Advances in Bringing High-order Methods to Practical Applications in Computational Fluid Dynamics

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This article surveys recent developments in the design and analysis of high order discretization methods for unstructured meshes, with an emphasis on Huynh's flux reconstruction method, and energy stability. The article also highlights some open issues with respect to energy stability for nonlinear problems, and methods for time-integration. Some representative results of simulations of transitional and turbulent flows are included to illustrate the current state of the art.

I. Introduction

Following pioneering research by mathematicians such as Lax and Godunov, the early development of computational fluid dynamics (CFD) between 1970 and 1995 was particularly pursued in the Aerospace Community, driven by the compelling need for robust and accurate methods to predict compressible flows with shock waves in the transonic, supersonic and hypersonic regimes. While satisfactory methods of solving the transonic potential flow equation for realistic geometries were developed between 1970 and 1980, it proved harder to find accurate and efficient methods for solving the Euler and Navier Stokes equations. A major achievement of this early period was the development of "high resolution" shock capturing schemes, which were second order accurate in regions of smooth flow and provably non-oscillatory across shock waves at the expense of reverting to first order accuracy in their vicinity. For a review of salient aspects of these developments see van Leer.¹ Many of them were closely associated with the golden years of ICAE at NASA Langley which brought together many of the leading researchers. The methodology of high resolution schemes has spread out to other areas of science where there is a need to capture discontinuous fronts. A notable example is astrophysics where these methods have been applied to simulations of galaxy formation and violent events such as galactic jets.

The parallel development of methods for simulating incompressible viscous and turbulent flows in a broad range of industrial applications gave birth to the commercial CFD industry. CFD is now ubiquitous in myriad applications, and it is routinely used in the engineering design of numerous products including cars, ships, aircraft and turbo-machinery. For the last 15 years however, it has been on a plateau where simulations have been based on the Reynolds averaged Navier Stokes (RANS) equations.² The current state of the art is exemplified by steady RANS simulations of Formula 1 racing cars. Typically these are performed on hybrid unstructured meshes with around 100 million polyhedral cells including prisms, hexahedra and tetrahedra and pyramids. The algorithms used for these simulations are normally second order accurate. In reality turbulent flows over such complex shapes are both separated and unsteady, often dominated by vortex interactions. It turns out that second order accurate algorithms are generally too dissipative to resolve vortices as they are convected downstream more than a very short distance (a few chord lengths behind a wing or rotor blade, for example). Accordingly it has emerged that while correct RANS methods can provide sufficient accuracy to enable the low risk design of a wing in the cruise condition, they are quite inadequate for the design of rotorcraft or high lift systems, or a VTOL aircraft in the vicinity of the ground. Accurate and reliable simulations of such flows requires advancing to the higher level of fidelity provided by large eddy simulation (LES). It has been long been known that this level of fidelity will ultimately be needed to enable low risk engineering design,³ but hitherto LES has been limited by computational costs to simplified model problems at, by industrial standards, rather low Reynolds numbers. The very rapid ongoing advances in

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computer hardware now provide the opportunity to make the next step to LES of realistic and complete configurations.

The foundation of much modern research on high-order methods stems from a series of papers by Cockburn and Shu,⁴⁻⁸ in which they reformulated the discontinuous Galerkin (DG) method in order to treat conservation laws of the form

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} f(u) \quad (1)$$

where u is the solution and $f(u)$ is the flux. Their formulation provides a theoretical basis for rigorous stability proofs and error estimates, while accommodating the techniques of upwind and Godunov type scheme in a very natural manner through the use of Riemann or approximate Riemann solutions to resolve the discontinuities in the numerical solution at the cell interfaces. The computational complexity of this approach grows quite rapidly with the order of the scheme, and this has motivated the development of a variety of approaches which aim to eliminate the cost of repeated quadratures or otherwise reduce the computational complexity. A particularly simple and efficient range of DG schemes utilize high-order Lagrange polynomial basis functions inside each element with the solution defined by values at the corresponding distinct nodal points. With each element mapped to a universal reference element the required quadratures can be pre-calculated and stored, reducing the computational operations other than the Riemann solutions to a sequence of matrix vector multiplications within each element. An exposition of nodal DG (NDG) schemes of this type can be found in the recent textbook by Hesthaven and Warburton,⁹ as well as various articles by the same authors. Wang's spectral volume scheme¹⁰ is another approach which has proved successful, but still suffers a rapid growth of complexity with increasing order.

In the last few years spectral difference (SD) methods, which directly approximate the differential form of the equations, have emerged as a promising alternative. The foundation of such schemes was first put forward by Kopriva and Kolias¹¹ in 1996 under the name 'staggered grid chebyshev multi-domain method'. In 2006 Liu, Vinokur and Wang^{12,13} presented a more general formulation for both triangular and quadrilateral elements which they called the SD method, and this name has been generally accepted. In SD methods the solution u is represented in each element by a polynomial of degree p defined by values at $p + 1$ interior nodes, while the flux $f(u)$ is represented by a polynomial of degree $p + 1$ defined by values of $f(u)$ at p interior nodes interspersed with the solution nodes and values at the left and right boundaries of the element defined by an exact or approximate Riemann solution for the discontinuity between the element and the left or right neighbor.

In 2007 Huynh first presented his flux reconstruction (FR) method,^{14,15} which further simplifies the treatment of the equations in differential form. Instead of calculating the flux at a separate set of flux collocation points he proposed simply to modify the flux $f(u)$ calculated from the solution at the interior nodal points by corrections from the left and right boundaries based on the difference between the Riemann flux $\hat{f}$ at the interface and the value $f(u)$ calculated from the internal solution polynomial in the element. These corrections are propagated from each boundary by polynomials of degree $p + 1$ which vanish at the opposite boundary. Thus the corrected flux is represented by a polynomial of degree $p + 1$, so that its derivative $\frac{\partial f}{\partial x}$ is a polynomial of degree p , consistent with the polynomial representing the solution. For the linear case Huynh was able to show that by appropriate choices of the correction polynomials he could recover both the standard NDG scheme and the SD scheme as well as a variety of hitherto unexplored variations which might have some potential advantages. He also used Fourier analysis to verify the stability of some of these schemes for third order accuracy. In the non-linear case the FR schemes with appropriate correction polynomials are no longer exactly equivalent to the corresponding NDG and SD schemes. However the FR methodology provides a rich framework for the design of high-order schemes of minimal complexity.

Sections II-V of this article review recent developments in the formulation and analysis of high order schemes for unstructured meshes, with an emphasis on the flux reconstruction method and energy stability. In comparison with the Fourier method of stability analysis, the energy method is general and rigorous, enabling proofs of stability for all orders of accuracy for entire classes of schemes. Moreover these proofs are valid for non-uniform meshes. While the Fourier method requires a separate analysis for each particular case, and normally assumes a uniform mesh, it has the advantages that it provides more detailed information about the distribution of dispersive and diffusive errors, and it also enables the identification of the property of super accuracy for linear problems. Section VI and VII present the results of some representative high order simulations of transitional and turbulent flows, while illustrate the current state of the art.

For a more complete survey of high order methods for unstructured meshes the reader is referred to the recent articles by Wang,¹⁶ and Vincent and Jameson.¹⁷

II. Review of Stability Analysis for High-order Schemes

Here the overall ideas behind the energy estimate of high order methods and the flux reconstruction scheme are sketched. The details of each scheme can be found in Hesthaven,⁹ Jameson¹⁸ and Huynh¹⁴ respectively. Here we consider the linear advection equation:

$$\frac{\partial u}{\partial t} + a \frac{\partial u}{\partial x} = 0$$

The energy estimate for linear advection is obtained by multiplying the equation by u and integrating over the domain,

$$\int_a^b u \left(\frac{\partial u}{\partial t} + a \frac{\partial u}{\partial x} \right) dx = 0$$

to give

$$\frac{d}{dt} \int_a^b \frac{u^2}{2} dx = \frac{1}{2} a (u_a^2 - u_b^2)$$

II.A. Energy Stability Proof for NDG

The following section outlines the proof of the stability of the DG method for the linear advection equation in the energy norm

$$||u|| = \int \left(\frac{u^2}{2} \right) dx$$

Nodal DG Discretization for Linear Advection

As a first step, we derive the nodal DG discretization for the linear advection equation in strong form. In nodal DG, linear advection is discretized by introducing n collocation points x_j in each element and define the local solution by the Lagrange polynomial of degree $p = n - 1$.

$$u_h^k = \sum_{j=1}^n u_j l_j(x)$$

where u_h^k is the discrete solution in element k . For convenience, the elements have been transformed to reference elements covering the interval $[-1,1]$. At the same time, the discrete residual in the element k is written as

$$R_h^k = \frac{\partial u_h^k}{\partial t} + \frac{\partial f(u_h^k)}{\partial x}$$

The Galerkin method requires the residual of the equation to be orthogonal to the basis functions. The strong form for the nodal Galerkin can be obtained by firstly integrating by parts to obtain the weak form, and then integrating the middle term of the weak form back by parts. The final result is

$$\int_{-1}^1 \frac{\partial u_h^k}{\partial t} l_j dx + \int_{-1}^1 l_j \frac{\partial f(u_h^k)}{\partial x} dx + [\hat{f} - f(u_h^k)] l_j \Big|_{-1}^1 = 0$$

where $\hat{f}$ is the numerical flux at the element interface. The same equation can be cast in a matrix form by defining the reference mass and stiffness matrices

$$\mathbf{M}^k \frac{d\mathbf{u}}{dt} + \mathbf{S}^k \mathbf{f} + \mathbf{l}(\hat{f} - f) \Big|_{-1}^1 = 0 \quad \text{where} \quad \mathbf{M}^k_{ij} = \int_{-1}^1 \frac{(x_R - x_L)}{2} l_i l_j dx \quad \text{and} \quad \mathbf{S}^k_{ij} = \int_{-1}^1 l_i l'_j dx$$

