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Kinetic Energy Conserving Discontinuous Galerkin Scheme

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In this document, we present a proof that generalizes Jameson's Kinetic Energy Preserving Finite Volume scheme to higher order Discontinuous Garlerkin method. We then present some results and discuss the method derived.

Introduction

The successful simulation of complex unsteady vortical flows requires the use of numerical methods that are stable yet not too dissipative. The kinetic energy conserving scheme introduced by Jameson[4,5] in 2007 for finite volumes proved efficient at solving these kinds of problems[2,3]. In this paper we present an extension of this work to high order discontinuous Galerkin methods and show how an identical flux leads to a method that is almost kinetic energy conserving.

In a first part, the discontinuous Galerkin method for 1D conservation equation is briefly described. In a second part, the kinetic energy conserving flux is derived for the Burger's equation. In a third part, the flux is derived for the Euler's equations and numerical experiments are conducted to test the method. In both cases, the flux derived is identical to the one obtained for the special case of finite volumes. Finally, we discuss the advantages and pitfalls of the method.

I. Discontinuous Galerkin Method for 1D conservation equations

We consider the Discontinuous Galerkin (DG) method to solve the equation $\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} f(u) = 0$ on the domain $\Omega = [a, b]$. Ω is decomposed in N elements

$$\begin{aligned}\Omega &= \bigcup_{k=0}^{N-1} [x_k, x_{k+1}], \quad a = x_0 < x_1 < ... < x_N = b \\ &= \bigcup_{k=0}^{N-1} \Omega_k\end{aligned}$$

The function space considered is the set of functions piecewise H^1:

$$V^h(\Omega) = \left\{ \phi : [a,b] \mapsto \mathbb{R} \quad / \quad \forall k, \ \phi_{|\Omega_k} \in V_k^h = H^1(\Omega_k) \right\}$$

This space is a bit ill-defined at points $x_1, \ldots, x_{N-1}$ as a function ϕ^h in V^h takes two different values $\phi^- = \phi^h(x_k)$ on Ω_k and $\phi^+ = \phi^h(x_k)$ on Ω_{k+1}. Also, if $\phi^h \in V^h$, we denote by ϕ_k^h the restriction of ϕ^h to Ω_k. Therefore, ϕ_k^h is in $H^1(\Omega^k)$. In what follows and to lighten the notations, we will drop the superscript ^{h} to denote the elements of V^h.

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1. DG formulation

The conservation problem we are trying to solve, associated with a set of appropriate initial and boundary conditions can be expressed in DG weak form as

> Find $u \in V^h$ s.t. $\forall \phi \in V^h$
>
> $$\forall k, \int_{\Omega_k} \frac{\partial u_k}{\partial t}\phi_k - f(u_k)\frac{\partial \phi_k}{\partial x}\ dx = -\left(f(u)\right)^\star_{k,k+1} \cdot \phi(x_{k+1}) + \left(f(u)\right)^\star_{k-1,k} \cdot \phi(x_k)$$

where $f(u)^\star_{k,k+1}$ is the flux between cell k and cell $k+1$ and needs to be defined more carefully. One can integrate by parts one more time to obtain the DG strong form

> Find $u \in V^h$ s.t. $\forall \phi \in V^h$
>
> $$\forall k, \int_{\Omega_k} \frac{\partial u_k}{\partial t}\phi_k + \frac{\partial f(u_k)}{\partial x}\phi_k\ dx = \left[f(u_k(x_{k+1})) - \left(f(u)\right)^\star_{k,k+1}\right] \cdot \phi(x_{k+1}) - \left[f(u_k(x_k)) - \left(f(u)\right)^\star_{k-1,k}\right] \cdot \phi(x_k)$$

The derivation with respect to x is made in the sens of distributions (or we can consider the restriction of H^1 to a space of smooth functions). Both weak and strong forms are mathematically equivalent but not numerically for the integrations are not necessarily performed exactly, as described in the following.

2. Polynomial Approximation

Of course, the above formulation is not really helpful to numerically solve the equation and we need to introduce a finite dimension approximation of the space V^h. To do so, we consider the space of piecewise polynomials of order p, namely V^h_p

$$V^h_p(\Omega) = \left\{\tilde{\phi} : [a,b] \mapsto \mathbb{R} \quad / \quad \forall k,\ \tilde{\phi}_{|_{\Omega^k}} \in V^h_k = \mathbb{R}_p[X]\right\}$$

If $\tilde{u} \in V^h_p(\Omega)$ is the approximation of u on Ω, $\tilde{u}_k$ is the restriction of the approximation on Ω_k. Therefore, $\tilde{u}_k$ is a polynomial of degree p and can be represented in a lagrange polynomial basis $\{\ell^i_k\}_{i\in[0,p]}$ defined by the points $\{x^i_k\}_{i\in[0,p]} \in (\Omega_k)^{p+1}$

$$\forall x \in \Omega_k,\ \ \tilde{u}_k(x) = \sum_{i=0}^{p} \tilde{u}_k(x^i_k)\ell^i_k(x)$$

Following the notations introduced by Hesthaven and Warburton[1], we will identify $\tilde{u}_k : x \mapsto \tilde{u}_k(x)$ and $\tilde{u}_k = [\tilde{u}_k(x^0_k)\ \tilde{u}_k(x^1_k)\ \cdots\ \tilde{u}_k(x^p_k)]^T$. For simplicity, we introduce the notation $\tilde{u}^i_k = \tilde{u}_k(x^i_k)$. We can now formulate the approximate problem in V^h_p

> Find $\tilde{u} \in V^h_p$ s.t. $\forall i$ and $\forall k$
>
> $$\int_{\Omega_k} \frac{\partial \tilde{u}_k}{\partial t}\ell^i_k + \frac{\partial f(\tilde{u}_k)}{\partial x}\ell^i_k\ dx = \left[f(\tilde{u}_k(x_{k+1}) - \left(f(u)\right)^\star_{k,k+1}\right] \cdot \ell^i_k(x_{k+1}) - \left[f(\tilde{u}_k(x_k) - \left(f(u)\right)^\star_{k-1,k}\right] \cdot \ell^i_k(x_k)$$

Therefore, we obtain $p+1$ equations per element k (for each i) and we can solve for the $p+1$ unknowns $\{u^i_k\}$.

