An Extension of Energy Stable Flux Reconstruction to Unsteady, Non-linear, Viscous Problems on Mixed Grids

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This paper extends the high-order Flux Reconstruction (FR) approach to the treatment of non-linear diffusive fluxes on triangles. The FR approach for solving diffusion problems is reviewed on quadrilaterals and extended for triangles, allowing the treatment of mixed grids. In particular, this paper examines a subset of FR schemes, referred to as Vincent-Castonguay-Jameson-Huynh (VCJH) schemes, which are provably stable across all orders of accuracy for linear fluxes in first order systems. The correction fields of the VCJH schemes are shown to represent a family of lifting operators which are used to enforce inter-element continuity of the solution and the diffusive flux. For diffusion problems, the lifting operators of nodal DG schemes are shown to be a subset of this family. Finally, numerical results are used to show the effectiveness of VCJH schemes for a range of problems, including the model diffusion equation and the compressible Navier-Stokes equations. Optimal orders of accuracy are obtained on unstructured mixed meshes of triangular and quadrilateral elements.

I. Introduction

In recent times, a variety of unstructured high-order methods have become popular. These methods are compelling because of their ability to produce high-fidelity results on unstructured grids in the vicinity of complex geometries. In general, high-order methods are less dissipative than their low-order counterparts,¹ which makes them ideal for simulating vortex-dominated flows in the vicinity of flapping wings,²⁻⁴ rotorcraft blades,⁵⁻⁷ and turbomachinery.^{8,9} High-order methods are well-suited for any application in which there are significant interactions between a trailing body and the wake of a foregoing body. In addition, high-order methods are ideally-suited for applications with low-error tolerances, including aeroacoustics¹⁰⁻¹² and the DNS of turbulent flows.^{13,14} The benefits of high-order methods are partially tempered by several factors. Relative to their low-order counterparts, high-order methods are more computationally costly, less robust, and more difficult to implement. Further research efforts are required to facilitate their wide-spread adoption.¹⁵

Currently, the majority of research efforts are focused on Discontinuous Galerkin (DG) methods. The DG approach was first used by Reed and Hill¹⁶ in 1973 for the solution of the hyperbolic neutron transport equation. Since that time, DG methods have been extended to elliptic problems by researchers, principally Wheeler, Arnold, and Brezzi.¹⁷⁻²⁰ In addition, Cockburn and Shu rigorously examined DG methods in the context of the advection-diffusion equation.²¹ Other researchers (Bassi, Rebay, Baumann, and Oden),^{22,23} extended the DG approach to solving the compressible Navier-Stokes equations. In this context, recent times have seen the invention of new treatments for shock-capturing and viscous terms.^{24,25} The last few years have also seen the emergence of potentially more efficient and intuitive DG schemes, referred to as

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nodal DG schemes (see the work of Hesthaven and Warburton).²⁶ Also, other high-order methods have arisen as alternatives to DG. The Spectral Difference (SD) approach was originally proposed by Kopriva and Kolias²⁷ and later generalized by Liu, Vinokur, and Wang.²⁸ It has since been shown (by Jameson)²⁹ that one particular SD scheme is provably stable for linear fluxes in first order systems. (Note that in this context, the stability of SD was shown for 1D and hypothesized for tensor-product extensions to 2D and 3D via quadrilateral and hexahedral elements). SD approaches have proved successful in providing solutions to a variety of problems including several involving the compressible Navier-Stokes equations.^{30–33} The particular SD approach identified by Jameson is attractive due to its larger CFL limit (relative to DG).

A number of high-order schemes (including DG and SD) have been unified into a single framework called Flux Reconstruction (FR). The FR approach is an effort to address the disadvantages of high-order methods by attempting to make them simultaneously more robust, intuitive, and cost-efficient. In 2007, Huynh introduced FR,³⁴ and showed that (for linear advection) it was capable of recovering SD type schemes and collocation-based nodal DG schemes. Validation was presented in the form of numerical results for model problems in 1D. Huynh also demonstrated that the FR approach could be used to form several new schemes with improved time step limits (relative to DG).³⁴ Following this, Vincent, Castonguay, and Jameson proved the energy-stability of a subset of FR schemes.³⁵ These schemes (henceforth referred to as Vincent-Castonguay-Jameson-Huynh (VCJH) schemes) are stable in a broken-Sobolev type norm for all orders of accuracy, for fluxes of the form $f = f(u)$ where f is linear. The VCJH schemes constitute an infinite range of schemes, parameterized by a single constant c . With the proper choice of c , stable DG and SD schemes are recovered. Several of these schemes have improved stability properties, with larger CFL limits than standard DG schemes.³⁵ Since their inception, VCJH schemes have been predominantly applied to first order systems of partial differential equations. In this context, they have been extended to triangles³⁶ and applied to the treatment of linear fluxes, and non-linear fluxes (via application to the Euler equations).³⁷

In 2009, Huynh began preliminary efforts³⁸ towards extending general FR schemes to solve systems with second order partial derivatives (systems with fluxes of the form $f = f(u, \nabla u)$). These systems typically arise in the study of diffusive (viscous) processes. Preliminary efforts in this area have been limited to linear diffusion problems in 1D and in 2D (using quadrilaterals) on structured grids.^{38,39} In an attempt to extend FR to simplex elements, Wang, Gao, and Haga have modified the FR approach, referring to the modified form as the Lifting Collocation Penalty (LCP) approach. The LCP framework has been applied with success to second-order systems on mixed grids.^{40–42} However, the mathematical stability of this approach has not been fully explored. The purpose of this paper is to extend the provably stable VCJH schemes to the treatment of non-linear time-dependent diffusion problems on triangles and unstructured mixed grids. The paper is structured as follows. Section II presents a review of the FR approach in 1D for solving second order PDEs. Then it presents the extension of the FR approach to 2D quadrilaterals (via tensor products) and to triangles (via the use of VCJH correction fields). Section III presents both the general form of the VCJH schemes, and the stability results for schemes on quadrilaterals and triangles. The paper concludes with Section IV, which demonstrates the effectiveness of different VCJH schemes on a series of test problems: time-dependent diffusion of a scalar, Couette flow, and viscous compressible flow over an airfoil.

II. The Flux Reconstruction Approach

This section describes the basic principles of flux reconstruction and its application to time-dependent diffusion problems on quadrilaterals and triangles.