For the purpose of better illustrating the SD and Flux Reconstruction energy stability proof in the later part of the paper, we follow the flux reconstruction pocedure proposed by Huynh and rewrite the flux at each boundary as

$$\hat{f}(-1) = a u_h(-1) + f_{CL}, \quad \hat{f}(1) = a u_h(1) + f_{CR}$$

where f_{CL} and f_{CR} are boundary corrections

$$f_{CL} = \hat{f}(-1) - au_h(-1), \quad f_{CR} = \hat{f}(1) - au_h(1)$$

With these, the strong form of the nodal DG scheme can alternatively be expressed as

$$\mathbf{M}^k \frac{d\mathbf{u}}{dt} + \mathbf{S}^k \mathbf{f} - f_{CL}l(-1) - f_{CR}l(1) = 0$$

Energy Stability Proof for Nodal DG

In Hesthaven and Warburton,⁹ the stability of the nodal discontinuous Galerkin method has been proved for the linear advection equation, by showing that the solution discretized by nodal DG satisfies a similar estimate as the continuous linear advection, shown previously. The proof is sketched as follows. Taking the strong form of the discretized linear advection equation, and multiply it with the local solution to obtain

$$\mathbf{u}^T \mathbf{M}^k \frac{d\mathbf{u}}{dt} + \mathbf{u}^T \mathbf{S}^k \mathbf{f} + \mathbf{u}^T \mathbf{l}(\hat{f} - f) \Big|_{-1}^1 = 0$$

Since M^k and S^k have been pre-integrated exactly, this is equivalent to

$$\frac{d}{dt} \int_{x_L}^{x_R} \frac{u_h^2}{2} dx + a \int_{x_L}^{x_R} u_h \frac{\partial u_h}{\partial x} dx + u_h(\hat{f} - au_h) \Big|_{x_L}^{x_R} = 0$$

where the superscript k denoting the element has been omitted to simplify the notation. Here the middle term can be integrated and combined with the last term

$$\frac{d}{dt} \int_{x_L}^{x_R} \frac{u_h^2}{2} dx = - \left(u_h \hat{f} - a \frac{u_h^2}{2} \right) \Big|_{x_L}^{x_R} \quad (2)$$

Now on summing over the elements, energy stability follows if the contributions from each interior interface are strictly non-positive, while proper inflow and out flow boundary conditions are enforced at the left and right boundaries.

Let u_L and u_R be values of u_h on the left and right sides of a cell interface. For the numerical flux we now take

$$\hat{f} = \frac{1}{2}a(u_R + u_L) - \frac{1}{2}\alpha|a|(u_R - u_L), 0 \leq \alpha \leq 1$$

where if $\alpha = 0$ we have a central flux, and if $\alpha = 1$ we have the upwind flux. Now on summing (2) over the elements, the left side yields $\frac{d}{dt} \int_a^b \frac{u_h^2}{2} dx$, while at each interior interface, collecting the contributions from the elements on the left and right sides, there is a total contribution

$$\begin{aligned} & u_R \hat{f} - a \frac{u_R^2}{2} - (u_L \hat{f} - a \frac{u_L^2}{2}) \\ &= \frac{1}{2}a(u_R^2 - u_L^2) - \frac{1}{2}\alpha|a|(u_R - u_L)^2 - \frac{1}{2}a(u_R^2 - u_L^2) \\ &= -\frac{1}{2}\alpha|a|(u_R - u_L)^2 \end{aligned}$$

If we set the numerical flux to the true value au_a at the inflow boundary, and to the extrapolated upwind value au_h at the outflow boundary, it now follows that there is a negative contribution at every element boundary except the inflow boundary, where the contribution is

$$au_a u_h - \frac{1}{2}au_h^2 = \frac{1}{2}au_a^2 - \frac{1}{2}a(u_a - u_h)^2$$

which is strictly less than the boundary contribution $a \frac{u_a^2}{2}$ in the true solution. Thus the use of an upwind biased numerical flux is sufficient to ensure the stability of the DG scheme in an energy norm.

II.B. Energy Stability Proof for SD

The analysis in this section outlines the proof of the stability of the SD method. Because of the difference between the SD method and the nodal DG method, the proof that the SD method is stable for 1D linear advection equation for all orders of accuracy is established in an energy norm of Sobolev type. Specifically using solution polynomials of degree p , which are expected to yield accuracy of order $p + 1$, the norm is

$$||u|| = \int \left(u^2 + c \left(\frac{\partial^p u}{\partial x^p} \right)^2 \right) dx$$

where the coefficient c will be discussed later.

Spectral Difference Discretization for Linear Advection

Like the nodal DG scheme, in the SD scheme the discrete solution is locally represented by Lagrange polynomial on the solution collocation points x_j as

$$u_h = \sum_{j=1}^n u_j l_j(x)$$

where for polynomials of degree p , $n = p + 1$. u_h is the discrete solution in a reference element spanning $[-1, 1]$. However, in contrast to the nodal DG scheme, the flux is represented by a separate Lagrange polynomial, $\hat{l}_j(x)$, of degree $p + 1$, defined by the $n + 1$ flux collocation points $\hat{x}_j$

$$f_h = \sum_{j=1}^{n+1} f_j \hat{l}_j(x)$$

For this discrete flux, the interior values at the flux collocation points f_j are set equal to $f(u_h(\hat{x}_j))$ where $u_h(\hat{x}_j)$ is interpolated from $u_h(x)$. At the element boundaries $f(-1)$ and $f(1)$ are defined to be the single valued numerical flux $\hat{f}$. Following the flux reconstruction procedure proposed by Huynh and rewriting the boundary flux in terms of boundary corrections f_{CL} and f_{CR} , the discrete flux can be expanded as

$$f_h(x) = f_{CL} \hat{l}_1(x) + \sum_{j=1}^{n+1} f_j \hat{l}_j(x) + f_{CR} \hat{l}_{n+1}(x)$$

For linear advection $f = au$, and also since $au_h(x)$ is a polynomial of degree p , it is exactly represented by the sum in the middle term. Hence

$$f_h(x) = f_{CL} \hat{l}_1(x) + au_h(x) + f_{CR} \hat{l}_{n+1}(x)$$

Finally by differentiating the flux polynomial at the solution collocation we arrive at the SD scheme as

$$\frac{\partial u_h}{\partial t} + \left[a \frac{\partial u_h}{\partial x} + f_{CL} \hat{l}'_1(x) + f_{CR} \hat{l}'_{n+1}(x) \right] = 0$$

To show more clearly the distinction between the nodal DG and SD schemes, we cast the SD scheme in matrix form as

$$\frac{du_i}{dt} + \left[a \sum_{j=1}^n \mathbf{D}_{ij} u_j + f_{CL} \frac{d\hat{l}_1}{dx}(x_i) + f_{CR} \frac{d\hat{l}_{n+1}}{dx}(x_i) \right] = 0, \quad \text{where } \mathbf{D} = \mathbf{M}^{-1} \mathbf{S}$$

In order to convert the SD scheme to a form that resembles the nodal DG method, we multiply the equation by the mass matrix to get

$$\sum_j \mathbf{M}_{ij} \frac{du_i}{dt} + a \sum_{j=1}^n \mathbf{S}_{ij} u_j + f_{CL} \sum_j \mathbf{M}_{ij} \hat{l}'_1(x_i) = 0$$

where for simplicity we assume a pure upwind flux so that f_{CR} is equal to zero.

Expanding the mass matrix in the last term, this can be further reduced to

$$\sum_j \mathbf{M}_{ij} \frac{du_i}{dt} + a \sum_{j=1}^n \mathbf{S}_{ij} u_j - f_{CL} l_i(-1) = f_{CL} \int_{-1}^1 l'_i(x) \hat{l}_1(x) dx \quad (3)$$

It is apparent that the SD scheme differs from the DG scheme only by the right hand side term!

Energy Stability Proof for SD

To prove the energy stability of the SD scheme, it is most effective to leverage the nodal DG stability proof presented above. To achieve this, we need to eliminate the last term in equation (3), while at the same time retaining the basic form of the two schemes. To this end, a new matrix $\mathbf{Q}$ is proposed in place of the mass matrix $\mathbf{M}$ such that its introduction brings the SD scheme to the following form,

$$\sum_j \mathbf{Q}_{ij} \frac{du_i}{dt} + a \sum_{j=1}^n \mathbf{S}_{ij} u_j - f_{CL} l_i(-1) = 0$$

If a suitable $\mathbf{Q}$ can be identified as above so that the basic form is retained while the term that differentiates DG and SD is removed or absorbed, we can attain an energy estimate for the SD scheme with the norm $\mathbf{u}^T \mathbf{Q} \mathbf{u}$ replacing the norm $\mathbf{u}^T \mathbf{M} \mathbf{u}$ in the nodal DG scheme in each element. The requirements for $\mathbf{Q}$ can be summarized as follows,

1. It has the form $\mathbf{Q} = \mathbf{M} + \mathbf{C}$
2. It retains the function of the mass matrix and satisfies $\mathbf{Q} \mathbf{D} = \mathbf{S}$
3. The last two requirements lead to the third requirement that $\mathbf{C} \mathbf{D} = \mathbf{0}$
4. The expansion of the following term eliminates the term that distinguishes the SD and DG schemes such that $f_{CL} \sum_j \mathbf{Q}_{ij} l'_1(x_i) = f_{CL} l_i(-1)$

It is shown by Jameson¹⁸ that the above requirements can indeed be satisfied by choosing the matrix

$$\mathbf{Q} = \mathbf{M} + c d d^T$$

where d^T is the p^{th} difference operator. The first, second and third requirements are satisfied, since for any polynomial $R_p(x)$ of degree p , the combined operation by d and $\mathbf{D}$ on $R_p(x)$ leads to $(p+1)^{th}$ derivative on p^{th} degree polynomial, which is zero.

$$\sum_j d_i \sum_j D_{ij} R_p(x_j) = R_p^{p+1} = 0$$

This leaves only the parameter c to be determined which can satisfy the last requirement. It is shown that this is possible by picking c as

$$c = \frac{2p}{2p+1} \frac{1}{c_p^2} \frac{1}{p!(p+1)!} > 0$$

where $c_p = \frac{1 \cdot 3 \cdot 5 \dots (2p-1)}{p!}$ is the leading coefficients of the Legendre polynomial $L_p(x)$ of degree p .