Some attention needs to be given to the way integrations are performed. It is evident that $\int \ell^i_k \ell^j_k dx$ and $\int \ell^i_k \frac{\partial \ell^j_k}{\partial x} dx$ can be performed exactly. However, the point is not so clear when considering $\int \ell^i_k \frac{\partial f(\tilde{u}_k)}{\partial x} dx$, as

$f(\tilde{u}_k)$ is not necessarily a polynomial of degree p. One way to overcome this difficulty is to consider $\widetilde{f(u)}_k$, the degree p polynomial approximation of $f(\tilde{u}_k)$ defined as

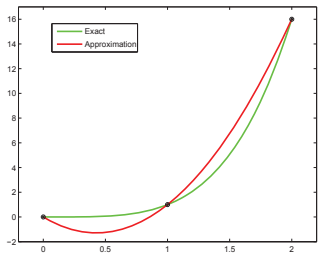
$$\widetilde{f(u)}_k = \sum_{i=0}^{p} f(\tilde{u}_k^i)\ell_k^i$$
$$\Leftrightarrow \quad \widetilde{f(u)}_k = [f(\tilde{u}_k^0) \quad f(\tilde{u}_k^1) \quad \cdots \quad f(\tilde{u}_k^p)\ ]^T$$

Now the integration can be performed exactly on the approximation of f. Another strategy would be to transform the term in the integral $\frac{\partial f(\tilde{u})}{\partial x}\ell = \frac{\partial f}{\partial u}\frac{\partial \tilde{u}}{\partial x}\ell$ and use more than $(p+1)/2$ quadrature points to get a better approximation of the integral. However, this strategy would be extremely expensive as a lot of new function evaluations would have to be done.

Here is an example illustrating the difference between $f(\tilde{u}_k)$ and $\widetilde{f(u)}_k$ in the case $p = 2$:

$\Omega_k = [0, 2] \quad x_k^0 = 0, \ \ x_k^1 = 1, \ \ x_k^2 = 2$

$$f : x \mapsto x^2$$
$$\tilde{u}_k = x^2$$
$$f(\tilde{u}_k) = x^4$$
$$\widetilde{f(u)}_k = 7x^2 - 6x$$



II. Burgers' Equation

The scalar Burgers' equation on $[a, b]$ is given by

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(\frac{u^2}{2}) = 0 \tag{1}$$

Again, the problem is only fully defined with an associated set of initial and boundary conditions. Multiplying (1) by u and integrating over the domain gives a global equation for the conservation of energy (assuming that there are no shocks)

$$\int_\Omega u\frac{\partial u}{\partial t} + u\frac{\partial}{\partial x}(\frac{u^2}{2})dx = 0$$
$$\Rightarrow \int_\Omega \frac{\partial}{\partial t}(\frac{u^2}{2})dx = \int_\Omega \frac{\partial}{\partial x}(\frac{u^3}{3})dx$$
$$\Rightarrow \boxed{\frac{\partial}{\partial t}\int_\Omega (\frac{u^2}{2})dx = \frac{u_a^3}{3} - \frac{u_b^3}{3}}$$

Now, let us consider the DG method in its general strong formulation. The problem is to find $u \in V$ (u piecewise H^1) such that for all $\phi \in V$ and for all element k,

$$\int_{\Omega_k} \frac{\partial u_k}{\partial t}\phi_k + \frac{\partial}{\partial x}(\frac{u_k^2}{2})\phi_k \ dx = \left[\frac{u_k^2(x_{k+1})}{2} - \left(\frac{u^2}{2}\right)^\star_{k,k+1}\right]\phi_k(x_{k+1}) - \left[\frac{u_k^2(x_k)}{2} - \left(\frac{u^2}{2}\right)^\star_{k-1,k}\right]\phi_k(x_k)$$

Since $u \in V$, we can substitude ϕ by u to obtain

$$\int_{\Omega_k} \frac{\partial u_k}{\partial t}u_k + \frac{\partial}{\partial x}(\frac{u_k^2}{2})u_k \ dx = \left[\frac{u_k^2(x_{k+1})}{2} - \left(\frac{u^2}{2}\right)^\star_{k,k+1}\right]u_k(x_{k+1}) - \left[\frac{u_k^2(x_k)}{2} - \left(\frac{u^2}{2}\right)^\star_{k-1,k}\right]u_k(x_k)$$

To make things clearer, we introduce the notation

$$u_k^- = u_{k-1}(x_k)$$
$$u_k^+ = u_k(x_k)$$

We then have

$$\int_{\Omega_k} \frac{\partial u_k}{\partial t} u_k + \frac{\partial}{\partial x}(\frac{u_k^2}{2}) u_k \; dx = \left[\frac{{u_{k+1}^-}^2}{2} - \left(\frac{u^2}{2}\right)^\star_{k,k+1}\right] u_{k+1}^- - \left[\frac{{u_k^+}^2}{2} - \left(\frac{u^2}{2}\right)^\star_{k-1,k}\right] u_k^+$$

$$\Leftrightarrow \int_{\Omega_k} \frac{\partial}{\partial t}(\frac{u_k^2}{2}) + \frac{\partial}{\partial x}(\frac{u_k^3}{3}) \; dx = \left[\frac{{u_{k+1}^-}^2}{2} - \left(\frac{u^2}{2}\right)^\star_{k,k+1}\right] u_{k+1}^- - \left[\frac{{u_k^+}^2}{2} - \left(\frac{u^2}{2}\right)^\star_{k-1,k}\right] u_k^+$$

$$\Leftrightarrow \int_{\Omega_k} \frac{\partial}{\partial t}(\frac{u_k^2}{2}) \; dx + \frac{{u_{k+1}^-}^3}{3} - \frac{{u_k^+}^3}{3} = \left[\frac{{u_{k+1}^-}^2}{2} - \left(\frac{u^2}{2}\right)^\star_{k,k+1}\right] u_{k+1}^- - \left[\frac{{u_k^+}^2}{2} - \left(\frac{u^2}{2}\right)^\star_{k-1,k}\right] u_k^+$$

$$\Leftrightarrow \int_{\Omega_k} \frac{\partial}{\partial t}(\frac{u_k^2}{2}) \; dx = \left[\frac{{u_{k+1}^-}^2}{6} - \left(\frac{u^2}{2}\right)^\star_{k,k+1}\right] u_{k+1}^- - \left[\frac{{u_k^+}^2}{6} - \left(\frac{u^2}{2}\right)^\star_{k-1,k}\right] u_k^+$$

