h}(w_{2h}^{(0)}) \quad (10)$$

where R is the residual of the difference scheme. To update the solution on a coarse grid the multistage scheme is reformulated as

$$\begin{aligned} w^{(1)}_{2h} &= w^{(0)}_{2h} - \alpha_1 \Delta t (R^{(0)}_{2h} + P_{2h}) \\ \dots \\ w^{(q+1)}_{2h} &= w^{(0)}_{2h} - \alpha_q \Delta t (R^{(q)}_{2h} + P_{2h}) \\ \dots \end{aligned} \quad (11)$$

where $R^{(q)}$ is the residual of the q^{th} stage. In the first stage of the scheme, the addition of P_{2h} cancels $R_{2h}(w^{(0)}_{2h})$ and replaces it by $\sum R_h(w_h)$, with the result that the evolution on the coarse grid is driven by the residuals on the fine grid. This process is repeated on successively coarser grids. Finally the correction calculated on each grid is passed back to the next finer grid by bilinear interpolation.

Since the evolution on a coarse grid is driven by residuals collected from the next finer grid, the final solution on the fine grid is independent of the choice of boundary condition on the coarse grids. The surface boundary condition is treated in the same way on every grid by using the normal pressure gradient to extrapolate the surface pressure from the pressure in the cells adjacent to the wall. The far field conditions can either be transferred from the fine grid or recalculated by the procedure described in the next section.

The success of a multigrid scheme is critically dependent on the shape of the amplification factor. In particular it is necessary for the scheme to provide rapid damping of high frequency modes. To see how the multistage schemes can be adapted to meet this requirement we consider the model problem

$$u_t + u_x + \mu \Delta x^3 u_{xxxx} = 0 \quad (12)$$

With centered differences the residual has the form

$$\begin{aligned} \Delta t R(u)_j &= \frac{\lambda}{2} (u_{j+1} - u_{j-1}) \\ &+ \lambda \mu (u_{j+2} - 4u_{j+1} + 6u_j - 4u_{j-1} + u_{j-2}) \end{aligned} \quad (13)$$

where $\lambda = \Delta t / \Delta x$ is the Courant number. If we consider a Fourier mode $\hat{u} = e^{ipx}$ the discretization in space yields

$$\Delta t \frac{d\hat{u}}{dt} = z\hat{u}$$

where z is the Fourier symbol of the residual. When the residual is given by equation (13) we obtain

$$z = -\lambda i \sin \xi - 4\lambda \mu (1 - \cos \xi)^2 \quad (14)$$

where $\xi = p\Delta x$. The imaginary part corresponds to the centered difference approximation to the flux term and the real part corresponds to the dissipative term.

If we evaluate the dissipation at every stage, the amplification factor $g(z)$ of the scheme will be a function of the single complex variable z . For an explicit scheme this function will be analytic in any finite domain. It follows from the maximum modulus theorem that the maximum value of $|g(z)|$ over a finite domain will occur on the boundary. Thus, if we consider contours of $|g(z)|$, then within any contour, say $|g(z)|=1$, the value of $|g(z)|$ will always be less than one. The region bounded by this particular contour corresponds to values of z for which the scheme will be stable. With the coefficients given by (9) we obtain

$$g(z) = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24} \quad (15)$$

The stability region of this scheme is shown in Figure 1(a). The Figure also shows the locus of z as the wave number ξ is varied between 0 and 2π for a Courant number $\lambda = 2.8$ and a dissipation coefficient $\mu = 1/32$. The corresponding variation of $|g|$ with ξ is presented in Figure 1(b).

If we do not evaluate the dissipation at every time step, the amplification factor and hence the stability region of the scheme undergoes an interesting modification. Suppose we have a k stage scheme and we evaluate the dissipation once at the first stage. Then for the last stage we have

$$u(k) = u(0) - \Delta t (Q u(k-1) - D u(0))$$

Now writing $z = x + iy$ we find that

$$g(x,y) = 1 + x + iy f(x,y) \quad (16)$$

where $f(x,y)$ arises from the Fourier transform of $u(k-1)$. We now find the g is no longer a function of the single complex variable z and it follows that $|g|$ may exceed the value one at interior points of the stability region. Although care must be taken when deriving schemes without evaluation of the dissipation at every stage, it is not difficult to devise schemes which still have well behaved stability regions. For example, the standard coefficients (9) now produce a scheme with the stability region shown in Figure 2(a). This also shows that this form of scheme is stable for a Courant number of 2.6 with the same amount of dissipation. The corresponding variation of the amplification factor $|g|$ with ξ is shown in 2(b).