A. The FR Approach for Diffusive Fluxes in 1D

Consider the 1D time-dependent second order equation

$$\frac{\partial u}{\partial t} - \frac{\partial f}{\partial x} = 0. \quad (2.1)$$

In general, the flux is a non-linear function which takes the form $f = f(u, \frac{\partial u}{\partial x})$. For model problems (such as the diffusion equation or the heat equation) the flux takes the linear form $f = \alpha(\frac{\partial u}{\partial x})$, where α is a measure of the diffusivity. For convenience, the general governing equation (Eq. (2.1)) is rewritten as the following first order system,

$$\frac{\partial u}{\partial t} - \frac{\partial}{\partial x} [f(u, q)] = 0, \quad (2.2)$$

$$q - \frac{\partial u}{\partial x} = 0. \quad (2.3)$$

Consider solving this system within an arbitrary periodic domain Ω , where x is the spatial coordinate, t is time, $u = u(x, t)$ is the quantity undergoing diffusion, $q = q(u, \frac{\partial u}{\partial x})$ is an auxiliary variable, and $f = f(u, q)$ is a measure of the flux of u in the direction of x . Furthermore, consider partitioning Ω into N non-overlapping, distinct elements each denoted $\Omega_n = \{x | x_n < x < x_{n+1}\}$.

$$\Omega = \bigcup_{n=1}^N \Omega_n, \quad \bigcap_{n=1}^N \Omega_n = \emptyset. \quad (2.4)$$

Once Ω has been partitioned into separate elements, approximate representations for u , f , and q can be introduced. This amounts to applying the FR procedure separately to each equation in the first order system. In brief, the FR procedure consists of forming a quantity which is piecewise discontinuous between elements and updating it with a quantity which is piecewise continuous between elements. (Consult references 34 and 35 for details of this procedure for linear first-order systems). In the first equation of the system (Eq. (2.2)), the solution u is a piecewise discontinuous quantity that must be updated based on a piecewise continuous flux f . In the second equation (Eq. (2.3)), the auxiliary variable q is a piecewise discontinuous quantity that must be updated based on a piecewise continuous quantity u . These two observations guide the FR formulation for the approximate representations of u , f , and q .

In Eq. (2.2), the solution u within each Ω_n is represented by a polynomial of degree p denoted $u_n^{\delta D} = u_n^{\delta D}(x, t)$ (which is piecewise discontinuous between elements), and the flux f within each Ω_n is represented by a polynomial of degree $p + 1$ denoted $f_n^\delta = f_n^\delta(x, t)$ (which is piecewise continuous between elements). In Eq. (2.3), the auxiliary variable q within each Ω_n is represented by a polynomial of degree p denoted $q_n^{\delta D} = q_n^{\delta D}(x, t)$ (which is piecewise discontinuous between elements), and the solution u within each Ω_n is represented by a polynomial of degree $p + 1$ denoted $u_n^\delta = u_n^\delta(x, t)$ (which is piecewise continuous between elements). Within Ω , the total (domain-wide) approximate quantities $u^\delta = u^\delta(x, t)$, $f^\delta = f^\delta(x, t)$, $u^{\delta D} = u^{\delta D}(x, t)$, and $q^{\delta D} = q^{\delta D}(x, t)$ are given by

$$u^\delta = \sum_{n=1}^N u_n^\delta, \quad f^\delta = \sum_{n=1}^N f_n^\delta, \quad (2.5)$$

$$u^{\delta D} = \sum_{n=1}^N u_n^{\delta D}, \quad q^{\delta D} = \sum_{n=1}^N q_n^{\delta D}. \quad (2.6)$$

Note that the polynomials in the n th element are confined to that element and assume the value of zero outside. Within each element Ω_n , the approximation to the first order system becomes,

$$\frac{\partial u_n^{\delta D}}{\partial t} - \frac{\partial f_n^\delta}{\partial x} = 0, \quad (2.7)$$

$$q_n^{\delta D} - \frac{\partial u_n^\delta}{\partial x} = 0. \quad (2.8)$$

Note that, throughout this paper, quantities which are piecewise discontinuous between elements will appear with a superscript D and those which are piecewise continuous between elements will appear sans the superscript. For simplicity, it is useful to rewrite equations (2.7) and (2.8) with respect to a reference coordinate system. This involves transforming each Ω_n to a reference element $\Omega_s = \{r | -1 \leq r \leq 1\}$ via the mapping

$$x = \Theta_n(r) = \left(\frac{1-r}{2}\right) x_n + \left(\frac{1+r}{2}\right) x_{n+1}, \quad (2.9)$$

which has as its inverse

$$r = \Theta_n^{-1}(x) = 2 \left(\frac{x - x_n}{x_{n+1} - x_n} \right) - 1. \quad (2.10)$$

Following this transformation, the evolution of u^δ within any individual $\mathbf{\Omega}_n$ can be determined by solving the transformed governing system of equations within the reference element $\mathbf{\Omega}_s$

$$\frac{\partial \hat{u}^{\delta D}}{\partial t} = \frac{\partial \hat{f}^\delta}{\partial r}, \quad (2.11)$$

$$\hat{q}^{\delta D} = \frac{\partial \hat{u}^\delta}{\partial r}. \quad (2.12)$$

Here the transformed quantities are defined as

$$\hat{u}^{\delta D} = \hat{u}^{\delta D}(r, t) = u_n^{\delta D}(\Theta_n(r), t), \quad (2.13)$$

$$\hat{q}^{\delta D} = \hat{q}^{\delta D}(r, t) = q_n^{\delta D}(\Theta_n(r), t) \quad (2.14)$$

where both quantities ($\hat{u}^{\delta D}$ and $\hat{q}^{\delta D}$) are polynomials of degree p , and

$$\hat{u}^\delta = \hat{u}^\delta(r, t) = \frac{u_n^\delta(\Theta_n(r), t)}{J_n}, \quad (2.15)$$

$$\hat{f}^\delta = \hat{f}^\delta(r, t) = \frac{f_n^\delta(\Theta_n(r), t)}{J_n} \quad (2.16)$$

where both quantities ($\hat{u}^\delta$ and $\hat{f}^\delta$) are polynomials of degree $p + 1$. Here J_n is the jacobian defined as $J_n = (x_{n+1} - x_n)/2$.

There are seven stages in the FR approach for solving equations (2.11) and (2.12) within the standard element $\mathbf{\Omega}_s$. The first stage involves defining a specific form for $\hat{u}^{\delta D}$. To this end, it is assumed that the value of $\hat{u}^{\delta D}$ is known at a set of $p + 1$ solution points inside $\mathbf{\Omega}_s$, with each point located at a distinct position r_i ($i = 0$ to p). Lagrange polynomials $l_i = l_i(r)$ defined as

$$l_i = \prod_{j=0, j \neq i}^p \left(\frac{r - r_j}{r_i - r_j} \right) \quad (2.17)$$

can then be used to construct the following expression for $\hat{u}^{\delta D}$

$$\hat{u}^{\delta D} = \sum_{i=0}^p \hat{u}_i^{\delta D} l_i, \quad (2.18)$$

where $\hat{u}_i^{\delta D} = \hat{u}_i^{\delta D}(t)$ are the known values of $\hat{u}^{\delta D}$ at the solution points r_i .

The second stage of the FR approach involves calculating numerical interface fluxes for $\hat{u}^{\delta D}$ at either end of the standard element $\mathbf{\Omega}_s$ (at $r = \pm 1$). In order to calculate these fluxes, one must first obtain values for the approximate solution $u^{\delta D}$ at either end of the standard element via Eq. (2.18). Once these values have been obtained they can be used in conjunction with analogous information from adjoining elements to calculate numerical interface fluxes. The exact methodology for calculating such numerical interface fluxes has been thoroughly researched. Well-known approaches for computing these fluxes include the Central Flux (CF),²⁶ Local Discontinuous Galerkin (LDG),²¹ Compact Discontinuous Galerkin (CDG),²⁵ and Internal Penalty (IP) approaches.¹⁸ (Note that the Central Flux approach is a form of the commonly used Bassi-Rebay approach,⁴³ often denoted BR2). In what follows, the numerical fluxes associated with the left and right hand ends of $\mathbf{\Omega}_s$ (and transformed appropriately for use in $\mathbf{\Omega}_s$) will be denoted $\hat{u}_L^{\delta I}$ and $\hat{u}_R^{\delta I}$ respectively.