Now, summing over the entire domain gives

$$\frac{\partial}{\partial t}\int_\Omega (\frac{u^2}{2}) \; dx = \sum_{k=0}^{N-1} \left[\frac{{u_{k+1}^-}^2}{6} - \left(\frac{u^2}{2}\right)^\star_{k,k+1}\right] u_{k+1}^- + \frac{u_b^3}{6} - \left(\frac{u^2}{2}\right)^\star_b u_b$$
$$- \sum_{k=1}^{N} \left[\frac{{u_k^+}^2}{6} - \left(\frac{u^2}{2}\right)^\star_{k-1,k}\right] u_k^+ - \frac{u_a^3}{6} - \left(\frac{u^2}{2}\right)^\star_a u_a$$

Let us take

$$\left(\frac{u^2}{2}\right)^\star_a = \frac{u_a^2}{2}$$
$$\text{and} \quad \left(\frac{u^2}{2}\right)^\star_b = \frac{u_b^2}{2}$$

and change the bounds of the first summation

$$\frac{\partial}{\partial t}\int_\Omega (\frac{u^2}{2}) \; dx = \frac{u_a^3}{3} - \frac{u_b^3}{3} + \sum_{k=1}^{N} \left[\left(\frac{u^2}{2}\right)^\star_{k-1,k} (u_k^+ - u_k^-) + \frac{{u_k^+}^3}{6} - \frac{{u_k^-}^3}{6}\right]$$
$$= \frac{u_a^3}{3} - \frac{u_b^3}{3} + \sum_{k=1}^{N} \left[(u_k^+ - u_k^-)\left(\left(\frac{u^2}{2}\right)^\star_{k-1,k} - \left(\frac{{u_k^-}^2 + u_k^- u_k^+ + {u_k^+}^2}{6}\right)\right)\right]$$

If we take $\left(\frac{u^2}{2}\right)^\star_{k-1,k} = \left(\frac{{u_k^-}^2 + u_k^- u_k^+ + {u_k^+}^2}{6}\right)$, then we recover the correct conservation of energy for the Burgers equation solved using DG, for the summation terms vanish. This flux can be used in an actual implementation of the DG method but one should remember that then, we only have an approximate conservation of energy. Indeed, as mentioned before, u does not satisfy exactly the equations as the term $\int_k \ell_i^k \frac{\partial}{\partial x} f(u) dx$ is evaluated only approximatively.

We stated earlier that we assumed the solution u was shock free and smooth. If a shock occurs in the solution, the global conservation of energy is modified as each jump dissipates energy at the rate $-\frac{1}{12}(u_L - u_R)^3$, where u_L and u_R are the states on the left side and right side of the shock ($u_L > u_R$). However the idea is that if kinetic energy is conserved in a shock-less case, stability of computations should be improved.

Therefore, we expect that this extra stability should allow us to perform viscous computations without the need of adding shock capturing operators.

III. Euler's Equations

The 1D Euler equations are given on $\Omega = [a, b]$ by

$$\frac{\partial w}{\partial t} + \frac{\partial}{\partial x} f(w) = 0 \tag{2}$$

$$\text{where} \quad w = \begin{bmatrix} \rho \\ \rho u \\ \rho E \end{bmatrix}, \quad \text{and} \quad f(w) = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ u(\rho E + p) \end{bmatrix}$$

By combining the continuity and momentum equations, one can obtain an equation for kinetic energy $k = \frac{1}{2}\rho u^2$:

$$\begin{aligned} \frac{\partial k}{\partial t} &= \frac{\partial k}{\partial w} \cdot \frac{\partial w}{\partial t}, \quad \text{with} \quad \frac{\partial k}{\partial w} = k_{,w} = \begin{bmatrix} -\frac{u^2}{2} & u & 0 \end{bmatrix} \\ &= -\frac{\partial k}{\partial w} \cdot \frac{\partial f}{\partial x} \\ &= -\frac{\partial}{\partial x}\left[u\left(p + \rho\frac{u^2}{2}\right)\right] + p\frac{\partial u}{\partial x} \end{aligned}$$

Integrating this relation over the entire domain Ω gives a global conservation equation for kinetic energy:

$$\boxed{\frac{\partial}{\partial t}\int_\Omega k \; dx = \left[u\left(p + \rho\frac{u^2}{2}\right)\right]_a - \left[u\left(p + \rho\frac{u^2}{2}\right)\right]_b + \int_\Omega p\frac{\partial u}{\partial x} \; dx}$$

Now, we consider the general DG strong formulation to solve the Euler equations. We seek $w \in V^3$ such that for all $\phi \in V$ and for all elements k,

$$\int_{\Omega_k} \phi_k \left(\frac{\partial w_k}{\partial t} + \frac{\partial f(w_k)}{\partial x}\right) \; dx = \phi_k(x_{k+1}) \left[f(w_k(x_{k+1})) - f^\star_{k,k+1}\right] - \phi_k(x_k) \left[f(w_k(x_k) - f^\star_{k-1,k}\right]$$

where we assume that $f^\star_{k,k+1}$ takes the form

$$f^\star_{k,k+1} = \begin{bmatrix} (\rho u)^\star_{k,k+1} \\ (\rho u^2 + p)^\star_{k,k+1} \\ (u(\rho E + p))^\star_{k,k+1} \end{bmatrix}$$

Since $u \in V$ and $\frac{u^2}{2} \in V$, we have for all k

$$\int_{\Omega_k} k_{,w} \left(\frac{\partial w_k}{\partial t} + \frac{\partial f(w_k)}{\partial x}\right) \; dx = k_{,w}(x_{k+1}) \left[f(w_k(x_{k+1})) - f^\star_{k,k+1}\right] - k_{,w}(x_k) \left[f(w_k(x_k) - f^\star_{k-1,k}\right] \tag{3}$$