So finally, with this choice of $\mathbf{C} = c d d^T$ and $\mathbf{Q} = \mathbf{M} + \mathbf{C}$, the SD scheme can be written in the following form,

$$\sum_j \mathbf{Q}_{ij} \frac{du_i}{dt} + a \sum_{j=1}^n \mathbf{S}_{ij} u_j - f_{CL} l_i(-1) = 0$$

and now the same argument that was used to prove the energy stability of the nodal DG scheme establishes the energy stability of the SD scheme with the norm

$$\|u\| = \int \left(u^2 + c \left(\frac{\partial^p u}{\partial x^p} \right)^2 \right) dx$$

in each element, provided the interior flux collocation points are the zeros of the Legendre polynomial $L_p(x)$.

II.C. Energy Stability Proof for FR

Flux Reconstruction Scheme for Linear Advection

For completeness, the essential ideas of Huynh's flux reconstruction approach are outlined here. Like the previous two schemes, the discrete solution is locally represented by Lagrange polynomial on the solution collocation points x_j as

$$u_h = \sum_{j=1}^n u_j l_j(x)$$

where for polynomials of degree p , $n = p + 1$, and u_h is the discrete solution in a reference element covering $[-1, 1]$. However, the representation of the discrete flux with the current approach is different. The discrete flux $f_j(x)$ is required to be continuous across elements. In one hand, it should take on the single valued numerical fluxes at the element interfaces, while on the other it should represent the interior discrete solutions as much as possible.

In the FR scheme, the continuous $f_j(x)$ is now made up of an interior flux term f_j^D and a correction flux term f_j^C . The interior flux represents the constructed flux using the discrete solution at the interior of the element. Its interpolated values at the element interfaces are in general discontinuous, hence we label this discrete flux f_h^D , and it is written as

$$f_h^D = \sum_{j=1}^{n+1} f_j^D l_j(x)$$

For this discrete flux, the interior values at the solution collocation points f_j^D are set equal to $f^D(u_h(x_j))$ where $u_h(x_j)$ is the discrete solution at the solution collocation points x_j . The discrete flux at the element interfaces $f_h^D(-1)$ and $f_h^D(1)$ are the interpolated values from $f_h^D(x)$, and, in general, are discontinuous across the interfaces.

While f_h^D best represents the interior solutions, at the element interface it differs from the desired continuous flux f_j by, taking the left boundary as an example,

$$f_h^C(-1) = f_{CL} = f_h(-1) - f_h^D(-1) = \hat{f}(-1) - f_h^D(-1)$$

The introduction of the correction flux f_j^C aims to enforce the numerical flux, and hence continuity, at the element boundaries, while trying to minimize the difference between f_h and f_h^D in the interior of the element. In order to define a f^C with such properties, consider a degree $k + 1$ correction function $g_L = g_L(r)$ and $g_R = g_R(r)$ that approximate zero (in some sense) within the element while satisfying

$$g_L(-1) = 1, \quad g_L(1) = 0 \quad \& \quad g_R(-1) = 0, \quad g_R(1) = 1$$

A suitable expression for f^C can now be written in terms of g_L and g_R as

$$f_h^C = f_{CL} g_L + f_{CR} g_R$$

The discrete flux f_h that has continuous values across the element can now be constructed as follows

$$f_h = f_h^D + f_h^C$$

Finally we differentiate the flux at the solution collocation points to obtain

$$\frac{du_i}{dt} + \left[\sum_{j=1}^{n+1} f_j^D \frac{dl_j}{dx}(x_i) + f_{CL} \frac{dg_L}{dx}(x_i) + f_{CR} \frac{dg_R}{dx}(x_i) \right] = 0$$

For linear advection with $f = au$, this is equivalent to

$$\frac{\partial u_h}{\partial t} + a \frac{\partial u_h}{\partial x} + f_{CL} \frac{dg_L}{dx} + f_{CR} \frac{dg_R}{dx} = 0 \quad (4)$$

This final form of the Flux Reconstruction scheme also quite closely resembles the form of the SD scheme shown previously.

Energy Stability Proof for FR

From the earlier section, we derived an energy estimate for the SD scheme using boundary flux corrections in a way very similar to the Flux Reconstruction scheme. The connection between the two schemes is (to some degree) implied by the similarity of the final forms of the schemes. Hence it is also plausible to approach the energy estimate of the FR scheme using a similar norm of the form

$$||u|| = \int u^2 + \frac{c}{2} \left(\frac{\partial^p u}{\partial x^p} \right)^2 dx$$

where c for FR scheme is to be determined.

Starting with the FR scheme, we can derive the explicit expressions for the two terms that make up the norm. The results are quoted below, while details can be found in Vincent, Castonguay and Jameson.¹⁹ For the first term, multiplying (4) by u_h and integrating, to get

$$\frac{d}{dt} \int_{-1}^1 u_h^2 dx = -\hat{a}[u_R^2 - u_L^2] - 2f_{CL} \left(-u_L - \int_{-1}^1 g_L \frac{\partial u_h}{\partial x} dx \right) - 2f_{CR} \left(u_R - \int_{-1}^1 g_R \frac{\partial u_h}{\partial x} dx \right)$$

and for the second term,

$$\frac{1}{2} \frac{d}{dt} \int_{-1}^1 c \left(\frac{\partial^p u_h}{\partial x^p} \right)^2 dx = -2cf_{CL} \left(\frac{\partial^p u_h}{\partial x^p} \right) \left(\frac{d^{p+1} g_L}{dx^{p+1}} \right) - 2cf_{CR} \left(\frac{\partial^p u_h}{\partial x^p} \right) \left(\frac{d^{p+1} g_R}{dx^{p+1}} \right)$$

and lastly adding the two equations together we obtain the desired explicit expression as

$$\begin{aligned} \frac{d}{dt} \int_{-1}^1 u_h^2 + \frac{c}{2} \left(\frac{\partial^p u_h}{\partial x^p} \right)^2 dx = & + 2(f_{CL} u_L - \hat{a} \frac{u_L^2}{2}) - 2(f_{CR} u_R - \hat{a} \frac{u_R^2}{2}) \\ & + 2f_{CL} \left[\int_{-1}^1 g_L \frac{\partial u_h}{\partial x} dx - c \left(\frac{\partial^p u_h}{\partial x^p} \right) \left(\frac{d^{p+1} g_L}{dx^{p+1}} \right) \right] \\ & + 2f_{CR} \left[\int_{-1}^1 g_R \frac{\partial u_h}{\partial x} dx - c \left(\frac{\partial^p u_h}{\partial x^p} \right) \left(\frac{d^{p+1} g_R}{dx^{p+1}} \right) \right] \end{aligned}$$

If the last two terms of the above equation are set equal to zero, from the section where energy stability of DG scheme is presented, we see that we recover the same form of the DG scheme. This is also the similar approach used to prove SD scheme energy stability. From the previous section, the energy estimate for the SD scheme depends on the choice of the parameter c . Hence we expect the same for the FR scheme. Vincent, Castonguay and Jameson¹⁹ show that provided the following three conditions are met,

1. Condition a: $\int_{-1}^1 g_L \frac{\partial u_h}{\partial x} dx - c \left(\frac{\partial^p u_h}{\partial x^p} \right) \left(\frac{d^{p+1} g_L}{dx^{p+1}} \right) = 0$
2. Condition b: $\int_{-1}^1 g_R \frac{\partial u_h}{\partial x} dx - c \left(\frac{\partial^p u_h}{\partial x^p} \right) \left(\frac{d^{p+1} g_R}{dx^{p+1}} \right) = 0$
3. Condition c: $c_0(k) < c < \infty$, where $c_0(k) = \frac{-2}{(2k+1)(a_k k!)^2}$

the Flux Reconstruction scheme will be energy stable. Since both g_L and g_R are functions of c , the single parameter c determines the stability as well as the type of Flux Reconstruction scheme. These three equations can be solved and the final expressions for the left and right flux correction functions in terms of Legendre polynomials are written as

$$\begin{aligned} g_L &= \frac{(-1)^p}{2} \left[L_p - \left(\frac{\eta_p L_{p-1} + L_{p+1}}{1 + \eta_p} \right) \right] \\ g_R &= \frac{(+1)^p}{2} \left[L_p + \left(\frac{\eta_p L_{p-1} + L_{p+1}}{1 + \eta_p} \right) \right] \end{aligned}$$

where $\eta_p = \frac{c(2p+1)(a_p p!)^2}{2}$. Use of these correction functions leads to energy stable FR schemes with a suitable c . The choices of c that lead various schemes are discussed in the next section.