Before constructing the numerical flux for u , the following definitions must be introduced. Let the values of the solution on either side of the interface I be denoted as u_- and u_+ . Represent the average of u and the jump in u as $\{u\}$ and $[u]$ respectively, where

$$\{u\} = \frac{u_- + u_+}{2}, \quad [u] = u_+ \mathbf{n}_+ + u_- \mathbf{n}_-. \quad (2.19)$$

Here $\mathbf{n}_-$ denotes the outward pointing unit normal from the element on the left side of the interface and $\mathbf{n}_+$ denotes the outward pointing unit normal for the element on the right side of the interface. Using these definitions, the numerical flux for the CF approach becomes

$$u^{\delta I} = \{u^{\delta D}\}, \quad (2.20)$$

and the numerical flux for the LDG approach becomes

$$u^{\delta I} = \{u^{\delta D}\} + \beta \cdot [u^{\delta D}], \quad \beta = \pm(1/2)\mathbf{n}_+. \quad (2.21)$$

Generally, the numerical flux formulation determines the size of the stencil for the scheme. While the CF approach is the most intuitive, it also has the largest stencil (information from four neighboring cells is required).²⁶ Depending on the sign of β , the LDG flux (in 1D) reduces to upwinding or downwinding on u . The result is that the LDG flux produces a compact stencil (involving only the two nearest neighboring cells).²⁶ For alternative formulations of the numerical flux, the reader is referred to Ref. 25 and Ref. 18 for details of constructing the CDG and IP fluxes.

Before proceeding with the remaining steps in the FR approach, the untransformed numerical flux $u^{\delta I}$ is transformed back to the reference coordinate system $\hat{u}^{\delta I}$. The necessary transformation appears in Eq. (2.15).

The third stage of the FR approach involves adding a degree $p + 1$ transformed solution correction $\hat{u}^{\delta C} = \hat{u}^{\delta C}(r, t)$ to the approximate transformed discontinuous solution $\hat{u}^{\delta D}$, such that their sum equals the transformed numerical flux at $r = \pm 1$. In order to define $\hat{u}^{\delta C}$ such that it satisfies the above requirements, consider first defining degree $p + 1$ correction functions $h_L = h_L(r)$ and $h_R = h_R(r)$ that approximate zero (in some sense) within Ω_s , as well as satisfying

$$h_L(-1) = 1, \quad h_L(1) = 0, \quad (2.22)$$

$$h_R(-1) = 0, \quad h_R(1) = 1, \quad (2.23)$$

and, based on symmetry considerations

$$h_L(r) = h_R(-r). \quad (2.24)$$

A suitable expression for $\hat{u}^{\delta C}$ (the solution correction) can now be written in terms of h_L and h_R as

$$\hat{u}^{\delta C} = (\hat{u}_L^{\delta I} - \hat{u}_L^{\delta D})h_L + (\hat{u}_R^{\delta I} - \hat{u}_R^{\delta D})h_R, \quad (2.25)$$

where $\hat{u}_L^{\delta D} = \hat{u}^{\delta D}(-1, t)$ and $\hat{u}_R^{\delta D} = \hat{u}^{\delta D}(1, t)$. Using this expression, a continuous degree $p + 1$ transformed solution $\hat{u}^{\delta} = \hat{u}^{\delta}(r, t)$ within Ω_s can be constructed from the discontinuous solution and $\hat{u}^{\delta C}$ as follows

$$\hat{u}^{\delta} = \hat{u}^{\delta D} + \hat{u}^{\delta C} = \hat{u}^{\delta D} + (\hat{u}_L^{\delta I} - \hat{u}_L^{\delta D})h_L + (\hat{u}_R^{\delta I} - \hat{u}_R^{\delta D})h_R. \quad (2.26)$$

The fourth stage of FR involves applying the gradient operator $\partial/\partial r$ to the piecewise continuous solution $\hat{u}^{\delta}$. The resulting expression is equivalent to $\hat{q}^{\delta D}$ by Eq. (2.12).

$$\hat{q}^{\delta D} = \frac{\partial \hat{u}^{\delta}}{\partial r} = \left[\frac{\partial \hat{u}^{\delta D}}{\partial r} + (\hat{u}_L^{\delta I} - \hat{u}_L^{\delta D}) \frac{\partial h_L}{\partial r} + (\hat{u}_R^{\delta I} - \hat{u}_R^{\delta D}) \frac{\partial h_R}{\partial r} \right] \quad (2.27)$$

By this formula, the value of $\hat{q}^{\delta D}$ is known at a set of $p + 1$ solution points inside Ω_s . It can be represented using a nodal form (i.e. the same form as $\hat{u}^{\delta D}$ in Eq. (2.18)). Written in terms of Lagrange polynomials, it becomes

$$\hat{q}^{\delta D} = \sum_{i=0}^p \hat{q}_i^{\delta D} l_i, \quad (2.28)$$

where $\hat{q}_i^{\delta D} = \hat{q}_i^{\delta D}(t)$ are the known values of $\hat{q}^{\delta D}$ at the solution points r_i . With the fourth stage of FR completed, work on Eq. (2.12) is finished. The remaining steps of the algorithm involve the evaluation of the first equation in the system (Eq. (2.11)).

Since $\hat{q}^{\delta D}$ is known, the transformed discontinuous flux can be computed by $\hat{f}^{\delta D} = \hat{f}^{\delta D}(\hat{u}^{\delta D}, \hat{q}^{\delta D})$. (Note that $\hat{u}^{\delta D}$ is known from Eq. (2.18)). The transformed discontinuous flux can be approximately constructed

from a degree p polynomial $\hat{f}^{\delta D} = \hat{f}^{\delta D}(r, t)$ within Ω_s . A collocation projection at the $p + 1$ solution points is employed to obtain $\hat{f}^{\delta D}$, which can hence be expressed as

$$\hat{f}^{\delta D} = \sum_{i=0}^p \hat{f}_i^{\delta D} l_i \quad (2.29)$$

where the coefficients $\hat{f}_i^{\delta D} = \hat{f}_i^{\delta D}(\hat{u}_i, \hat{q}_i)$ are simply values of the transformed flux at each solution point r_i evaluated directly from the approximate solution and the auxiliary variable. The flux $\hat{f}^{\delta D}$ is termed discontinuous since it is calculated directly from quantities which are in general piecewise discontinuous between elements.

The fifth stage of the FR approach involves calculating numerical interface fluxes for $\hat{f}^{\delta D}$ at either end of the standard element Ω_s . In order to calculate these fluxes, one must first obtain values for the auxiliary variable $q^{\delta D}$ at either end of the standard element via Eq. (2.28). The auxiliary variable $q^{\delta D}$ combines with $u^{\delta D}$ (computed at the element edges in stage 2) to form the discontinuous flux $f^{\delta D}$ at the element boundary. Once these discontinuous flux values have been obtained, they can be used in conjunction with analogous information from adjoining elements to calculate numerical interface fluxes. In what follows the transformed numerical interface fluxes associated with the left and right hand ends of Ω_s will be denoted $\hat{f}_L^{\delta I}$ and $\hat{f}_R^{\delta I}$ respectively. To preserve stability and accuracy, numerical interface fluxes (from stages 2 and 5) must be computed using a consistent formulation.²⁶ This means that $\hat{u}^{\delta I}$ and $\hat{f}^{\delta I}$ should both be computed using the same formulation: CF, LDG, CDG, IP or a similar approach.