Again, to gain in clarity, we reintroduce the following compact notation:

$$w_k^- = w_{k-1}(x_k)$$
$$w_k^+ = w_k(x_k)$$

We can now expand both sides of equality (3):

- Left hand side

$$\begin{aligned}
\text{LHS} &= \int_{\Omega_k} k_{,w}\left(\frac{\partial w_k}{\partial t} + \frac{\partial f(w_k)}{\partial x}\right)\ dx \\
&= \int_{\Omega_k} \frac{\partial k_k}{\partial t} + \frac{\partial}{\partial x}\left[u_k\left(p_k + \rho\frac{u_k^2}{2}\right)\right] - p_k\frac{\partial u_k}{\partial x}\ dx \\
&= \int_{\Omega_k} \frac{\partial k_k}{\partial t}\ dx + \left[u\left(p+\rho\frac{u^2}{2}\right)\right]_{k+1}^- - \left[u\left(p+\rho\frac{u^2}{2}\right)\right]_k^+ - \int_{\Omega_k} p_k\frac{\partial u_k}{\partial x}\ dx
\end{aligned}$$

- Right hand side

$$\begin{aligned}
\text{RHS} =& k_{,w}(x_{k+1})\left[f(w_k(x_{k+1})) - f^\star_{k,k+1}\right] - k_{,w}(x_k)\left[f(w_k(x_k) - f^\star_{k-1,k}\right] \\
=& \left[u\left(\rho u^2+p\right) - u\Big((\rho u^2+p)^\star_{k,k+1}\Big)\right]_{k+1}^- - \left[\frac{u^2}{2}(\rho u) - \frac{u^2}{2}(\rho u)^\star_{k,k+1}\right]_{k+1}^- \\
& - \left[u\left(\rho u^2+p\right) - u\Big((\rho u^2+p)^\star_{k-1,k}\Big)\right]_k^+ + \left[\frac{u^2}{2}(\rho u) - \frac{u^2}{2}(\rho u)^\star_{k-1,k}\right]_k^+
\end{aligned}$$

Therefore, equating LHS and RHS gives

$$\begin{aligned}
\int_{\Omega_k}\frac{\partial k_k}{\partial t}\ dx - \int_{\Omega_k} p_k\frac{\partial u_k}{\partial x}\ dx =& -\cancel{\left[u\left(p+\rho\frac{u^2}{2}\right)\right]_{k+1}^-} + \cancel{\left[u\left(p+\rho\frac{u^2}{2}\right)\right]_k^+} \\
&+ \cancel{\left[u\left(\rho u^2+p\right)} - u\Big((\rho u^2+p)^\star_{k,k+1}\Big)\right]_{k+1}^- - \left[\cancel{\frac{u^2}{2}(\rho u)} - \frac{u^2}{2}(\rho u)^\star_{k,k+1}\right]_{k+1}^- \\
&- \cancel{\left[u\left(\rho u^2+p\right)} - u\Big((\rho u^2+p)^\star_{k-1,k}\Big)\right]_k^+ + \left[\cancel{\frac{u^2}{2}(\rho u)} - \frac{u^2}{2}(\rho u)^\star_{k-1,k}\right]_k^+ \\
=& \left[-u^-_{k+1}\left(\rho u^2+p\right)^\star_{k,k+1} + \frac{{u^-_{k+1}}^2}{2}(\rho u)^\star_{k,k+1}\right] \\
& - \left[-u_k^+\left(\rho u^2+p\right)^\star_{k-1,k} + \frac{{u_k^+}^2}{2}(\rho u)^\star_{k-1,k}\right]
\end{aligned}$$

Now summing over the entire domain gives

$$\begin{aligned}
\int_\Omega\frac{\partial k}{\partial t}\ dx - \sum_{k=1}^N\int_{\Omega_k} p_k\frac{\partial u_k}{\partial x}\ dx =& \sum_{k=0}^{N-1}\left[-u^-_{k+1}(\rho u^2+p)^\star_{k,k+1} + \frac{{u^-_{k+1}}^2}{2}(\rho u)^\star_{k,k+1}\right] + \left[-u_b(\rho u^2+p)^\star_b + \frac{{u_b}^2}{2}(\rho u)^\star_b\right] \\
& - \sum_{k=1}^N\left[-u_k^+(\rho u^2+p)^\star_{k-1,k} + \frac{{u_k^+}^2}{2}(\rho u)^\star_{k-1,k}\right] - \left[-u_a(\rho u^2+p)^\star_a + \frac{{u_a}^2}{2}(\rho u)^\star_a\right]
\end{aligned}$$

Let us take

$$\begin{aligned}
&\left(\rho u^2+p\right)^\star_a = \rho_a u_a^2 + p_a \\
\text{and}\quad &\left(\rho u^2+p\right)^\star_b = \rho_b u_b^2 + p_b
\end{aligned}$$

and change the bounds of the first summation on the right hand side of the relation:

$$\int_\Omega \frac{\partial k}{\partial t}\ dx - \sum_{k=1}^{N} \int_{\Omega_k} p_k \frac{\partial u_k}{\partial x}\ dx = \left[u\left(p+\rho\frac{u^2}{2}\right)\right]_a - \left[u\left(p+\rho\frac{u^2}{2}\right)\right]_b + \sum_{k=1}^{N}\left[-u_k^-(\rho u^2+p)^\star_{k-1,k} + \frac{u_k^{-2}}{2}(\rho u)^\star_{k-1,k} + u_k^+(\rho u^2+p)^\star_{k-1,k} - \frac{u_k^{+2}}{2}(\rho u)^\star_{k-1,k}\right]$$

Eventually, if we assume that $(\rho u^2+p)^\star_{k-1,k} = (\rho u^2)^\star_{k-1,k} + p^\star_{k-1,k}$, we obtain

$$\int_\Omega \frac{\partial k}{\partial t}\ dx = \left[u\left(p+\rho\frac{u^2}{2}\right)\right]_a - \left[u\left(p+\rho\frac{u^2}{2}\right)\right]_b + \left[\sum_{k=1}^{N}\int_{\Omega_k} p_k\frac{\partial u_k}{\partial x}\ dx + \sum_{k=1}^{N} p^\star_{k-1,k}(u_k^+ - u_k^-)\right] + \sum_{k=1}^{N}(u_k^+ - u_k^-)\cdot\left[(\rho u^2)^\star_{k-1,k} - (\rho u)^\star_{k-1,k}\left(\frac{u_k^+ + u_k^-}{2}\right)\right]$$