Nodal DG scheme

By the choice of c , if the flux correction functions are reduced to the right and left Radau polynomials, then a nodal scheme is recovered. The corresponding flux correction functions are

$$g_L = \frac{(-1)^p}{2}(L_p - L_{p+1})$$

$$g_R = \frac{(+1)^p}{2}(L_p - L_{p+1})$$

which is a result of picking $c = 0$.

SD scheme

The SD scheme has a set of flux collocation points within the element. At those points, the flux correction functions should assume zero values. By choosing $c = \frac{2p}{(2p+1)(p+1)(a_p p!)^2}$, the resulting flux correction functions are

$$g_L = \frac{(-1)^p}{2}(1-x)L_p$$

$$g_R = \frac{(+1)^p}{2}(1+x)L_p$$

Huynh Type scheme

If the value of c is set equal to $\frac{2(p+1)}{(2p+1)p(a_p p!)^2}$, the resulting flux correction functions are

$$g_L = \frac{(-1)^p}{2} \left[L_p - \left(\frac{(p+1)L_{p-1} + pL_{p+1}}{2p+1} \right) \right]$$

$$g_R = \frac{(+1)^p}{2} \left[L_p + \left(\frac{(p+1)L_{p-1} + pL_{p+1}}{2p+1} \right) \right]$$

This recovers a particular scheme that was proposed by Huynh, which he found to be very stable.

III. DG, SD and FR Schemes for Multi-dimensional Problems

III.A. Background

The flux reconstruction family of schemes (including NDG and SD) can readily be extended to quadrilateral and hexahedral elements using a tensor product formulations as suggested by Huynh.¹⁴ This was already used by Kopriva and Kolias,¹¹ and it has proved successful in practice.^{20, 21, 23} While the formulation of the nodal DG method for simplex elements is well established,⁹ the original SD scheme for simplex elements¹² was found to be unstable when the order was increased. This has motivated the search for stable variants of the FR formulation for simplex elements, and three approaches are currently being pursued. Wang and Gao have proposed the LCP method,²⁴ while Balan, May and Schoberl²⁵ have reformulated the SD scheme using Raviart Thomas basis polynomials and have used Fourier analysis to verify stability for the case of third order accuracy. Recently Castonguay, Vincent and Jameson have extended the formulation of energy stable flux reconstruction schemes to multi-dimensional problems, including the nodal DG scheme as a special case.²⁶ The next section summarizes the derivation.

III.B. Energy Estimate of Flux Reconstruction Scheme

We represent the solution using interior nodal values, with $\mathbf{x} = (x, y)$

$$u_h(\mathbf{x}) = \sum u_i l_i(\mathbf{x})$$

Represent the correction flux using flux mismatch at the interface and a correction function that propagates the difference into the interior.

$$g(\mathbf{x}) = \sum f_{ck} g_k(\mathbf{x})$$

where f_{ck} represents the difference between the common interface flux $\hat{u}$ derived from a Riemann solution and the flux calculated from the interior solution.

The governing equation can then be written in terms of the divergence of the interior and correction fluxes as

$$\frac{\partial u_h}{\partial t} + \nabla \cdot (\mathbf{a}u_h) + \nabla \cdot \mathbf{g} = 0 \quad (5)$$

where $\mathbf{a}=(a, b)$ is the wave velocity vector. Note that here we can represent $\nabla \cdot \mathbf{g}$ directly as a polynomial of the same degree as u_h .

In order to choose a correction function that will ensure energy stability, first obtain the energy estimate in the usual way by multiplying by u_h and integrating over the domain

$$\int_D u_h \left[\frac{\partial u_h}{\partial t} + \nabla \cdot (\mathbf{a}u_h) + \nabla \cdot \mathbf{g} \right] dA = 0$$

to get

$$\frac{d}{dt} \int_D \frac{u_h^2}{2} dA + a \int_D u_h \frac{\partial u_h}{\partial x} dA + b \int_D u_h \frac{\partial u_h}{\partial y} dA + \int_D u_h \nabla \cdot \mathbf{g} dA = 0$$

Further integration by parts yields

$$\frac{d}{dt} \int_D \frac{u_h^2}{2} dA + \int_B (\mathbf{n} \cdot \mathbf{a}) \frac{u_h^2}{2} dS + \int_B \mathbf{n} \cdot \mathbf{g} u_h dS - \int_D \mathbf{g} \cdot \nabla u_h dA = 0$$

Compared to the discrete energy estimate of the nodal DG, we now have the extra term

$$\left(\frac{d}{dt} \int_D \frac{u_h^2}{2} dA - \underbrace{\int_D \mathbf{g} \cdot \nabla u_h dA}_{\text{additional term}} \right) + \int_B (\mathbf{n} \cdot \mathbf{a}) \frac{u_h^2}{2} dS + \int_B \mathbf{n} \cdot \mathbf{g} u_h dS = 0 \quad (6)$$

First, we note the second derivative of the governing equation

$$\frac{\partial^2}{\partial \mathbf{x}^2} \left[\frac{\partial u_h}{\partial t} + \nabla \cdot (\mathbf{a}u_h) + \nabla \cdot \mathbf{g} \right] = 0$$

reduce to

$$\frac{\partial}{\partial t} \mathbf{u}_{hxx} + 0 + \frac{\partial^2}{\partial x^2} \nabla \cdot \mathbf{g} = 0$$

Hence if we choose $\mathbf{g}$ such that

$$\int_D \mathbf{g} \cdot \nabla u_h dA = c_1 A u_{hxx} \frac{\partial^2}{\partial x^2} \nabla \cdot \mathbf{g} + c_2 A u_{hxy} \frac{\partial^2}{\partial xy} \nabla \cdot \mathbf{g} + c_3 A u_{hyy} \frac{\partial^2}{\partial y^2} \nabla \cdot \mathbf{g} \quad (7)$$

then

$$\begin{aligned} - \int_D \mathbf{g} \cdot \nabla u_h dA &= c_1 A u_{hxx} \frac{\partial}{\partial t} u_{hxx} + c_2 A u_{hxy} \frac{\partial}{\partial t} u_{hxy} + c_3 A u_{hyy} \frac{\partial}{\partial t} u_{hyy} \\ &= \frac{1}{2} \frac{d}{dt} \int (c_1 u_{hxx}^2 + c_2 u_{hxy}^2 + c_3 u_{hyy}^2) dA \end{aligned}$$

Hence the energy estimate of the flux reconstruction scheme becomes

$$\underbrace{\frac{1}{2} \frac{d}{dt} \int_D \left(u_h^2 + c_1 u_{hxx}^2 + c_2 u_{hxy}^2 + c_3 u_{hyy}^2 \right) dA}_{\text{energy estimate for FR}} + \int_B (\mathbf{n} \cdot \mathbf{a}) \frac{u_h^2}{2} dS + \int_B \mathbf{n} \cdot \mathbf{g} u_h dS = 0$$

The Flux Reconstruction scheme is stable in this new norm

III.C. ESFR Schemes for Simplex Elements with Second Degree Polynomials

In order to satisfy equation (7) suppose that u_h and $\nabla \cdot \mathbf{g}$ are represented as

$$u_h = a_1 + a_2x + a_3y + a_4x^2 + a_5xy + a_6y^2$$

and

$$\nabla \cdot \mathbf{g} = b_1 + b_2x + b_3y + b_4x^2 + b_5xy + b_6y^2$$

Then

$$\nabla u_h = \begin{bmatrix} a_2 + 2a_4x + a_5y \\ a_3 + a_5x + 2a_6y \end{bmatrix}$$

while

$$u_{h \ xx} = 2a_4$$

$$u_{h \ xy} = a_5$$

$$u_{h \ yy} = a_6$$

and

$$\frac{\partial^2}{\partial x^2} \nabla \cdot \mathbf{g} = 2b_4$$

$$\frac{\partial^2}{\partial xy} \nabla \cdot \mathbf{g} = b_5$$

$$\frac{\partial^2}{\partial y^2} \nabla \cdot \mathbf{g} = b_6$$

Also, denoting the components of $\mathbf{g}$ by g_x and g_y ,

$$\int_D \mathbf{g} \cdot \nabla u_h dA = a_2 \int_D g_x dA + a_3 \int_D g_y dA + 2a_4 \int_D g_x x dA + a_5 \int_D (g_x y + g_y x) dA + 2a_6 \int_D g_y y dA$$

while

$$c_1 \int_D u_{hxx} \frac{\partial^2}{\partial x^2} \nabla \cdot \mathbf{g} + c_2 \int_D u_{hxy} \frac{\partial^2}{\partial xy} \nabla \cdot \mathbf{g} + c_3 \int_D u_{hyy} \frac{\partial^2}{\partial y^2} \nabla \cdot \mathbf{g} = A(4c_1 a_4 b_4 + c_2 a_5 b_5 + 4c_3 a_6 b_6)$$

Hence to satisfy equation (7) the moments $g\mathbf{g}$ must satisfy

$$\begin{aligned} \int_D g_x dA &= 0 \\ \int_D g_y dA &= 0 \\ \int_D g_x x dA &= 2c_1 b_4 \\ \int_D (g_x y + g_y x) dA &= c_2 b_5 \\ \int_D g_y y dA &= 2c_3 b_6 \end{aligned}$$

while $\mathbf{g} \cdot \mathbf{n}$ is defined on the element boundary B .