The numerical flux can be constructed from the average of f and the jump in f . The average $\{f\}$ and the jump $[f]$ can be computed from previous definitions (see Eq. (2.19)). For the CF approach, the numerical flux becomes

$$f^{\delta I} = \{f^{\delta D}\} - \gamma[u^{\delta D}] \quad (2.30)$$

and for the LDG approach it becomes

$$f^{\delta I} = \{f^{\delta D}\} - \beta \cdot [f^{\delta D}] - \gamma[u^{\delta D}]. \quad (2.31)$$

Here γ is a penalty factor which is ordinarily chosen to be $O(1)$ (see Ref. 44 for guidelines governing how γ is chosen). β and γ can be viewed as penalty factors which control the jumps in u and f . These penalty factors ensure that the scheme converges to the optimal order of accuracy. Failure to include these penalty terms in the numerical flux can result in a loss of accuracy and stability.²¹

Before proceeding with the remaining steps in the FR approach, the untransformed numerical flux $f^{\delta I}$ is transformed back to the reference coordinate system $\hat{f}^{\delta I}$. The necessary transformation appears in Eq. (2.16).

The penultimate stage of the FR approach involves adding a degree $p + 1$ transformed correction flux $\hat{f}^{\delta C} = \hat{f}^{\delta C}(r, t)$ to the transformed discontinuous flux $\hat{f}^{\delta D}$, such that their sum equals the transformed numerical interface flux at $r = \pm 1$. A suitable expression for $\hat{f}^{\delta C}$ can now be written in terms of h_L and h_R as

$$\hat{f}^{\delta C} = (\hat{f}_L^{\delta I} - \hat{f}_L^{\delta D})h_L + (\hat{f}_R^{\delta I} - \hat{f}_R^{\delta D})h_R, \quad (2.32)$$

where $\hat{f}_L^{\delta D} = \hat{f}^{\delta D}(-1, t)$ and $\hat{f}_R^{\delta D} = \hat{f}^{\delta D}(1, t)$. Using this expression, a continuous degree $p + 1$ transformed flux $\hat{f}^{\delta} = \hat{f}^{\delta}(r, t)$ within Ω_s can be constructed from the discontinuous and corrective fluxes as follows

$$\hat{f}^{\delta} = \hat{f}^{\delta D} + \hat{f}^{\delta C} = \hat{f}^{\delta D} + (\hat{f}_L^{\delta I} - \hat{f}_L^{\delta D})h_L + (\hat{f}_R^{\delta I} - \hat{f}_R^{\delta D})h_R. \quad (2.33)$$

The final stage of the FR approach involves calculating the divergence of $\hat{f}^{\delta}$ at each solution point r_i using the expression

$$\frac{\partial \hat{f}^{\delta}}{\partial r}(r_i) = \sum_{j=0}^p \hat{f}_j^{\delta D} \frac{dl_j}{dr}(r_i) + (\hat{f}_L^{\delta I} - \hat{f}_L^{\delta D}) \frac{dh_L}{dr}(r_i) + (\hat{f}_R^{\delta I} - \hat{f}_R^{\delta D}) \frac{dh_R}{dr}(r_i). \quad (2.34)$$

These values can then be used to advance the approximate transformed solution $\hat{u}^{\delta D}$ in time via a suitable temporal discretization of the following semi-discrete expression

$$\frac{d\hat{u}_i^{\delta D}}{dt} = \frac{\partial \hat{f}^{\delta}}{\partial r}(r_i). \quad (2.35)$$

The behavior of a particular FR scheme depends solely on three factors, namely the location of the solution collocation points r_i , the methodology for calculating the transformed numerical interface fluxes ($\hat{u}_L^{\delta I}$, $\hat{u}_R^{\delta I}$, $\hat{f}_L^{\delta I}$, and $\hat{f}_R^{\delta I}$), and finally the form of the flux correction functions h_L and h_R .³⁵

The FR scheme presented above constitutes a variant of the FR approach presented by Huynh in 38. (The approach presented in this paper allows for a more general form for the numerical fluxes). Huynh's numerical experiments on linear diffusion problems in 1D and on quadrilaterals (in 2D) suggest that the FR approach can recover both SD and DG type methods by the choice of certain correction functions h_L and h_R .³⁸ These numerical experiments are supported by theory. For linear diffusion problems, it can be mathematically shown that a collocation based (under integrated) nodal DG scheme is recovered in 1D if the correction functions h_L and h_R are Radau polynomials. (A proof for this result will be presented in a subsequent publication). In addition, Huynh has suggested several alternative forms for h_L and h_R , leading to schemes with various stability and accuracy properties. For details of these schemes the reader is referred to 34 and 38.

Flux reconstruction in 1D can be straightforwardly extended to quadrilateral elements in 2D, via the construction of tensor product bases as described in 34. With a tensor product construction, the transformed approximate solution (of degree p) within the reference element is represented by a set of $(p+1)^2$ solution points as demonstrated in figure (1). Along each element edge, the transformed approximate solution can be uniquely represented by its values at $p+1$ flux points. The location of the solution points along 1D lines in each element automatically fixes the location of the flux points along each edge.

Figure 1: Solution points (circles) and flux points (squares) in the reference quadrilateral element for $p = 2$

B. Extension of FR Approach to Triangles

There are several different FR approaches for solving first-order systems on triangles (see Huynh,⁴⁵ Castonguay et al.,³⁶ and Gao et al.⁴⁰). To the author's knowledge, the approach of Castonguay et al. is the only extension of FR that is provably stable for all orders of accuracy (in the context of linear first-order systems on triangles). This section, and the remainder of the paper, will focus on extending Castonguay et al.'s approach to the treatment of second-order systems.

Begin by considering the 2D time-dependent second order equation

$$\frac{\partial u}{\partial t} - \nabla \cdot \mathbf{f}(u, \nabla u) = 0. \quad (2.36)$$

In general, the flux is a non-linear function which takes the form $\mathbf{f} = \mathbf{f}(u, \nabla u)$. For model problems (such as the diffusion equation or the heat equation) the flux takes the linear form $\mathbf{f} = \alpha \nabla u$, where (as in the 1D case) α is a measure of the diffusivity. For convenience, the general governing equation (Eq. (2.36)) is rewritten as a first order system,

$$\frac{\partial u}{\partial t} - \nabla \cdot \mathbf{f}(u, \mathbf{q}) = 0, \quad (2.37)$$

$$\mathbf{q} - \nabla u = 0. \quad (2.38)$$

This system can be solved within an arbitrary domain Ω , where x and y are spatial coordinates, t is time, $u = u(x, y, t)$ is the diffused quantity, $\mathbf{q} = (q_r, q_s)$ where $q_r = q_r(u, \nabla u)$ and $q_s = q_s(u, \nabla u)$ are the x and y components of the auxiliary variable, and $\mathbf{f} = (f, g)$ where $f = f(u, \mathbf{q})$ and $g = g(u, \mathbf{q})$ are the x and y components of the flux. Consider partitioning the domain Ω into N non-overlapping, conforming linear triangular elements Ω_n such that

$$\Omega = \bigcup_{n=1}^N \Omega_n, \quad \bigcap_{n=1}^N \Omega_n = \emptyset. \quad (2.39)$$