We note from (16) that $g(x,y)$ has a zero on the real axis at $x = -1$. It follows that if we adjust the dissipation or Courant number we can arrange for the locus of z to pass through this zero. Since the intercept on the real axis corresponds to $\xi = \pi$, we thus achieve a scheme with good damping properties for the high frequency components. From equation (14) it is apparent that the choice $\lambda = 2$, $\mu = 1/32$ will cause the locus of z to pass through this zero.

However we are still free to adjust the coefficients of the scheme and can therefore further improve its high frequency damping behaviour. For example, when the four stage scheme is used with a single evaluation of the dissipative terms, a good choice of the coefficients is

$$\alpha_1 = .25, \alpha_2 = .5, \alpha_3 = .55 \quad (17)$$

The resulting stability region for the model problem (12) is shown in Figure 3(a). From Figure 3(b) it can also be seen that with a Courant number $\lambda = 2$ and a dissipation $\mu = 1/32$, the magnitude $|g|$ of the amplification factor is less than .35 for all wave numbers in the range $1.0 < \xi < \pi$. Another scheme which has proved very effective has three stages with a single evaluation of the dissipation. For this scheme the coefficients

$$\alpha_1 = .6, \alpha_2 = .6 \quad (18)$$

generate the stability region shown in Figure 4(a). With $\lambda = 1.5$ and $\mu = 1/25$ we obtain an amplification factor whose magnitude variation with wave number is shown in Figure 4(b).

Further variations on this theme are possible. For example, with evaluation of the dissipation at the first and second stages only, we obtain the following form for the amplification factor

$$g(x,y) = 1 + x + \alpha_1 x^2 + iy [f(x,y) + \alpha_1 x]$$

where again $f(x,y)$ arises from the Fourier transform of $u(k-1)$. In this case there are two zeros on the real axis. The particular choice $\alpha_1 = 1/4$ leads to a double zero at $x = -2$ which might be expected to produce a scheme with good damping over a broad neighborhood of this zero. For example the 5 stage scheme with two evaluations of dissipation and coefficients

$$\alpha_1 = 1/4, \alpha_2 = 1/6, \alpha_3 = 3/8, \alpha_4 = 1/2 \quad (19)$$

generates a scheme with the stability region shown in Figure 5(a). With $\lambda = 3$ and $\mu = 1/25$ we obtain an amplification factor whose magnitude varies with ξ as shown in Figure 5(b).

In addition we can also include implicit smoothing⁷ which permits stable operation at Courant numbers beyond the ordinary Courant number limit of the explicit scheme. In fact, the results presented in Section 5 have been calculated using the five stage scheme with coefficients given by Equation (19), two evaluations of dissipation and a Courant number of around seven. The smoothing coefficient described in Reference 7 has been generalised to allow different amounts of smoothing in each coordinate direction. The values used to run at a Courant number of around seven were $\epsilon_i = 0.5$ and $\epsilon_j = 0.3$ where the subscript i refers to the mesh line which wraps around the airfoil and j refers to the radial mesh line. In the three dimensional calculations, the additional smoothing coefficient ϵ_k for the spanwise direction was given the value 0.2. The range of possible schemes is considerable and the best choice for multigrid is still under investigation.

4. Boundary Conditions

At a solid boundary the only contribution to the flux balance comes from the pressure. The normal pressure gradient $\partial p / \partial n$ at the wall can be estimated from the condition that $\partial / \partial t (p q_n) = 0$, where q_n is the normal velocity component. The pressure at the wall is then estimated by extrapolation from the pressure at the adjacent cell centers, using the known value of $\partial p / \partial n$.

The rate of convergence to a steady state will be impaired if outgoing waves are reflected back into the flow from the outer boundaries. The treatment of the far field boundary condition is based on the introduction of Riemann invariants for a one dimensional flow normal to the boundary. Let subscripts ∞ and e denote free stream values, and values extrapolated from the interior cells adjacent to the boundary, and let q_n and c be the velocity component normal to the boundary and the speed of sound. Assuming that the flow is subsonic at infinity, we introduce fixed and extrapolated Riemann invariants

$$R_\infty = q_{n\infty} - \frac{2c_\infty}{\gamma - 1}$$

and

$$R_e = q_{ne} + \frac{2c_e}{\gamma - 1}$$

corresponding to incoming and outgoing waves. These may be added and subtracted to give

$$q_n = \frac{1}{2} (R_e + R_\infty)$$

and

$$c = \frac{\gamma - 1}{4} (R_e - R_\infty)$$

where q_n and c are the actual normal velocity component and speed of sound to be specified in the far field. At an outflow boundary, the tangential velocity component and entropy are extrapolated from the interior, while at an inflow boundary they are specified as having free stream values. These four quantities provide a complete definition of

the flow in the far field. If the flow is supersonic in the far field, all the flow quantities are specified at an inflow boundary and they are extrapolated from the interior at an outflow boundary.