It is not necessary, however, to solve for the moments of $\mathbf{g}$ directly, since they can be expressed in terms of the moments of $\nabla \cdot \mathbf{g}$ using integration by parts. Thus

$$\begin{aligned} \int_D \nabla \cdot \mathbf{g} dA &= \int_B \mathbf{g} \cdot \mathbf{n} dS \\ \int_D g_x dA &= \int_D \mathbf{g} \cdot \nabla x dA = \int_B x \mathbf{g} \cdot \mathbf{n} dS - \int_D x \nabla \cdot \mathbf{g} dA \end{aligned}$$

etc.

A polynomial ϕ_p of degree p is fully determined by its moments up to degree p since if all the moments were zero it would imply that

$$\int \phi_p^2 dA = 0$$

Thus we can directly solve for $\nabla \cdot \mathbf{g}$ by satisfying

$$\begin{aligned}\int_D \nabla \cdot \mathbf{g} dA &= \int_B \mathbf{g} \cdot \mathbf{n} dS \\ \int_D \nabla \cdot \mathbf{g} x dA &= \int_B \mathbf{g} \cdot \mathbf{n} x dS \\ \int_D \nabla \cdot \mathbf{g} y dA &= \int_B \mathbf{g} \cdot \mathbf{n} y dS \\ \int_D \nabla \cdot \mathbf{g} x^2 dA &= \int_B \mathbf{g} \cdot \mathbf{n} x^2 dS - 4c_1 b_4 \\ \int_D \nabla \cdot \mathbf{g} xy dA &= \int_B \mathbf{g} \cdot \mathbf{n} xy dS - c_2 b_5 \\ \int_D \nabla \cdot \mathbf{g} y^2 dA &= \int_B \mathbf{g} \cdot \mathbf{n} y^2 dS - 4c_3 b_6\end{aligned}$$

The nodal DG scheme is obtained by setting

$$c_1 = c_2 = c_3 = 0$$

In order to produce a rotationally invariant norm the coefficient so the derivatives in the norm should correspond to the binomial coefficients

$$c_1 = c, \quad c_2 = 2c, \quad c_3 = c$$

giving a one parameter family of energy stable schemes.

The derivation is valid for an arbitrary triangle. If a given triangle is mapped to a reference triangle one can precalculate $\nabla \cdot \mathbf{g}_k$ for correction function satisfying

$$\mathbf{g}_k \cdot \mathbf{n}_l = \delta_{kl}$$

III.D. ESFR Schemes on Quadrilateral Elements

In the case of quadrilateral elements a natural approach is to use a tensor product to represent the discrete solution

$$u_h = \sum_{j=1}^{p+1} \sum_{n=1}^{p+1} u_{jn} l_j(x) n_n(y)$$

where $l_j(x)$ and $n_n(y)$ are Lagrange basis polynomials of degree p . In the case $p = 2$ this can be expanded as

$$u_h = a_1 + a_2 x + a_3 y + a_4 x^2 + a_5 xy + a_6 y^2 + a_7 x^2 y + a_8 xy^2 + a_9 x^2 y^2$$

Now

$$\nabla u_h = \begin{bmatrix} a_2 + 2a_4 x + a_5 y + 2a_7 xy + a_8 y^2 + 2a_9 xy^2 \\ a_3 + a_5 x + 2a_6 y + a_7 x^2 + 2a_8 xy + 2a_9 x^2 y \end{bmatrix}$$

As before flux correction can be propagated from the boundaries with a vector correction function $\mathbf{g}$ to produce the discrete equation (5). Now we assume that the divergence of the correction function can be represented by a polynomial with the same form as u_h :

$$\nabla \cdot \mathbf{g} = b_1 + b_2 x + b_3 y + b_4 x^2 + b_5 xy + b_6 y^2 + b_7 x^2 y + b_8 xy^2 + b_9 x^2 y^2$$

Now, differentiating twice in both x and y , we find that

$$\frac{\partial}{\partial t} u_{hxyy} + \frac{\partial^2}{\partial x^2} \frac{\partial^2}{\partial y^2} \nabla \cdot \mathbf{g} = 0 \quad (8)$$

Also multiplying by u_h and integrating by parts over x and y yields equation (6) as before. Now choose $\mathbf{g}$ such that

$$\int_D \mathbf{g} \cdot \nabla u_h dA = c u_{hxyy} \frac{\partial^2}{\partial x^2} \frac{\partial^2}{\partial y^2} \nabla \cdot \mathbf{g} = \int \frac{c}{4} u_{hxyy} \frac{\partial^2}{\partial x^2} \frac{\partial^2}{\partial y^2} \nabla \cdot \mathbf{g} dA$$

since both terms are constants. Thus using equation (8)

$$\int \mathbf{g} \cdot \nabla u_h dA = -\frac{c}{4} \int u_{hxyy} \frac{\partial}{\partial t} u_{hxyy} dA = -\frac{c}{4} \frac{d}{dt} \int u_{hxyy}^2 dA$$

and substituting for $\int \mathbf{g} \cdot \nabla u_h dA$ in equation (6),

$$\frac{d}{dt} \int_D \left(\frac{u_h^2}{2} + \frac{c}{4} u_{hxyy}^2 \right) dA + \int_B \mathbf{n} \cdot \mathbf{a} \frac{u_h^2}{2} dS + \int_B \mathbf{n} \cdot \mathbf{g} u_h dS = 0$$

Now on summing over the elements, a proper choice of the interface fluxes yields energy stability as before.

Hence

$$\begin{aligned} u_{hxyy} &= 4a_9 \\ \frac{\partial^2}{\partial x^2} \frac{\partial^2}{\partial y^2} \nabla \cdot \mathbf{g} &= 4b_9 \end{aligned}$$

and

$$\begin{aligned} \int_D \mathbf{g} \cdot \nabla u_h dA &= a_2 \int g_x dA + a_3 \int g_y dA \\ &\quad \dots \\ &\quad + 2a_9 \int (g_x x y^2 + g_y x^2 y) dA \end{aligned}$$

Hence to satisfy (8) all lower moments of $\mathbf{g}$ must vanish while

$$\int_D (g_x x y^2 + g_y x^2 y) dA = 8cb_9$$

Now we can integrate by parts to obtain conditions directly for the moments of $\nabla \cdot \mathbf{g}$

$$\begin{aligned} \int_D \nabla \cdot \mathbf{g} dV &= \int \mathbf{g} \cdot \mathbf{n} dS + 0 \\ \int_D \nabla \cdot \mathbf{g} x dV &= \int \mathbf{g} \cdot \mathbf{n} x dS + 0 \\ \int_D \nabla \cdot \mathbf{g} y dV &= \int \mathbf{g} \cdot \mathbf{n} y dS + 0 \\ &\quad \dots \\ \int_D \nabla \cdot \mathbf{g} x^2 y^2 dV &= \int \mathbf{g} \cdot \mathbf{n} x^2 y^2 dS + 16cb_9 \end{aligned}$$

These moment conditions are sufficient to define $\nabla \cdot \mathbf{g}$ uniquely. The nodal DG scheme is recovered by setting $c = 0$. In this case the equations for the coefficients of $\nabla \cdot \mathbf{g}$ can be solved with $\nabla \cdot \mathbf{g}$ expressed in the form of a tensor product. However, if $c \neq 0$ the solution for $\nabla \cdot \mathbf{g}$ can no longer be expressed as a tensor product.

III.E. ESFR Schemes for 3D Pyramids

A general solver for unstructured meshes in 3D should provide for mixed elements, including tetrahedra, hexahedra, prisms and pyramids, which may be needed to connect tetrahedra and hexahedra. The present approach has the flexibility to enable the treatment of all these elements. For example in the case of pyramids, using polynomials of degree 2, we can represent u_h and $\nabla \cdot \mathbf{g}$ as

$$u_h = a_1 + a_2x + a_3y + a_4x^2 + a_5xy + a_6y^2 + a_7x^2y + a_8xy^2 + a_9x^2y^2 + a_{10}z + a_{11}xz + a_{12}yz + a_{13}z^2 + a_{14}xyz$$

and

$$\nabla \cdot \mathbf{g} = b_1 + b_2x + b_3y + b_4x^2 + b_5xy + b_6y^2 + b_7x^2y + b_8xy^2 + b_9x^2y^2 + b_{10}z + b_{11}xz + b_{12}yz + b_{13}z^2 + b_{14}xyz$$

The energy stability for 3D pyramids requires all moments of $\mathbf{g}$ to vanish except

$$\begin{aligned} \int_D (g_xxy^2 + g_yx^2y) dV &= 8c_1b_9 \\ \int_D (g_zz) dV &= 2c_2b_{13} \\ \int_D (g_xyz + g_yxz + g_zxy) dV &= c_3b_{14} \end{aligned}$$

Then

$$\begin{aligned} \frac{\partial}{\partial t} \mathbf{u}_{hxyy} + \frac{\partial^2}{\partial x^2} \frac{\partial^2}{\partial y^2} \nabla \cdot \mathbf{g} &= 0 \\ \frac{\partial}{\partial t} \mathbf{u}_{hzz} + \frac{\partial^2}{\partial z^2} \nabla \cdot \mathbf{g} &= 0 \\ \frac{\partial}{\partial t} \mathbf{u}_{hxyz} + \frac{\partial}{\partial x} \frac{\partial}{\partial y} \frac{\partial}{\partial z} \nabla \cdot \mathbf{g} &= 0 \end{aligned}$$

The requirement to ensure energy stability becomes

$$\int_D \mathbf{g} \cdot \nabla u_h dA = \frac{1}{2} \frac{d}{dt} \int_D \left(c_1 u_{hxyy}^2 + c_2 u_{hzz}^2 + c_3 u_{hxyz}^2 \right) dV$$