Following the partitioning of Ω into separate elements, approximations to equations (2.37) and (2.38) can be constructed. In Eq. (2.37), the exact solution u within each Ω_n can be replaced by an approximate solution $u_n^{\delta D} = u_n^{\delta D}(x, y, t)$, which is a polynomial of degree p within Ω_n and is identically zero outside the element. Similarly, the flux $\mathbf{f}$ can be replaced by a function $\mathbf{f}_n^{\delta} = (f_n^{\delta}, g_n^{\delta}) = \mathbf{f}_n^{\delta}(x, y, t)$, which is a polynomial of degree $p+1$ within Ω_n and is identically zero outside the element. In Eq. (2.38), the auxiliary variable $\mathbf{q}$ can be replaced by a function $\mathbf{q}_n^{\delta D} = (q_r^{\delta D}, q_s^{\delta D})_n = \mathbf{q}_n^{\delta D}(x, y, t)$ which is a polynomial of degree p within Ω_n and is identically zero outside. Similarly, the exact solution u can be replaced by an approximate solution $u_n^{\delta} = u_n^{\delta}(x, y, t)$, which is a polynomial of degree $p+1$ within Ω_n and is identically zero outside. Within Ω , the total (domain-wide) approximate quantities $u^{\delta} = u^{\delta}(x, y, t)$, $\mathbf{f}^{\delta} = \mathbf{f}^{\delta}(x, y, t)$, $u^{\delta D} = u^{\delta D}(x, y, t)$, and $\mathbf{q}^{\delta D} = \mathbf{q}^{\delta D}(x, y, t)$ are given by

$$u^{\delta} = \sum_{n=1}^N u_n^{\delta}, \quad \mathbf{f}^{\delta} = \sum_{n=1}^N \mathbf{f}_n^{\delta}, \quad (2.40)$$

$$u^{\delta D} = \sum_{n=1}^N u_n^{\delta D}, \quad \mathbf{q}^{\delta D} = \sum_{n=1}^N \mathbf{q}_n^{\delta D}. \quad (2.41)$$

Within in each element Ω_n , the approximate first order system becomes

$$\frac{\partial u_n^{\delta D}}{\partial t} - \nabla \cdot \mathbf{f}_n^{\delta} = 0, \quad (2.42)$$

$$\mathbf{q}_n^{\delta D} - \nabla u_n^{\delta} = 0. \quad (2.43)$$

To aid the implementation, each element Ω_n in physical space is mapped to a reference equilateral triangle Ω_s using a mapping Θ_n as shown in figure (2).

Figure 2: Mapping between the physical space (x, y) and the computational space (r, s)

For a linear triangular element, the mapping Θ_n is

$$\mathbf{x} = \Theta_n(r) = \frac{(-3r + 2 - \sqrt{3}s)}{6} \mathbf{x}_{1,n} + \frac{(2 + 3r - \sqrt{3}s)}{6} \mathbf{x}_{2,n} + \frac{(2 + 2\sqrt{3}s)}{6} \mathbf{x}_{3,n} \quad (2.44)$$

where $\mathbf{x}_{1,n}$, $\mathbf{x}_{2,n}$, and $\mathbf{x}_{3,n}$ are the coordinates of the vertices of the triangular element Ω_n in physical space.

Now define $\hat{\nabla}$ as the gradient in the reference domain taken with respect to r and s (where thus far ∇ has been the gradient in the physical domain taken with respect to x and y). The governing equations ((2.42) and (2.43)) in the physical domain can be transformed to the following equivalent equations in the reference domain

$$\frac{\partial \hat{u}^{\delta D}}{\partial t} = \hat{\nabla} \cdot \hat{\mathbf{f}}^{\delta}, \quad (2.45)$$

$$\hat{\mathbf{q}}^{\delta D} = \hat{\nabla} \hat{u}^{\delta}. \quad (2.46)$$

Here the transformed piecewise discontinuous quantities take the form,

$$\hat{u}^{\delta D} = \hat{u}^{\delta D}(\mathbf{r}, t) = J_n u_n^{\delta D}(\Theta_n(\mathbf{r}), t), \quad (2.47)$$

$$\hat{\mathbf{q}}^{\delta D} = \hat{\mathbf{q}}^{\delta D}(\mathbf{r}, t) = (\hat{q}_r^{\delta D}, \hat{q}_s^{\delta D}) \quad (2.48)$$

$$= \left(\frac{\partial x}{\partial r} q_r^{\delta D} + \frac{\partial y}{\partial r} q_s^{\delta D}, \frac{\partial x}{\partial s} q_r^{\delta D} + \frac{\partial y}{\partial s} q_s^{\delta D} \right)_n \quad (2.49)$$

and the transformed piecewise continuous quantities take the form,

$$\hat{u}^{\delta} = \hat{u}^{\delta}(\mathbf{r}, t) = u_n^{\delta}(\Theta_n(\mathbf{r}), t), \quad (2.50)$$

$$\hat{\mathbf{f}}^{\delta} = \hat{\mathbf{f}}^{\delta}(\mathbf{r}, t) = (\hat{f}^{\delta}, \hat{g}^{\delta}) \quad (2.51)$$

$$= \left(\frac{\partial y}{\partial s} f^{\delta} - \frac{\partial x}{\partial s} g^{\delta}, -\frac{\partial y}{\partial r} f^{\delta} + \frac{\partial x}{\partial r} g^{\delta} \right)_n. \quad (2.52)$$

Finally, the jacobian in each of the above expressions takes the form,

$$J_n = \frac{\partial x}{\partial r} \frac{\partial y}{\partial s} - \frac{\partial x}{\partial s} \frac{\partial y}{\partial r}. \quad (2.53)$$

The metric terms J_n , $\frac{\partial x}{\partial r}$, $\frac{\partial x}{\partial s}$, $\frac{\partial y}{\partial r}$, and $\frac{\partial y}{\partial s}$ (which depend on the shape of the element n) can be evaluated from Eq. (2.44). These terms are valid everywhere within the domain of the element. To make the domains of the n th element and the reference element more precise, the following definitions are introduced. Let Γ_n and Γ_s refer to the boundary of the physical element Ω_n and the reference element Ω_s , respectively.

Prior to extending the FR approach to triangular elements, two more definitions are required. First, let $P_p(\Omega_s)$ define the space of polynomials of degree $\leq p$ on Ω_s , where the dimension of $P_p(\Omega_s)$ is $\frac{1}{2}(p+1)(p+2)$. Also, let the polynomial space $R_p(\Gamma_s)$ on the edges of the reference element be defined as

$$R_p(\Gamma_s) = \{ \phi | \phi \in L^2(\Gamma_s), \phi|_{\Gamma_f} \in P_p(\Gamma_f), \forall \Gamma_f \} \quad (2.54)$$

where Γ_f is used to represent edge f of the reference element Ω_s . Functions of $R_p(\Gamma_s)$ are polynomials of degree $\leq p$ on each side of Ω_s , and are not necessarily continuous at the vertices.

Inline with the 1D FR approach, the 2D FR approach on triangles requires seven steps. In the first stage, the approximate solution $\hat{u}^{\delta D}$ within the reference element Ω_s is represented by a multi-dimensional polynomial of degree p , defined by its values at a set of $N_p = \frac{1}{2}(p+1)(p+2)$ solution points. These solution points are represented by hollow circles in figure (3).