Hence, it is clear that if the continuity and momentum fluxes are chosen such that $(\rho u^2)^\star_{k-1,k} = (\rho u)^\star_{k-1,k}\left(\frac{u_k^+ + u_k^-}{2}\right)$, the last summation vanishes, leading to

$$\int_\Omega \frac{\partial k}{\partial t}\ dx = \left[u\left(p+\rho\frac{u^2}{2}\right)\right]_a - \left[u\left(p+\rho\frac{u^2}{2}\right)\right]_b + \left[\sum_{k=1}^{N}\int_{\Omega_k} p_k\frac{\partial u_k}{\partial x}\ dx + \sum_{k=1}^{N} p^\star_{k-1,k}(u_k^+ - u_k^-)\right]$$

The last term between the brackets "looks like" an approximation of $\int_\Omega p\frac{\partial u}{\partial x}$. However things are not that simple as p is not a continuous function on the domain Ω.

First, let us consider the simple case of a finite volume approximation (that is $V_\infty^h = V_0^h$ is the space of piecewise constant functions). The pressure term reduces to

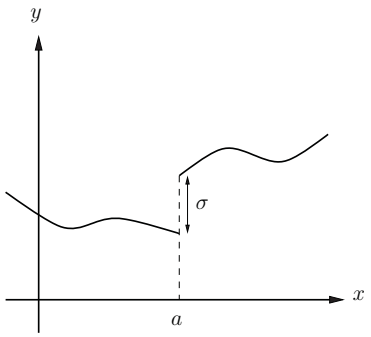
$$\sum_{k=1}^{N} p^\star_{k-1,k}(u_k - u_{k-1}) = \sum_{k=1}^{N} p^\star_{k-1,k}\left(\frac{u_k - u_{k-1}}{\Delta x}\right)\cdot\Delta x \approx \int_\Omega p\frac{\partial u}{\partial x}dx$$

This term corresponds to the integration of $p\frac{\partial u}{\partial x}$ on a grid staggered to the first one. Therefore, a reasonable choice would be to take $p^\star_{k-1,k} = \frac{1}{2}(p_{k-1}+p_k)$, an approximation of p_k on the staggered grid.

Now we consider the more general case of functions piecewise C^∞. We remember that, following the theory of distributions, if T_f is the distribution associated to the L^1 function f and if f is discontinuous in a:

$$T'_f = T_{f'} + \sigma\delta(a)$$

where δ is the dirac distribution and $\sigma = f(a^+) - f(a^-)$ the amplitude of the jump of function f in a.



It follows that if p is C^∞ on Ω we have exactly

$$\sum_{k=1}^{N} \int_{\Omega_k} p_k \frac{\partial u_k}{\partial x} \, dx + \sum_{k=1}^{N} p(x_k)(u_k^+ - u_k^-) = \int_\Omega p \frac{\partial u}{\partial x} dx$$

However, since p is discontinuous at each x_k, the relation is not valid and we only have an approximation of $\int_\Omega p \frac{\partial u}{\partial x} dx$. To remain consistent with the finite volume case, we will take $p^\star_{k-1,k} = \frac{1}{2}(p_k^- + p_k^+)$.

Eventually, we have

$$\boxed{\int_\Omega \frac{\partial k}{\partial t} \, dx \quad |exact \approx \int_\Omega \frac{\partial k}{\partial t} \, dx \quad |DG \;\; theoretical}$$

provided there are no shocks in the solution.

However the approximation does not only come from the pressure term. In the above demonstration, we assumed that all the integrations were performed exactly. When considering an actual implementation of DG, using piecewise polynomials of degree p, nothing guarantees that integrations will be exact (or at best we will have exact integrals of approximate functions, as explained earlier in this document). Also we assumed in the proof that $k_w \in V$. It is actually not true when considering $V = V_p^h$. Indeed, $u = \frac{\rho u}{\rho}$ is rational, not polynomial.

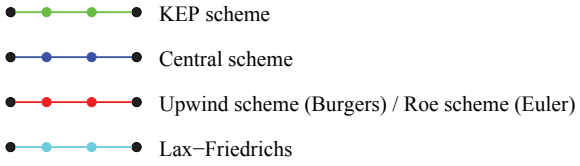
$$\boxed{\int_\Omega \frac{\partial k}{\partial t} \, dx \quad |exact \approx \int_\Omega \frac{\partial k}{\partial t} \, dx \quad |DG \;\; theoretical \approx \int_\Omega \frac{\partial k}{\partial t} \, dx \quad |DG \;\; implemented}$$

Numerical Results

It is interesting to test the Kinetic Energy Preserving (KEP) flux in a situation where a central flux is not expected to perform well. In what follows, we consider a viscous sod shocktube and compare the behavior of the regular central flux, the KEP flux, the Roe flux and the Lax-Friedrichs-Rusanov flux in our DG code. Viscous fluxes are computed using an Upwind/Downwind approach as described in the book by Hestaven[?]. All the simulations are performed without the addition of filtering or shock capturing operators to be able to clearly identify the effect of the fluxes. This is also the reason why we consider a (slightly) viscous test case and not an inviscid one. As the mesh is refined and the order of the interpolating polynomials increased, we expect to fully capture the continuous viscous solution even though no special operator is added.

The behavior of the various schemes applied to the Navier Stokes equations depends directly on the mesh's finess (in what follows, a fine mesh is considered to have a characteristic length of the order of the viscous shock thickness) and the order of the polynomials used in the DG approximation (a low order typically refering to the finite volume approximation $p = 0$). Therefore, we study 4 configurations : *coarse mesh-low order*, *coarse mesh-high order*, *fine mesh-low order*, *fine mesh-high order*.