5. Results

The multigrid scheme has been coded for two dimensional airfoil calculations in both an O-mesh and a C-mesh version and in three dimensions for calculating flows over wings and wing/body/tail/fin combinations. The airfoil calculations were all carried out using a mesh with 160 intervals in the direction around the airfoil and 32 intervals in the radial direction. In each case two indicators of the convergence rate are presented. The first is the decay of the logarithm of the error where the error is measured by the root mean square rate of change of density on the fine grid. The second is the build-up of the number of points in the supersonic zone. For a transonic flow this indicates how quickly the supersonic zone develops and is a useful measure of the global convergence of the flowfield.

Figure 6 presents the pressure distribution for the NACA 0012 airfoil at Mach .8 and an angle of attack of 1.25° . This particular case was calculated on a C-mesh. However the O-mesh result over plots it and gives a drag coefficient which differs by only two drag counts from the C-mesh calculation. The convergence history on the O-mesh for a run of 300 cycles is presented in Figure 7 and the corresponding convergence information for the C-mesh is shown in Figure 8. In both cases there is a steady reduction in the error and the supersonic zone is evidently frozen in less than 100 cycles. Convergence on the C-mesh is particularly fast showing a mean convergence rate of under .92 over 300 cycles with the number of supersonic points steady after no more than 50 cycles.

The next example is the KORN airfoil at a shock-free condition. This is known to be a sensitive test for flowfield methods since small changes in the boundary conditions can introduce a shockwave. Moreover, what appears to be a small numerical effect can also radically alter the pressure distribution, and relatively small amounts of artificial dissipation can prevent the appearance of a shock-free flow. The essentially shock-free character of the pressure distribution shown in Figure 9 with a predicted drag coefficient of only five counts is a gratifying indication of the accuracy that can be obtained. The convergence history is present in Figure 10 and the mean convergence rate of this O-mesh calculation is below .93 over 300 cycles. Over the first 100 cycles, however, the mean convergence rate is better than .9 for both the NACA 0012 and the KORN airfoil calculations.

We now consider 3-D calculations and as a first example show the pressure distribution over the ONERA M6 wing at Mach .84 and an angle of attack of 3.06° . The result which is displayed in Figure 11 was calculated on a mesh of $80 \times 16 \times 16$ cells. Convergence information appears in Figure 12 and although the error reduction rate deteriorates to some extent after the first 100 cycles, it is clear that the supersonic zone has frozen in just over 50 cycles.

Our last example is the Boeing 747 without nacelles; the distribution of mesh cell faces on the aircraft is shown in Figure 13. The construction of the C-H mesh, which has $96 \times 16 \times 16$ cells, is described in Reference 7. In Figure 14 the wing pressures obtained from a wing/body calculation are shown for Mach number of 0.84 and an angle of attack of 2.44° . The convergence history for this case is presented in Figure 15. The mean error reduction rate over 200 cycles is .95 and the supersonic zone has settled after only 100 cycles.

Conclusion

These results clearly demonstrate that the convergence of a time dependent hyperbolic system to a steady state can be substantially accelerated by the introduction of multiple grids. The steady state is determined by the space discretization on the fine grid, and is independent of both the time stepping scheme and the discretization procedure used on the coarse grids. The present fine grid discretization scheme is second order accurate in smooth regions of flow and the controlled addition of first order dissipation in the neighborhood of shockwaves leads to good shock capturing properties. The ability of the scheme to achieve a shock-free transonic flow at the design condition for a shockfree airfoil and the apparent insensitivity to mesh structure, demonstrated by O-mesh and C-mesh comparisons, provides clear evidence of the accuracy of this method. Calculations of the flow over three dimensional geometries including wing/body configurations show that the convergence speed of the multigrid technique combined with the accuracy of the space discretization extends to the treatment of complex aerodynamic shapes.