After integration by parts $\nabla \cdot \mathbf{g}$ can be determined from 14 moments

$$\begin{aligned} \int_D \nabla \cdot \mathbf{g} dV &= \int \mathbf{g} \cdot \mathbf{n} dS + 0 \\ \int_D \nabla \cdot \mathbf{g} x dV &= \int \mathbf{g} \cdot \mathbf{n} x dS + 0 \\ \int_D \nabla \cdot \mathbf{g} y dV &= \int \mathbf{g} \cdot \mathbf{n} y dS + 0 \\ \int_D \nabla \cdot \mathbf{g} z dV &= \int \mathbf{g} \cdot \mathbf{n} z dS + 0 \\ &\dots \\ \int_D \nabla \cdot \mathbf{g} x^2 y^2 dV &= \int \mathbf{g} \cdot \mathbf{n} x^2 y^2 dS + 16c_1b_9 \\ \int_D \nabla \cdot \mathbf{g} z^2 dV &= \int \mathbf{g} \cdot \mathbf{n} z^2 dS + 4c_2b_{13} \\ \int_D \nabla \cdot \mathbf{g} xyz dV &= \int \mathbf{g} \cdot \mathbf{n} xyz dS + c_3b_{14} \end{aligned}$$

IV. Remarks on Non-linear Stability

In the case of a nonlinear conservation law

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} f(u) = 0$$

similar issues arise for the flux reconstruction and nodal DG schemes.²⁷ Suppose that the discrete solution and flux are represented inside an element as

$$u_h = \sum_{i=1}^{p+1} u_i l_i(x), f_h = \sum_{i=1}^{p+1} f_i l_i(x)$$

using polynomials of degree p . The FR scheme is

$$\frac{\partial u_h}{\partial t} + \frac{\partial f_h}{\partial x} + f_{cl} g'_l(x) + f_{cr} g'_r(x) = 0$$

where f_{cl} and f_{cr} denote the corrections from each boundary as before and $g_l(x)$ and $g_r(x)$ are polynomials of degree $p+1$ which vanish at the opposite boundary.

Multiplying by u_h and integrating by parts over the element

$$\frac{d}{dt} \int_{-1}^1 \frac{u_h^2}{2} dx + \int_{-1}^1 f_h \frac{\partial u_h}{\partial x} dx - f_h(1)u_h(1) - f_h(-1)u_h(-1) = f_{cl}(u_h(-1)) - \int_{-1}^1 g_l \frac{\partial u_h}{\partial x} dx - f_{cr}(u_h(1)) - \int_{-1}^1 g_r \frac{\partial u_h}{\partial x} dx$$

Now suppose that f_h is constructed by projection so that $f_h - f$ is orthogonal to all polynomials of degree $< p$. Then

$$\int_{-1}^1 f_h \frac{\partial u_h}{\partial x} dx = \int_{-1}^1 f(u_h) \frac{\partial u_h}{\partial x} dx \quad (9)$$

Also if there exists an entropy function $F(u)$ such that

$$u \frac{\partial f}{\partial u} = \frac{\partial}{\partial u} (uf) - f = \frac{\partial F}{\partial u}$$

then

$$f = \frac{\partial G}{\partial u} \quad (10)$$

where

$$G(u) = uf - F(u)$$

and

$$\int_{-1}^1 f(u_h) \frac{\partial u_h}{\partial x} dx = \int_{-1}^1 \frac{\partial G}{\partial u} dx = G(u_h(1)) - G(u_h(-1))$$

Now let the correction functions satisfy the same conditions as before in section II. Then in one element

$$\frac{d}{dt} \int \left(\frac{u_h^2}{2} + c \frac{u_h^{(p)^2}}{4} \right) dx + f_h(1) - G(u_h(1)) + f_{cr}(u_h(1)) - (f_h(-1) - G(u_h(-1)) + f_{cl}(u_h(-1))) = 0$$

where

$$f_{cr} = \hat{f}(1) - f_h(1), f_{cl} = \hat{f}(-1) - f_h(-1).$$

Then on summing over the elements an upwind flux $\hat{f}$ can be defined at each interface such that the contributions from the interfaces are strictly non-positive, with the consequence that the FR scheme is stable in a broken norm of Sobolev type as before, while $c = 0$ recovers the nodal DG scheme. There remains the snag, as in the case of the DG scheme, that in general the coefficients such that f_h satisfies equation (9) cannot be exactly evaluated by numerical integration, and the best one can hope for is to minimize the integration errors, possibly by over-integration with additional collocation points.

IV.A. Gas Dynamics

In the case of a system of equations condition (10) implies that the Jacobian matrix $\frac{\partial \mathbf{f}}{\partial \mathbf{u}}$ is symmetric. This is not the case for the standard conservation form for the equations of gas dynamics. However, they can be symmetrized by a transformation to entropy variables.^{28, 29} Consider the three dimensional gas dynamics equations

$$\frac{\partial \mathbf{u}}{\partial t} + \sum_{i=1}^3 \frac{\partial}{\partial x_i} \mathbf{f}_i(u) = 0 \quad (11)$$

Here the state and flux vectors are

$$\mathbf{u} = \begin{bmatrix} \rho \\ \rho u_1 \\ \rho u_2 \\ \rho u_3 \\ \rho E \end{bmatrix}, \mathbf{f}_i = \begin{bmatrix} \rho u_i \\ \rho u_i u_1 + \delta_{i1} p \\ \rho u_i u_2 + \delta_{i2} p \\ \rho u_i u_3 + \delta_{i3} p \\ \rho u_i H \end{bmatrix}$$

where ρ is the density, u_i are the velocity components, E and H are the total energy and enthalpy and p is the pressure,

$$p = (\gamma - 1)\rho(E - \frac{1}{2} \sum u_i^2), H = E + \frac{p}{\rho}$$

Also the entropy is

$$S = \log\left(\frac{p}{\rho^\gamma}\right) = \log p - \gamma \log \rho$$

Also the equations can be represented in the quasilinear form

$$\frac{\partial \mathbf{u}}{\partial t} + \sum_{i=1}^3 A_i \frac{\partial \mathbf{u}}{\partial x_i} = 0$$

where the Jacobian matrices $A_i = \frac{\partial \mathbf{f}}{\partial \mathbf{u}}$ are not symmetric.

The simplest form of the entropy variables is as follows. Define the entropy function

$$U(\mathbf{u}) = \frac{-\rho S}{\gamma - 1} = \frac{\gamma}{\gamma - 1} \rho \log p - \frac{\rho}{\gamma - 1} \log p$$

which satisfies the entropy conservation equation

$$\frac{\partial U(\mathbf{u})}{\partial t} + \sum_{i=1}^3 \frac{\partial}{\partial x_i} F_i(\mathbf{u}) = 0 \quad (12)$$

where

$$F_i(\mathbf{u}) = u_i U$$

and

$$\frac{\partial F_i}{\partial \mathbf{u}} = u_i \frac{\partial \mathbf{f}_i}{\partial \mathbf{u}}$$

Then the entropy variables are

$$\mathbf{v}^T = \frac{\partial U}{\partial \mathbf{u}} = \left[\frac{\gamma - S}{\gamma - 1} \frac{\rho}{p} \frac{u^2}{2}, \frac{\rho u_1}{p}, \frac{\rho u_2}{p}, \frac{\rho u_3}{p}, -\frac{\rho}{p} \right]$$

where

$$u^2 = \sum_{i=1}^3 u_i^2$$

Now

$$\mathbf{u} = \frac{\partial Q}{\partial \mathbf{v}}, \mathbf{f}_i = \frac{\partial G_i}{\partial \mathbf{v}}$$

where

$$Q = \rho, G_i = \rho u_i$$

Accordingly $\frac{\partial \mathbf{u}}{\partial \mathbf{v}}$ and $\frac{\partial \mathbf{f}_i}{\partial \mathbf{v}}$ are the second derivatives of scalar functions and are therefore symmetric. The transformed equations have the symmetric hyperbolic form

$$\frac{\partial \mathbf{u}}{\partial \mathbf{v}} \frac{\partial \mathbf{v}}{\partial \mathbf{t}} + \sum_{i=1}^3 \bar{A}_i \frac{\partial \mathbf{v}}{\partial \mathbf{x}_i} = 0$$

where $\frac{\partial \mathbf{u}}{\partial \mathbf{v}}$ is positive definite and

$$\bar{A}_i = A_i \frac{\partial \mathbf{u}}{\partial \mathbf{v}} = \frac{\partial f_i}{\partial \mathbf{u}} \frac{\partial \mathbf{u}}{\partial \mathbf{v}} = \frac{\partial f_i}{\partial \mathbf{v}}$$

Also multiplying (11) by $\mathbf{v} = \frac{\partial U}{\partial \mathbf{v}}$ recovers the entropy conservation law (12). The entropy function U may be regarded as a generalized energy, and the application of an energy stable FR schemes would lead to solutions with non-increasing U , or non-decreasing entropy. Fidkowski and Roe³⁰ have recently observed that the entropy variables satisfy an adjoint equation for the flux of entropy through the domain, and thus can be used as an error indicator for mesh adaptation.³¹

V. Review of Time Integration Methods

V.A. Explicit Strong-Stability-Preserving Runge-Kutta Schemes

A significant impediment to the wider use of higher-order schemes has been their computational cost. This is partly due to their complexity and partly due to severe time step restrictions in the case of explicit time-integration schemes. The nodal DG and flux reconstruction schemes provide a path to minimizing complexity.

The stability limit for explicit time-stepping schemes scales roughly as $\frac{1}{p^2}$ for schemes of order p . However for an equal number of degrees of freedom the element size scales as p , so the actual time step should scale roughly as $\frac{1}{p}$ in practice, and this can be further alleviated by optimizing FR schemes to maximize the stable time step.