In all that follows, the plots will have the same legend that we shall give here to keep the figures uncluttered



Viscous Sod shocktube

The viscous Sod shocktube is defined on $\Omega = [0, 1]$ by a left ($x \in [0, .5[$) and a right ($x \in]. 5, 1]$) initial states:

$$p_L = 1 \qquad p_R = .1$$
$$\rho_L = 1 \qquad \rho_R = .125$$
$$u_L = 0 \qquad u_R = 0$$
$$T_L = 300K \qquad T_R = 300K$$

The case was considered at a Reynolds number $Re = \frac{\rho L V_L}{\mu(T=T_L)} = 25000$.

The following results are compared with a reference solution, obtained using $N = 800$ elements, $p = 5^{th}$ order polynomial approximation and the Lax-Friedrichs flux (black solid line).

For every run, the solution was driven explicitely in time using a TVD-RK3 time stepping scheme.

1. Coarse mesh - Low order

We first consider the solutions obtained for $N = 100$ and $p = 0$ (Finite volume approximation) at $t = 0.15$. Results are presented in figure 1. Visually and as expected for a simulation involving shockwaves, the diffusive fluxes give better results. This comes from the fact that both central schemes (classical central and KEP) are not able to capture the shock and generate important oscillations around it. First order diffusive methods are stable but their solutions are extremely dissipated and features of the flow are largely smeared. We can see on figure 2 that the total kinetic energy computed using these methods is well below the reference in black. On the other hand, the KEP scheme, although very oscillatory on such a coarse mesh is much more stable than the regular central scheme. For any time step, the central flux simulation blew up while it was possible to drive the KEP solution to the end (oscillatory results, but did not "explode"). Another very interesting point is that the estimation of total kinetic energy computed using the KEP scheme is much closer to the reference solution on figure 2. This is a bit surprising since conservation of kinetic energy is not enforced in the presence of a shockwave and the solution is very oscillatory.

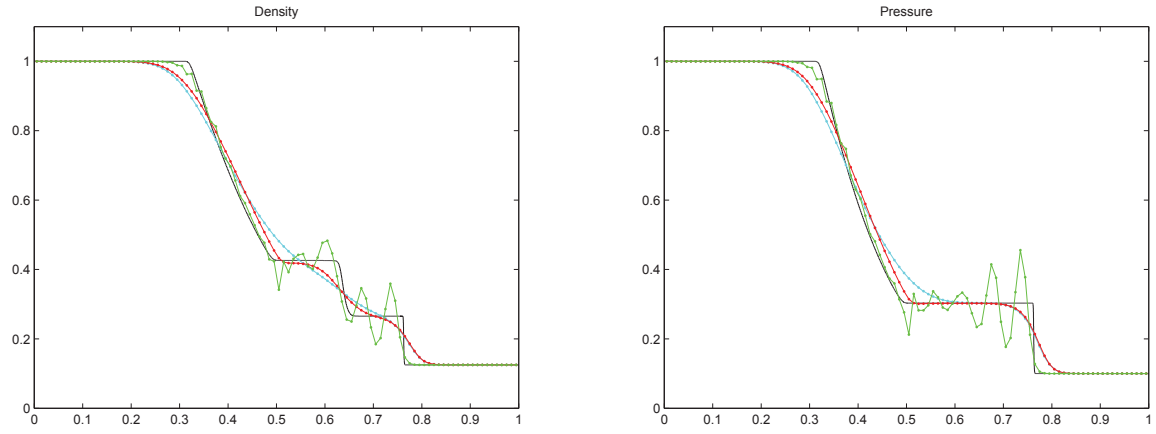


Figure 1. Viscous shocktube, coarse mesh ($N = 100$) and low order ($p = 0$)

2. Coarse mesh - High order

We condider a mesh of $N = 100$ elements and a polynomial interpolation of degree $p = 5$. Again, diffusive fluxes are performing the best (see figures 3 and 4). Although some oscillations are created in the vicinity of the shockwave (Gibbs phenomenon), first order diffusive fluxes provide enough dissipation to prevent the propagation of these wiggles to the rest of the solution. On the contrary, the KEP central flux and the regular central flux are unable to stop the propagation of these wiggles to the rest of the flow field. These oscillations

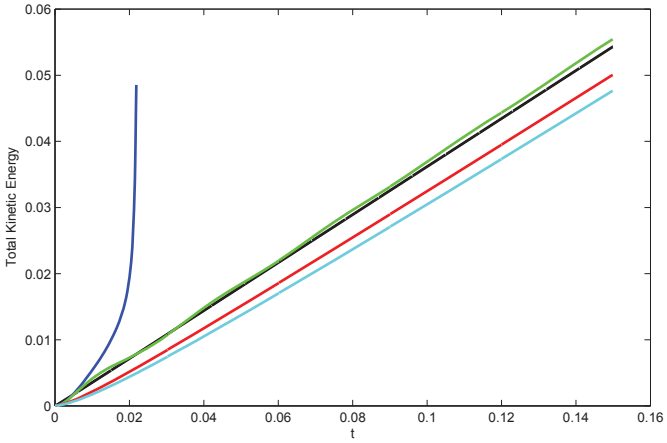


Figure 2. Viscous shocktube, Evolution of Kinetic Energy, coarse mesh ($N = 100$) and low order ($p = 0$)

are the exact analog of odd/even decoupling for finite volumes and finite differences. A DG method with first order diffusive fluxes will eliminate these odd/even modes while a DG method implemented with a central scheme is blind to them once they are created. However we still notice that the KEP central scheme behaves in a better fashion than the central scheme, especially at the begining of the expansion.