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Figure 1(a)
Stability Region of Standard 4 Stage Scheme
Contour lines $|g| = 1.0, .9, .8, \dots$
plus locus of $z(\xi)$ for $\lambda = 2.8, \mu = 1/32$
Coefficients $\alpha_1 = 1/4, \alpha_2 = 1/3, \alpha_3 = 1/2$

Figure 1(b)
Amplification Factor $|g|$ of Standard 4 Stage Scheme
for $\lambda = 2.8, \mu = 1/32$
Coefficients $\alpha_1 = 1/4, \alpha_2 = 1/3, \alpha_3 = 1/2$

Figure 2(a)
Stability Region of 4 Stage Scheme with Single
Evaluation of Dissipation
Contour lines $|g| = 1., .9, .8, \dots$
plus locus of $z(\xi)$ for $\lambda = 2.6, \mu = 1/32$
Coefficients $\alpha_1 = 1/4, \alpha_2 = 1/3, \alpha_3 = 1/2$

Figure 3(a)
Stability Region of 4 Stage Scheme with Single
Evaluation of Dissipation and Modified Coefficients
Contour lines $|g| = 1., .9, .8, \dots$
plus locus of $z(\xi)$ for $\lambda = 2., \mu = 1/32$
Coefficients $\alpha_1 = .25, \alpha_2 = .5, \alpha_3 = .55$

Figure 2(b)
Amplification Factor $|g|$ of 4 Stage Scheme with
Single Evaluation of Dissipation for
 $\lambda = 2.6, \mu = 1/32$
Coefficients $\alpha_1 = 1/4, \alpha_2 = 1/3, \alpha_3 = 1/2$

Figure 3(b)
Amplification Factor $|g|$ of 4 Stage Scheme with
Single Evaluation of Dissipation and Modified
Coefficients for
 $\lambda = 2, \mu = 1/32$
Coefficients $\alpha_1 = .25, \alpha_2 = .5, \alpha_3 = .55$

Figure 4(a)
Stability Region of 3 Stage Scheme with Single
Evaluation of Dissipation
Contour lines $|g| = 1, .9, .8, \dots$
plus locus of $z(\xi)$ for $\lambda = 1.5, \mu = 1/25$
Coefficients $\alpha_1 = .6, \alpha_2 = .6$

Figure 5(a)
Stability Region of 5 Stage Scheme with Two
Evaluations of Dissipation
Contour lines $|g| = 1, .9, .8, \dots$
plus locus of $z(\xi)$ for $\lambda = 3, \mu = 1/25$
Coefficients $\alpha_1 = 1/4, \alpha_2 = 1/6, \alpha_3 = 3/8, \alpha_4 = 1/2$

Figure 4(b)
Amplification Factor $|g|$ of 3 Stage Scheme with
Single Evaluation of Dissipation for
 $\lambda = 1.5, \mu = 1/25$
Coefficients $\alpha_1 = .6, \alpha_2 = .6$

Figure 5(b)
Amplification Factor $|g|$ of 5 Stage Scheme with Two
Evaluations of Dissipation for
 $\lambda = 3, \mu = 1/25$
Coefficients $\alpha_1 = 1/4, \alpha_2 = 1/6, \alpha_3 = 3/8, \alpha_4 = 1/2$

Figure 6

Figure 8
Convergence History on C-Mesh

Figure 7
Convergence History on O-Mesh

Figure 9

KORN AIRFOIL
 MACH 0.750 ALPHA 0.0
 RESID1 0.262D+01 RESID2 0.3500-09
 WORK 299.00 RATE 0.9266
 GRID 160X32

Figure 10

ONEA WING M6
 MACH 0.640 ALPHA 3.050
 RESID1 0.379E-01 RESID2 0.251E-05
 WORK 226.98 RATE 0.9352
 GRID 60X16X16

Figure 12

UPPER SURFACE PRESSURE LOWER SURFACE PRESSURE

ONEA WING M6
 MACH 0.640 ALPHA 3.050
 CL 0.2921 CD 0.0175 CM -0.2294
 GRID 60X16 N51UF 200 RESID 0.251E-05

Figure 11

Figure 13

Figure 13
(continued)

Figure 14
Upper and Lower Surface Wing Pressures
for 747 Wing/Body

747 WING/BODY / ALPHA=2.44 M=.84 DELTA=

MACH	0.840	ALPHA	2.440
RESID1	0.176E+01	RESID2	0.226E-04
WORK	227.37	RATE	0.9517
GRID	96X16X16		

Figure 15