Figure 3: Solution points (circles) and flux points (squares) in the reference element for $p = 2$.

In the figure above, the flux points are located at the Gauss quadrature points. Now that the reference element is completely defined, the approximate solution in that element takes the form

$$\hat{u}^{\delta D}(\mathbf{r}, t) = \sum_{i=1}^{N_p} \hat{u}_i^{\delta D} l_i(\mathbf{r}) \quad (2.55)$$

where $\hat{u}_i^{\delta D} = J_n \cdot \hat{u}^{\delta D}(\Theta_n^{-1}(\mathbf{r}_i), t)$ is the value of $\hat{u}^{\delta D}$ at the solution point i and $l_i(\mathbf{r})$ is the multi-dimensional Lagrange polynomial associated with the solution point i in the reference equilateral triangle Ω_s . This approximate solution $\hat{u}^{\delta D}$ lies in the space $P_p(\Omega_s)$.

The second stage of the FR approach involves calculating numerical interface fluxes for $\hat{u}^{\delta D}$ at flux points along the edges of Ω_s . Let expressions subscripted by the indices f, j correspond to a quantity at the flux point j of face f , where $1 \leq f \leq 3$ and $1 \leq j \leq N_{fp}$. The convention used to number faces and flux points is illustrated in figure (4) for $p = 2$.

Figure 4: Numbering convention for the faces and flux points on the reference triangular element.

As in the 1D FR method, the interface fluxes are computed by first evaluating the multiply defined values of $u^{\delta D}$ at each flux point using Eq. (2.55). At each flux point, $u_-^{\delta D}$ is defined as the value of $u^{\delta D}$ computed using the information local to the current element and $u_+^{\delta D}$ as the value of $u^{\delta D}$ computed using information from the adjoining element that shares the same flux point. Once both approximate solution values ($u_-^{\delta D}$ and $u_+^{\delta D}$) are evaluated at each flux point, an approach (CF, CDG, LDG, IP or something similar) can be used to compute a common numerical flux $u_{f,j}^{\delta I}$. For example, the numerical flux for the CF approach can be computed as

$$u_{f,j}^{\delta I} = \{u_{f,j}^{\delta D}\}, \quad (2.56)$$

and the numerical flux for the LDG approach can be computed as

$$u_{f,j}^{\delta I} = \{u_{f,j}^{\delta D}\} + \beta \cdot [u_{f,j}^{\delta D}], \quad \beta = \pm(1/2)\mathbf{n}_+. \quad (2.57)$$

Here the quantities $\{u\}$ and $[u]$ were computed using Eq. (2.19). Once the common numerical flux is computed, the transformed flux denoted by $\hat{u}_{f,j}^{\delta I}$ can be obtained from Eq. (2.50).

The third stage of the FR approach involves constructing a transformed solution correction $\hat{u}^{\delta C} = \hat{u}^{\delta C}(\mathbf{r}, t)$ to the approximate transformed discontinuous solution $\hat{u}^{\delta D}$, such that their sum equals the transformed numerical flux at the flux points. In order to define $\hat{u}^{\delta C}$ so that it satisfies this requirement, consider defining $\mathbf{h}_{f,j}(\mathbf{r})$, the vector correction function associated with flux point f, j . (Full details of the vector correction functions, including their motivation and construction, are presented in Ref. 36. Here only a brief summary is presented). Each vector correction function $\mathbf{h}_{f,j}(\mathbf{r})$ is restricted to lie in the Raviart-Thomas space⁴⁶ of order p , denoted by $RT_p(\Omega_s)$. Note that the space $RT_p(\Omega_s)$ contains some carefully selected terms of order $p + 1$. Because of this property,

$$\hat{\nabla} \cdot \mathbf{h}_{f,j} \in P_p(\Omega_s) \quad (2.58)$$

$$\mathbf{h}_{f,j} \cdot \hat{\mathbf{n}}|_{\Gamma_s} \in R_p(\Gamma_s). \quad (2.59)$$

This means that the divergence of each correction function ($\hat{\nabla} \cdot \mathbf{h}_{f,j}$) is a polynomial of degree $\leq p$ and the normal trace $\mathbf{h}_{f,j} \cdot \hat{\mathbf{n}}$ on Γ_s is also a polynomial of degree $\leq p$ along each edge. In what follows, the divergence of a correction function will be abbreviated as

$$\phi_{f,j} = \hat{\nabla} \cdot \mathbf{h}_{f,j}(\mathbf{r}). \quad (2.60)$$

For simplicity, the divergence of the correction function ($\phi_{f,j}$) will henceforth be referred to as the ‘correction field.’

The correction functions $\mathbf{h}_{f,j}$ satisfy

$$\mathbf{h}_{f,j}(\mathbf{r}_{f_2,j_2}) \cdot \hat{\mathbf{n}}_{f_2,j_2} = \begin{cases} 1 & \text{if } f = f_2 \text{ and } j = j_2 \\ 0 & \text{if } f \neq f_2 \text{ or } j \neq j_2 \end{cases} \quad (2.61)$$

An example of a vector correction function $\mathbf{h}_{f,j}$ is shown in figure (5) for the case $p = 2$. Note that for a single scheme, multiple vector correction functions are possible.³⁶ For the $p = 2$ case, only one of these possible functions is shown. The properties of a scheme are determined by the form of the correction field ($\phi_{f,j}$), and not by the particular form of the vector correction function ($\mathbf{h}_{f,j}$).³⁶ Thus, the non-uniqueness of $\mathbf{h}_{f,j}$ does not pose a problem.

Figure 5: Example of a vector correction function $\mathbf{h}_{f,j}$ associated with flux point $f = 2, j = 1$ for $p = 2$.

Using the vector correction functions ($\mathbf{h}_{f,j}$)’s, an expression for $\hat{u}^{\delta C}$ on each face f can now be constructed as follows

$$\begin{aligned} \hat{u}^{\delta C} \Big|_{\Gamma_f} &= \sum_{j=1}^{N_{fp}} [\hat{u}_{f,j}^{\delta I} - \hat{u}_{f,j}^{\delta D}] [\mathbf{h}_{f,j} \cdot \hat{\mathbf{n}}_{f,j}] \\ &= \sum_{j=1}^{N_{fp}} \Pi_{f,j} [\mathbf{h}_{f,j} \cdot \hat{\mathbf{n}}_{f,j}]. \end{aligned} \quad (2.62)$$

Here $\Pi_{f,j}$ is defined as the difference between the transformed numerical flux and the transformed discontinuous solution on face f at flux point j . Over the entire boundary Γ_s , $\hat{u}^{\delta C}$ becomes

$$\hat{u}^{\delta C} \Big|_{\Gamma_s} = \begin{cases} \hat{u}^{\delta C} \Big|_{\Gamma_1} & \text{on } \Gamma_1 \\ \hat{u}^{\delta C} \Big|_{\Gamma_2} & \text{on } \Gamma_2 \\ \hat{u}^{\delta C} \Big|_{\Gamma_3} & \text{on } \Gamma_3 \end{cases} \quad (2.63)$$

The fourth stage of FR involves constructing $\hat{\mathbf{q}}^{\delta D}$. This involves evaluating Eq. (2.46).