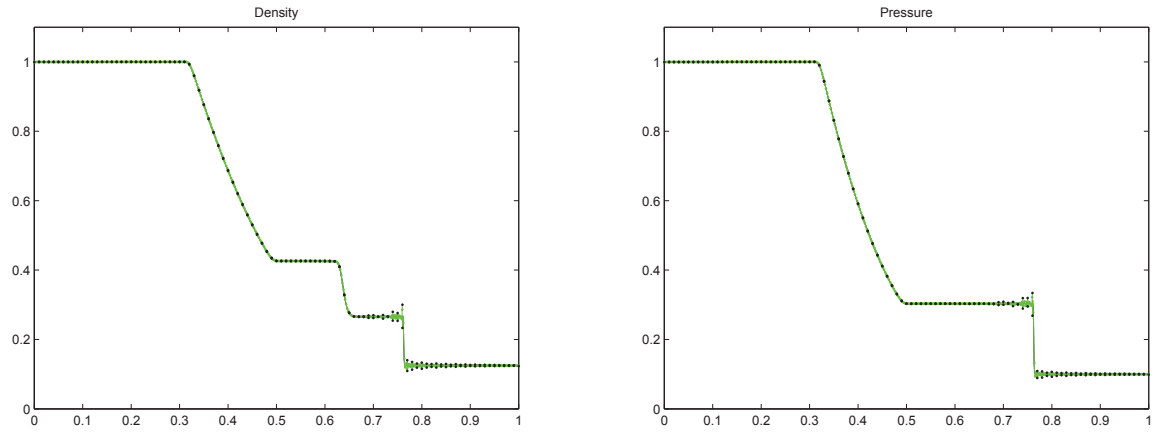
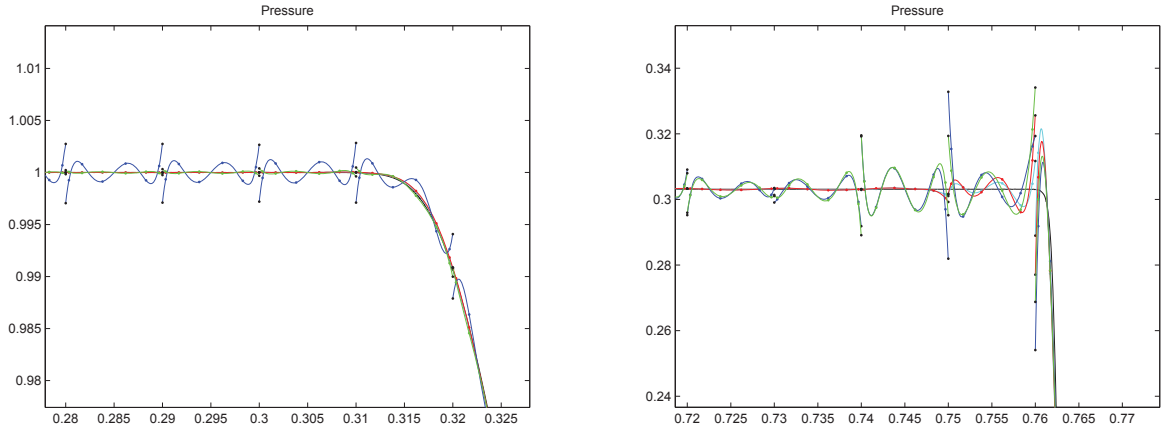


Figure 3. Viscous shocktube, coarse mesh ($N = 100$) and high order ($p = 5$), KEP flux

3. Fine mesh - Low order

Again we consider a finite volume approximation of the Euler equations but this time for $N = 4000$ cells. With such a large number of cells, we are able to capture the viscous structure of the shock and all schemes are expected to be oscillations free. As can be seen on figure 5 and 6-3, central fluxes give a sharper shock profile than first order diffusive fluxes, but they lead to a very small overshoot. When looking at the start of the expansion on figure 6-1, central fluxes also provide a result closer to the solid black line of the reference solution. This result is expected since central fluxes lead to 2nd order accurate schemes on uniform meshes while Lax-Friedrichs-Rusanov and Roe schemes remain 1st order. The interesting result is in the middle of the expansion, figure 6-2. The regular central flux exhibits a large amount of spurious oscillations that disappear completely when using the KEP flux.

This situation (fine mesh, finite volumes) is probably where the KEP scheme is the best. It is much cheaper than a classical Roe scheme yet it leads to non oscillatory solutions that are better than the ones



Start of the expansion Shockwave

Figure 4. Details of the Pressure distribution, coarse mesh ($N = 100$) and high order ($p = 5$)

obtained with a regular central scheme on a fine mesh. It is possible to obtain a second order solution on a large number of points in the time it would take to obtain a 1st order solution using a Roe scheme on a smaller number of points.