Within the class of explicit schemes an attractive choice is the family of strongly stability preserving (SSP) Runge Kutta schemes.³² If the discretization is total variation diminishing (TVD)³³ or local extremum diminishing (LED)^{34,35} for a forward Euler time stepping scheme, these properties are preserved by by SSP schemes through their construction as a convex combination of forward Euler steps. SSP schemes may be optimized for a given number of steps and have proved useful in practice.²⁰⁻²²

V.B. Implicit Schemes

In order to enable the use of large time steps consistent with the accuracy requirements of time dependent simulations, there have been numerous studies of implicit schemes, for which the bandwidth of the matrices to be inverted grow rapidly with the order, particularly for viscous simulations. Sun, Wang and Liu³⁶ and also Premasathan, Liang and Jameson²³ have had success with schemes using symmetric Gauss Seidel inner iterations. Peraire and Persson have developed a compact DG scheme which minimize the bandwidth.³⁷ Persson has also identified a third order implicit RK schemes³⁸ as cost effective in comparison to alternatives. Nevertheless the schemes are still expensive compared to standard second order accurate finite volume schemes, and this is an area of ongoing research.

V.C. Energy Stable Implicit Crank-Nicolson Type Scheme

In order to preserve true energy stability with the time integration scheme one ought to use an implicit scheme of Crank-Nicolson type. Using a superscript n to denote the time level, let

$$\bar{u}_h = \frac{1}{2}(u_h^{n+1} + u_h^n)$$

Suppose now that all terms in the space discretization are evaluated using $\bar{u}_h$. Then the scheme is

$$u_h^{n+1} = u_h^n - \Delta t \left(a \frac{\partial \bar{u}_h}{\partial x} + f_{cl} g'_l + f_{cr} g'_r \right)$$

where f_{cl} and f_{cr} are calculated from $\bar{u}_h$ in the adjacent cells. Moreover, differentiating p times

$$u_h^{(p)n+1} - u_h^{(p)n} = -\Delta t(f_{cl}g_l^{p+1} + f_{cr}g_r^{p+1})$$

Now, multiplying by $\bar{u}_h$ and integrating over the element

$$\begin{aligned} & \frac{1}{2} \int_{-1}^1 u_h^{n+1^2} dx - \frac{1}{2} \int_{-1}^1 u_h^{n^2} dx \\ &= -\Delta t \left[a \int_{-1}^1 \bar{u}_h \frac{\partial \bar{u}_h}{\partial x} dx + f_{cl} \int_{-1}^1 \bar{u}_h g_l' dx + f_{cr} \int_{-1}^1 \bar{u}_h g_r' dx \right] \\ &= -\Delta t \left[\frac{a}{2} (\bar{u}_h(1)^2 - \bar{u}_h(-1)^2) + f_{cr} \bar{u}_h(1) - f_{cl} \bar{u}_h(-1) \right] \\ & \quad + \Delta t \left[f_{cr} \int_{-1}^1 g_r \frac{\partial \bar{u}_h}{\partial x} dx + f_{cl} \int_{-1}^1 g_l \frac{\partial \bar{u}_h}{\partial x} dx \right] \end{aligned}$$

Choosing g_r and g_l as in section II such that

$$\int g_r \frac{\partial \bar{u}_h}{\partial x} = c g_r^{(p+1)} \bar{u}_h^{(p)}, \quad \int g_l \frac{\partial \bar{u}_h}{\partial x} = c g_l^{(p+1)} \bar{u}_h^{(p)}$$

The last term is equivalent to

$$-\frac{c}{2} (u_h^{(p)n+1} - u_h^{(p)n}) (u_h^{(p)n+1} + u_h^{(p)n}) = -\frac{c}{2} (u_h^{(p)n+1^2} - u_h^{(p)n^2})$$

Accordingly on summing over the elements with a proper choice of the interface flux, and transforming the reference element to the physical elements, the broken norm

$$\sum_{elements} \int_{x_n}^{x_{n+1}} \left(\frac{u_h^2}{2} + \frac{c}{4} h_n^{2p} u_h^{(p)^2} \right) dx$$

is strictly non-increasing.

V.D. Energy Stability of Space-Time High-order Schemes

An alternative approach is to use a flux reconstruction or nodal DG scheme with space-time elements.³⁹ In the case of linear advection

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} f(u) = 0 \quad (13)$$

with

$$f(u) = au$$

multiplication by u and integration of space and time yields

$$\int_0^T \int_0^L \left(\frac{\partial}{\partial t} \left(\frac{u^2}{2} \right) + a \frac{\partial}{\partial x} \left(\frac{u^2}{2} \right) \right) dx dt = 0$$

Hence if

$$u(0, t) = 0$$

at the inflow boundary,

$$\frac{1}{2} \int_0^L u(x, T)^2 dx = \frac{1}{2} \int_0^L u(x, 0)^2 dx - \frac{1}{2} a \int_0^T u(L, T)^2 dt$$

and the energy decreases due to escape of energy through the outflow boundary.

Now, assuming each element is mapped to a reference element $-1 \leq t \leq 1$, $-1 \leq x \leq 1$, let $l_i(x)$ and $n_n(t)$ be Lagrange basis polynomials of degree p and q respectively and represent the discrete solution as

$$u_h(x, t) = \sum_{j=1}^{p+1} \sum_{n=1}^{q+1} u_{jn} l_j(x) n_n(t)$$

Multiplying by test functions of the form

$$\phi(x, t) = l_i(x) n_n(t)$$

and integrating by parts over space and time yields $(p+1)(q+1)$ equations of the form

$$\begin{aligned} & - \int_{-1}^1 \int_{-1}^1 (u \phi_t + a u \phi_x) dx dt \\ & + \int_{-1}^1 \hat{u}(x, 1) \phi(x, 1) dx - \int_{-1}^1 \hat{u}(x, -1) \phi(x, -1) dx \\ & + a \int_{-1}^1 \hat{u}(1, t) \phi(1, t) dt - a \int_{-1}^1 \hat{u}(-1, t) \phi(-1, t) dt = 0 \end{aligned}$$

where $\hat{u}$ denotes the common interface flux as before. On the time interfaces it is natural to choose an upwind flux defined as

$$\hat{u}(-1, t) = u_h^{n-1}(1, t)$$

where the superscript indicates the previous element in time. In order to establish energy stability take u_h as the test function to yield

$$\begin{aligned} & - \int_{-1}^1 \int_{-1}^1 (u_h \frac{\partial u_h}{\partial t} + a u_h \frac{\partial u_h}{\partial x}) dx dt \\ & + \int_{-1}^1 \hat{u}(x, 1) u_h(x, 1) dx - \int_{-1}^1 \hat{u}(x, -1) u_h(x, -1) dx \\ & + a \int_{-1}^1 \hat{u}(1, t) u_h(1, t) dt - a \int_{-1}^1 \hat{u}(-1, t) u_h(-1, t) dt = 0 \end{aligned}$$

Now summing over the elements with a proper choice of the interface fluxes, the energy of the discrete solution is found to be non-increasing.

The flux reconstruction approach can also be adopted to space time elements by setting

$$\begin{aligned} f_{cl_n} &= \hat{u}(-1, t_n) - u_h(-1, t_n) \\ f_{cr_n} &= \hat{u}(1, t_n) - u_h(1, t_n) \\ f_{ct_j} &= \hat{u}^{n-1}(x_j, 1) - u_h^n(x_j, -1) \end{aligned}$$

Then the fluxes are defined as

$$\begin{aligned} f_h &= a u_h + \sum_{n=1}^{q+1} (f_{cl} g_l(x) + f_{cr} g_r(x)) n_n(t) \\ g_h &= u_h + \sum_{j=1}^{p+1} f_{ct} g_t(t) l_j(x) \end{aligned}$$

where g_l , g_r are polynomials of degree $p+1$ satisfying

$$g_l(-1) = 1, g_l(1) = 0$$

$$g_r(-1) = 0, g_r(1) = 1$$

and g_t is a polynomial of degree $q+1$ satisfying

$$g_t(-1) = 1, g_t(1) = 0$$

Now the discrete equation takes the form

$$\frac{\partial g_h}{\partial t} + \frac{\partial f_h}{\partial x} = 0$$

to be evaluated at each collocation point. Choosing g_l , g_r such that they are orthogonal to all polynomials of degree $< p$ and g_t orthogonal to all polynomials of degree $< q$ recovers the space-time DG scheme.

VI. Applications to Transitional Flows

Section VI and VII present some representative results obtained with the Stanford Aerospace Computing Laboratory’s three-dimensional SD Navier Stokes solver.

VI.A. Three-Dimensional Simulations of Transitional Flow Over an SD7003 Airfoil

Simulations of transitional flow over an SD7003 airfoil at a 4° angle of attack have been performed without the introduction of any turbulence or subgrid scale (SGS) model. The SD7003 airfoil was selected due to availability of existing experimental⁴⁰ and computational⁴¹ data. Forth-order accurate simulations were undertaken on meshes with 1.7×10^6 degrees of freedom at $Re = 1 \times 10^4$ and $Re = 6 \times 10^4$. When $Re = 1 \times 10^4$ the flow was essentially 2D with close-to-periodic vortex shedding (Fig. 1(a)). The computed average lift coefficient was 0.381 and the average drag coefficient was 0.0497, which are consistent with those obtained by.⁴¹ At a Reynolds number of $Re = 6 \times 10^4$ transition was observed to take place across a laminar separation bubble (Figure 1(b)).