$$\hat{\mathbf{q}}^{\delta D} = \hat{\nabla} \hat{u}^{\delta} = \hat{\nabla} \hat{u}^{\delta D} + \hat{\nabla} \hat{u}^{\delta C} \quad (2.64)$$

Here the 2D FR formulation deviates from the 1D formulation. In the 1D formulation $\hat{q}^{\delta D}$ was determined by applying the operator $\partial/\partial r$ to $\hat{u}^{\delta D}$ and $\hat{u}^{\delta C}$. Here in the 2D formulation, the situation is more complex. In particular, (as discussed previously) there is not a unique analytical form for the vector correction function ($\mathbf{h}_{f,j}$) that appears in $\hat{u}^{\delta C}$. In addition $\hat{u}^{\delta C}$ is only defined on the element boundary Γ_s , while $\hat{q}^{\delta D}$ is needed within the interior of the element (within Ω_s). Therefore, we cannot form $(\hat{\nabla} \hat{u}^{\delta C})$ by simply applying the $\hat{\nabla}$ operator to $\hat{u}^{\delta C}$. The proper form for $\hat{\mathbf{q}}^{\delta D}$ must be obtained by an alternative approach which involves forming the correction to the gradient $(\hat{\nabla} \hat{u}^{\delta C})$ as follows

$$\hat{\nabla} \hat{u}^{\delta C} = \sum_{f=1}^3 \sum_{j=1}^{N_{fp}} \Pi_{f,j} \psi_{f,j} \hat{\mathbf{n}}_{f,j}. \quad (2.65)$$

Here $\psi_{f,j}$ is a correction field or ‘lifting’ operator. This correction field must transform $\hat{u}^{\delta C}$ defined on Γ_s into $\hat{\nabla} \hat{u}^{\delta C}$ defined within Ω_s . The details of forming $\psi_{f,j}$ will be explained in section III. (Note that the choice for the definition of $\psi_{f,j}$ will play a role in determining the nature of the FR scheme). Using the result of Eq. (2.65), $\hat{\mathbf{q}}^{\delta D}$ becomes

$$\hat{\mathbf{q}}^{\delta D} = \hat{\nabla} \hat{u}^{\delta D} + \hat{\nabla} \hat{u}^{\delta C} = \sum_{i=1}^{N_p} \hat{u}_i^{\delta D} \hat{\nabla} l_i(\mathbf{r}) + \sum_{f=1}^3 \sum_{j=1}^{N_{fp}} \Pi_{f,j} \psi_{f,j} \hat{\mathbf{n}}_{f,j}. \quad (2.66)$$

Here $\hat{\mathbf{q}}^{\delta D} = (\hat{q}_r^{\delta D}, \hat{q}_s^{\delta D})$ is represented by a degree p polynomial for each of its components as follows

$$\hat{q}_r^{\delta D} = \sum_{i=1}^{N_p} (\hat{q}_r^{\delta D})_i l_i(\mathbf{r}) \quad \hat{q}_s^{\delta D} = \sum_{i=1}^{N_p} (\hat{q}_s^{\delta D})_i l_i(\mathbf{r}) \quad (2.67)$$

Once $\hat{\mathbf{q}}^{\delta D}$ is obtained, the approximate discontinuous flux can be computed by $\hat{\mathbf{f}}^{\delta D} = \hat{\mathbf{f}}^{\delta D}(\hat{u}^{\delta D}, \hat{\mathbf{q}}^{\delta D})$. (Note that $\hat{u}^{\delta D}$ is known from Eq. (2.55)). The approximate discontinuous flux $\hat{\mathbf{f}}^{\delta D} = (\hat{f}^{\delta D}, \hat{g}^{\delta D})$ can be expressed with a degree p polynomial for each of its components as

$$\hat{f}^{\delta D} = \sum_{i=1}^{N_p} \hat{f}_i^{\delta D} l_i, \quad \hat{g}^{\delta D} = \sum_{i=1}^{N_p} \hat{g}_i^{\delta D} l_i \quad (2.68)$$

where the coefficients $\hat{f}_i^{\delta D}$ and $\hat{g}_i^{\delta D}$ are the values of the transformed flux at the solution point i evaluated directly from the approximate solution $\hat{u}_i$ and the auxiliary gradient $\hat{\mathbf{q}}_i$ ($\hat{f}_i^{\delta D} = \hat{f}(\hat{u}_i, \hat{\mathbf{q}}_i)$ and $\hat{g}_i^{\delta D} = \hat{g}(\hat{u}_i, \hat{\mathbf{q}}_i)$).

The fifth stage of the FR approach involves calculating numerical interface fluxes for $\hat{\mathbf{f}}^{\delta D}$ at flux points along the edges of Ω_s . The interface fluxes are computed by first evaluating the multiply defined values of $\mathbf{q}^{\delta D}$ at each flux point using Eq. (2.67). At each flux point, $\mathbf{q}_-^{\delta D}$ and $\mathbf{q}_+^{\delta D}$ are obtained. The auxiliary variables $\mathbf{q}_-^{\delta D}$ and $\mathbf{q}_+^{\delta D}$ combine with $u_-^{\delta D}$ and $u_+^{\delta D}$ (computed at the element edges in stage 2) to form the discontinuous fluxes $\mathbf{f}_-^{\delta D}$ and $\mathbf{f}_+^{\delta D}$ at the element boundary. Once both approximate fluxes ($\mathbf{f}_-^{\delta D}$ and $\mathbf{f}_+^{\delta D}$) are evaluated at each flux point, an approach (the same as chosen in stage 2) can be used to compute a common numerical normal flux $(\mathbf{f} \cdot \mathbf{n})_{f,j}^{\delta I}$.

In particular, the numerical flux for the CF approach can be computed using $\{\mathbf{f}\}$ and $[\mathbf{f}]$. However, because $\mathbf{f}$ is a vector, the previous definitions of the average and jump operators for scalar quantities (from Eq. (2.19)) must be re-formulated in the following way

$$\{\mathbf{f}\} = \frac{\mathbf{f}_+ + \mathbf{f}_-}{2}, \quad [\mathbf{f}] = \mathbf{f}_+ \cdot \mathbf{n}_+ + \mathbf{f}_- \cdot \mathbf{n}_-. \quad (2.69)$$

For the CF approach, the numerical flux then becomes

$$\mathbf{f}_{f,j}^{\delta I} = \{\mathbf{f}_{f,j}^{\delta D}\} - \gamma[u_{f,j}^{\delta D}], \quad (2.70)$$

and for the LDG approach it becomes

$$\mathbf{f}_{f,j}^{\delta I} = \{\mathbf{f}_{f,j}^{\delta D}\} - \beta[\mathbf{f}_{f,j}^{\delta D}] - \gamma[u_{f,j}^{\delta D}]. \quad (2.71)$$

Note that in 2D the choice of an LDG flux does not guarantee a compact stencil.⁴⁷ Interface quantities are no longer only dependent on information from nearest neighbors. However, alternative approaches (such as the CDG approach) maintain compactness for higher dimensions. Refer to Ref. 25 for details.

Once the normal numerical flux is computed, the transformed normal numerical flux denoted by $(\hat{\mathbf{f}} \cdot \hat{\mathbf{n}})_{f,j}^{\delta I}$ can be obtained from Eq. (2.51).