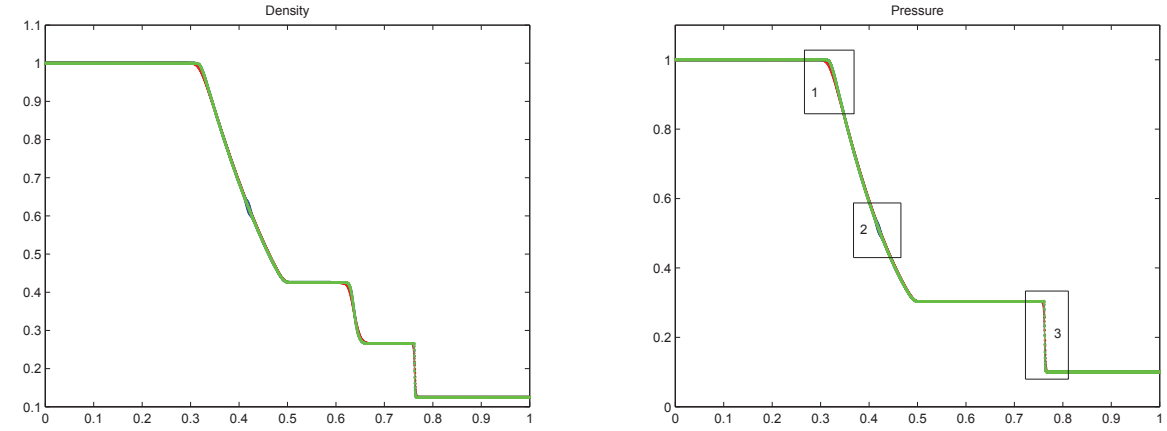


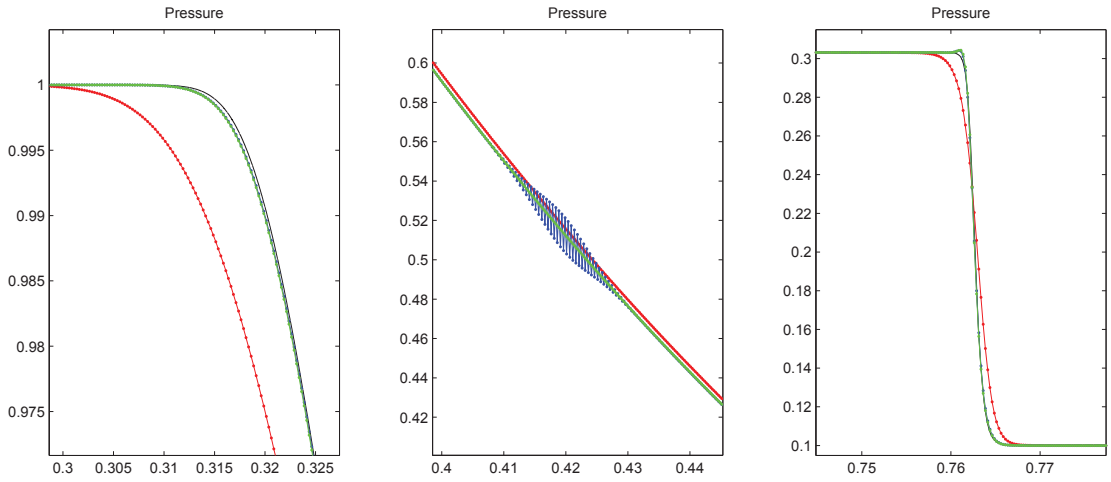
Figure 5. Viscous shocktube, fine mesh ($N = 4000$) and low order ($p = 0$)

4. Fine mesh - High order

Now, simulations are performed with $N = 800$ cells and $p = 5$. Results using the regular central scheme can be seen on figure 7. As expected, all the schemes behave the same way and all the solutions are superimposed. This comes from the fact that all the features of the flow are fully resolved (about one and a half cell through the shock) and that there are almost no discontinuity between the cells (due to the fact that interpolating polynomials are of high order). Therefore, since all the fluxes considered are consistent, they all provide the same results. The DG method is "almost continuous everywhere".

IV. Discussion - Conclusion

From the presented results, it appears that the Kinetic Energy Conserving scheme gives the best results when used for $p = 0$, i.e. for the finite volumes method. The interest is that on a regular mesh, the KEP flux



1- Start of the expansion 2- Expansion 3- Shockwave

Figure 6. Details of the Pressure distribution, fine mesh ($N = 4000$) and low order ($p = 0$)

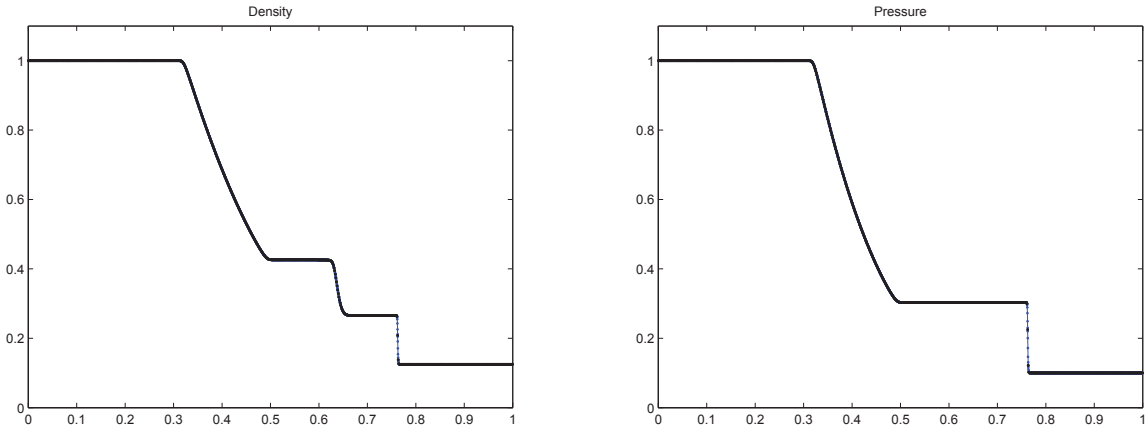


Figure 7. Viscous shocktube, fine mesh ($N = 800$) and high order ($p = 5$), central flux

leads to a 2^{nd} order accurate scheme that is clearly more stable than regular central scheme while also being more accurate than classical first order Riemann solvers. This difference in accuracy and stabily translates in better results in many simulation where the artificial dissipation needed to solve flow in a stable fashion deteriotates greatly the solution[2]. High resolution finite volume schemes however tend to have the same qualities as our Kinetic Energy Conserving scheme while being more expensive numerically. For viscous flows, the idea is that by using the KEP flux in a finite volume code and on a regular mesh, it is possible to obtain a 2^{nd} order accurate non oscillatory solution on a large number of points in the time it would take other high resolution schemes to solve the same problem on a much smaller set of points.

When considering higher order DG schemes, the order of the resulting scheme does not depend on the choice of the flux but only on the order of the approximation polynomials. The Kinetic Energy Conserving scheme is therefore of the same order as when using an approximate Riemann solver yet is less stable. This is mainly due to the fact that Riemann solvers are able to block spurious oscillations across interfaces, preventing them to spread over the entire computational domain. Central schemes on the other hand cannot eliminate the odd/even decoupling resulting from the Gibbs phenomenon at discontinuities. Although results are better than when using a regular central scheme, it should be noted that no actual DG implementation uses a central scheme for the numerical flux. The interest of the method is therefore more theoretical and

the flux can be used for mathematical purposes (proofs of energy stability often require a flux that conserve some energy quantity, see for example the proof of stability of DG by Jameson[6] or the Energy Stable Flux Reconstruction method by Vincent[7]).

Finally, the choice made to evaluate $p^\star$ in the momentum flux of the Euler equation is quite arbitrary (based uniquely on the result for $p = 0$) and maybe some improvements can be made. Also, it is possible to add a term to the KEP momentum flux so as to obtain a Kinetic Energy Decreasing scheme that might improve the stability of the scheme and reduce odd/even decoupling:

$$(\rho u^2)^\star_{k-1,k} = \frac{1}{2}(\rho u)^\star_{k-1,k}(u^+_k + u^-_k) - \frac{1}{2}\alpha|(\rho u)^\star_{k-1,k}|(u^+_k - u^-_k) \ , \quad \alpha \in [0,1]$$

The extension to multidimensions of all the results presented in this paper is straightforward and lead to the exact same fluxes. It should also be noted that the extension to any scalar conservation law can be made following the work of Jameson[4,5].

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