The Q-criterion Q provides a mean of visualizing vortex cores and identify turbulent structures. It can be calculated as

$$Q = \frac{1}{2}(\Omega_{ij}\Omega_{ij} - S_{ij}S_{ij}) \quad (14)$$

where

$$\Omega_{ij} = \left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right), \quad S_{ij} = \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \quad (15)$$

are the anti-symmetric and symmetric parts of the velocity gradient tensor respectively. Figure 1 shows instantaneous iso-surfaces of Q over the SD7003 airfoil for cases where $Re = 1 \times 10^4$ and $Re = 6 \times 10^4$.

Figure 1: Instantaneous iso-surfaces of Q above an SD7003 airfoil at a 4° angle of attack. When $Re = 1 \times 10^4$ (a) the flow is essentially 2D and remains laminar over the wing surface, while at $Re = 6 \times 10^4$ (b) transition takes place across a laminar separation bubble.

These results suggest that the SD method appears to be capable of accurately predicting the laminar separation and transition locations over an SD7003 airfoil at $Re = 6 \times 10^4$ using a mesh with insufficient resolution for DNS, and without the introduction of a SGS model. The effectiveness of this implicit LES is confirmed by Table 1. However, we believe that there are numerous industrial applications for which full LES using SGS modeling will be essential.

VI.B. Three-Dimensional Simulations of Transitional Flow Over an Eppler 61 Airfoil

As an effort towards a better understanding of ILES, three-dimensional simulations of flow over another airfoil, i.e. Eppler61 airfoil, at transitional Reynolds number have been conducted. The flow is solved using a 5th order spectral difference method on a $96 \times 16 \times 16$ mesh. We have investigated the flow over the Eppler61 airfoil at a range of angle of attacks. The angle of attacks studied include 6° , 8° , 10° , 12° , and 14° .

We compare the lift and drag coefficients with both wind tunnel and water tunnel experimental data.⁴⁵ The numerical results can fairly accurately predict the angle of stall and the drag rise point, as shown in Figure 3. The present results tend to overpredict the lift, but this overprediction is not unique in our case. Our computational results quite closely match the numerical results (not plotted here) by Mittal 2010.⁴⁶

Data Set	Freestream Turbulence	Separation x_{sep}	Transition x_{tr}/c	Reattachment x_r/c
Ref ⁴⁰	0.08%	0.30	0.53	0.64
Ref ⁴²	0.1%	0.18	0.47	0.58
Ref ⁴³	0%	0.23	0.55	0.65
Ref ⁴¹	0%	0.23	0.51	0.60
Present ILES ⁴⁴	0%	0.23	0.53	0.64

Table 1: Measured and computed properties of the flow over the SD7003 airfoil at $Re = 6 \times 10^4$ and 4° angle of attack.

Figure 2: Vorticity and Mach contours at airfoil midspan for flows over the Eppler61 airfoil at various angles of attack at Mach number=0.2 and $Re=46,000$.

However, note that the variation between the computational results and experimental results are well within the variation between the experiments themselves.

The flow patterns of the various cases are visualized by plotting the Mach and vorticity contours, and the vorticity isosurfaces, as shown in Figure 2 and Figure 4. For the flow condition with Mach=0.2 and $Re=46,000$, the 5th order SD solution seems to be able to capture the transition from laminar to turbulent flow, and resolve the small scale turbulent feature of flow separation. This is illustrated by the vorticity isosurface plots.

The 3D results obtained so far have been quite promising. The key objectives of the ongoing work are to qualitatively predict the separation, transition and reattachment locations and compare them with the experimental data (see Burns⁴⁷).

VII. Applications with Moving Boundaries

VII.A. 3D Simulation of Flow over a Plunging SD7003 Airfoil

Next we consider the flow past a plunging SD7003 wing at Reynolds number of 40,000. The flow physics and computational requirements of this case are more challenging. With experimental results available to compare with, this test case has been carefully examined by the referenced study.⁴⁸ We show here that similar results can be obtained with our present 3D solver to account for both the plunging flow and the

(a) CL vs AOA

(b) CD vs AOA

Figure 3: Comparison of CL and CD curves between SD3D and experiments. Experiment 1 is the water tunnel result. Experiment 2 is the wind tunnel result.

Figure 4: Vorticity isosurfaces of flows over the Epper61 airfoil at increasing angles of attack. The isosurfaces are colored by the magnitude of the Mach number.

transitional behavior. The computational grid used here is the same as the one used previously for the steady example, hence details are not included here. The key parameters that set the flow conditions are: $Mach=0.1$; $Re=40,000$; $k=3.93$; $\alpha_0=4deg$; $h_0/c=0.05$. We perform this simulation using a 4th order SD solution, with a total of 1.7 million degree of freedom. Figure 5 shows the process of flow transition from laminar to turbulent structure during the plunging motion, a key feature of flow over this particular airfoil at the transitional Reynolds number.

The aerodynamic forces are plotted in Figure (6). The aerodynamic lift and drag are compared with the corresponding results from the referenced study. Excellent agreements are obtained.

VII.B. 3D Simulation of Flow over a Flapping Wing

As demonstrated previously, the current high-order solver is well suited for simulating vortex dominated flow over complex and complete configurations. Further extending the capability of the existing flow solver to treat dynamically deforming meshes allows complex physical problems to be studied. Computational flapping wing aerodynamics is an area of application that is both fascinating and important. A computational tool

Figure 5: Vorticity isosurface over the plunging SD7003 airfoil. (a) to (c) show the process of flow transition from laminar to turbulent

(a) CL Time History with SD3D

(b) CL Time History from Visbal's study⁴⁸

(c) CD Time History with SD3D

(d) CD Time History from Visbal's study⁴⁸

Figure 6: Aerodynamic Lift and Drag Time Histories for the Transitional Flow over a Plunging SD7003 Airfoil at $Re=40,000$

for analyzing natural flyers compliments difficult experiments and allows us to better understand the physics of bird and insect flight. This can lead us to develop useful flapping wing aerial vehicles based on the same principles. Work towards this direction has been undertaken under the current numerical framework. Some preliminary results of flapping wing simulations are shown here. The simulations for flow over the flapping Naca0012 wing have been performed with 5th order SD method. With 37,704 cells in the starting mesh, this corresponds to 4,713,000 degrees of freedom. The flow Reynolds number is 2,000. Mach number is 0.1. In Figure 7 (a) the flow fields of the flapping wing motion is illustrated by plotting the flow entropy contours near the flapping wing in an instance about $\frac{2}{3}$ through a downstroke. The contours are colored by the magnitude of Mach number. In this case the wing is flapping at a Strouhal number of 0.3 and net thrust

Figure 7: (a) Entropy isosurface of flow over a flapping wing with naca0012 cross-section at Mach=0.1, Re=2000. (b) Time histories of the aerodynamic force coefficients. Negative mean value of drag indicates the generation of thrust.

is produced, as indicated by the negative value of the drag coefficient curve (green line) in Figure 7 (b).

VII.C. 3D Simulation of Flow over a Spinning Sphere

The flow over a spinning sphere has also been investigated using the high-order SD scheme and the results are outlined here. The flow is laminar. The Reynolds number based on sphere diameter is 300. We compare the present results with the existing results by Kim,⁴⁹ both quantitatively, in terms of streamline pattern, and qualitatively, in terms of aerodynamic force coefficients, and good agreements have been found. The comparisons of force coefficients are summarized in Figure 8 and 9.

Figure 8: Simulation results for laminar flow over a stationary sphere at Re=300. The streamline pattern is colored by Mach number level.

The non-dimensional rotational speed is defined as $\Omega = \frac{\omega R}{U_\infty}$. We consider the case of $\Omega = 0.6$, with the axis of rotation being in the transverse direction, i.e. perpendicular to the streamwise flow. The resulting flow with $\Omega = 0.6$ reaches a steady state. In the x-y plane that cuts through the center of the sphere, we plot the pressure and Mach contours together with the streamline pattern in Figure(9). The upward deflection of

(a) Pressure and Mach contours

(b) Surface pressure and streamline pattern

Figure 9: Simulation results for laminar flow over a spinning sphere at $Re=300$. The streamline pattern is colored by Mach number level.

the streamline is easily observed. In this plane we see the effect of spinning more clearly. The rotation of the sphere in the anti-clockwise direction leads high speed and low pressure region underneath the sphere, where the direction of rotation is in line with the freestream flow. This results in lift being generated perpendicular to the freestream flow, as expected. The present work can be extended in the future to study various aspects of sports aerodynamics, including flow simulations of tennis balls or golf balls.

VIII. Conclusion

The combined efforts of numerous investigators are laying the groundwork for a new era of computational fluid dynamics in which it will become feasible to perform accurate simulations of currently intractable problems involving vortex dominated transitional and turbulent flows. In the earlier phases of CFD development, advances in numerical algorithms and computer hardware made roughly equal contributions to the overall improvement in capability. This is likely to remain true. Advances in parallel computing such as the introduction of multi-core chips and general purpose graphical processing units (GPGPUs), are leading to a radical increase in the available computational power at affordable costs. The effective exploitation of the new hardware presents challenging programming problems.⁵⁰ By significantly reducing diffusive errors, high order methods should enable large eddy simulations (LES) which can accurately capture the bulk of the turbulent energy at Reynolds numbers representative of industrial flows. The extent to which it may be possible to perform accurate simulations without a subgrid scale model (implicit LES) is the subject of ongoing research. More research is also needed on subgrid scale models for wall bounded flows. But in any case we can anticipate wider adoption of LES in industrial applications. High order methods for unstructured meshes are also likely to be key to the improvement of aeroacoustic simulations of complex geometric configurations, such as high lift systems and landing gears.

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