In the penultimate stage of the FR approach, the transformed correction flux (for correcting $\hat{\mathbf{f}}^{\delta D}$) is constructed. The correction $\hat{\mathbf{f}}^{\delta C}$ takes the form

$$\begin{aligned} \hat{\mathbf{f}}^{\delta C}(\mathbf{r}) &= \sum_{f=1}^3 \sum_{j=1}^{N_{fp}} \left[(\hat{\mathbf{f}} \cdot \hat{\mathbf{n}})_{f,j}^{\delta I} - (\hat{\mathbf{f}}^{\delta D} \cdot \hat{\mathbf{n}})_{f,j} \right] \mathbf{h}_{f,j}(\mathbf{r}) \\ &= \sum_{f=1}^3 \sum_{j=1}^{N_{fp}} \Delta_{f,j} \mathbf{h}_{f,j}(\mathbf{r}). \end{aligned} \quad (2.72)$$

In Eq. (2.72), $\Delta_{f,j}$ is defined as the difference between the normal transformed numerical flux and the normal transformed discontinuous flux at the flux point f,j .

The final stage of the FR approach involves computing the divergence of the continuous transformed flux $\hat{\mathbf{f}}^{\delta}$. The result is then used to update the solution at the point $\mathbf{r}_i$ via

$$\frac{d\hat{u}_i^{\delta D}}{dt} = \left(\hat{\nabla} \cdot \hat{\mathbf{f}}^{\delta} \right) \Big|_{\mathbf{r}_i} \quad (2.73)$$

$$= \left(\hat{\nabla} \cdot \hat{\mathbf{f}}^{\delta D} \right) \Big|_{\mathbf{r}_i} + \left(\hat{\nabla} \cdot \hat{\mathbf{f}}^{\delta C} \right) \Big|_{\mathbf{r}_i} \quad (2.74)$$

$$= \sum_{k=1}^{N_p} \left(\hat{f}_k^{\delta D} \right) \frac{\partial l_k}{\partial r} \Big|_{\mathbf{r}_i} + \sum_{k=1}^{N_p} \left(\hat{g}_k^{\delta D} \right) \frac{\partial l_k}{\partial s} \Big|_{\mathbf{r}_i} + \sum_{f=1}^3 \sum_{j=1}^{N_{fp}} \Delta_{f,j} \phi_{f,j}(\mathbf{r}_i). \quad (2.75)$$

On triangles, the nature of the FR scheme depends on four factors: the location of the solution collocation points $\mathbf{r}_i$, the location of the flux points $\mathbf{r}_{f,j}$, the methodology for calculating the numerical fluxes $u_{f,j}^{\delta I}$ and $\mathbf{f}_{f,j}^{\delta I}$, and finally the form of the correction fields $\psi_{f,j}$ and $\phi_{f,j}$. The reader should consult references 37 and 48 for details regarding the selection of solution and flux point locations. Section III of this paper will present guidelines for selection of the correction fields.

C. The FR Approach for Inviscid Fluxes

The FR approach was originally applied to problems with fluxes of the form

$$\mathbf{f} = \mathbf{f}(u). \quad (2.76)$$

A large class of problems (ranging from linear advection up to the Euler equations) have fluxes that fit this general form. The FR approach in this context is very similar to what has been presented above, with two major exceptions: 1) $\mathbf{q}$ need not be formed as the flux does not depend on the gradient and 2) the numerical fluxes must take into account the direction of wave-propagation. The arithmetic averaging (from the CF approach) will produce instabilities. A number of researchers have discussed the application of the FR approach in this context. Consult references 34, 35, 36, 37, 45, or 48 for details on how the FR approach treats general inviscid fluxes of the form given by Eq. (2.76).

III. Vincent-Castonguay-Jameson-Huynh Schemes

If the Flux Reconstruction approach is to remain stable, a particular form is required for the correction functions and their associated correction fields. VCJH schemes specify a stability-preserving form for the correction functions and fields.^{35,36} VCJH schemes for quadrilaterals (via 1D tensor products) and triangles are presented in the following sections.

A. VCJH Schemes in 1D

Following the presentation in Ref. 35, VCJH schemes can be recovered if the left and right corrections functions h_L and h_R respectively are defined as

$$h_L = \frac{(-1)^p}{2} \left[\Upsilon_p - \left(\frac{\eta_p \Upsilon_{p-1} + \Upsilon_{p+1}}{1 + \eta_p} \right) \right], \quad (3.1)$$

and

$$h_R = \frac{1}{2} \left[\Upsilon_p + \left(\frac{\eta_p \Upsilon_{p-1} + \Upsilon_{p+1}}{1 + \eta_p} \right) \right], \quad (3.2)$$

where

$$\eta_p = \frac{c(2p+1)(a_p p!)^2}{2}, \quad a_p = \frac{(2p)!}{2^p (p!)^2}, \quad (3.3)$$

Υ_p is a Legendre polynomial of degree p , and c is a free scalar parameter that must lie within the range

$$\frac{-2}{(2p+1)(a_p p!)^2} < c < \infty. \quad (3.4)$$

Such correction functions satisfy

$$\int_{-1}^1 h_L \frac{\partial \hat{u}^{\delta D}}{\partial r} dr - c \left(\frac{\partial^p \hat{u}^{\delta D}}{\partial r^p} \right) \left(\frac{d^{p+1} h_L}{dr^{p+1}} \right) = 0, \quad (3.5)$$

and

$$\int_{-1}^1 h_R \frac{\partial \hat{u}^{\delta D}}{\partial r} dr - c \left(\frac{\partial^p \hat{u}^{\delta D}}{\partial r^p} \right) \left(\frac{d^{p+1} h_R}{dr^{p+1}} \right) = 0, \quad (3.6)$$

within the standard element Ω_s for any transformed solution $\hat{u}^{\delta D}$, and ensure that the resulting VCJH scheme will be linearly stable in the broken Sobolev type norm $\|u^{\delta D}\|_{p,2}$. Explicitly, VCJH correction functions ensure that

$$\frac{d}{dt} \|u^{\delta D}\|_{p,2}^2 \leq 0 \quad (3.7)$$

where

$$\|u^{\delta D}\|_{p,2} = \left[\sum_{n=1}^N \int_{x_n}^{x_{n+1}} (u_n^{\delta D})^2 + \frac{c}{2} (J_n)^{2p} \left(\frac{\partial^p u_n^{\delta D}}{\partial x^p} \right)^2 dx \right]^{1/2}. \quad (3.8)$$

This result (due to Vincent, Castonguay, and Jameson),³⁵ applies only to equations with fluxes of the form $f = f(u)$ where f is linear. It can be noted that several existing methods are encompassed by the new class of VCJH schemes. In particular if $c = 0$ then a collocation based nodal DG scheme is recovered. Alternatively, if

$$c = \frac{2p}{(2p+1)(p+1)(a_p p!)^2}, \quad (3.9)$$

an SD method is recovered (at least for a linear flux function). It is in fact the only SD type scheme that can be recovered from the range of VCJH schemes. Further, it is identical to the SD scheme that Jameson²⁹ proved to be linearly stable, which is the same as the only SD scheme that Huynh found to be devoid of weak instabilities.³⁴ Finally, if

$$c = \frac{2(p+1)}{(2p+1)p(a_p p!)^2}, \quad (3.10)$$

then a so called g_2 FR method is recovered, which was originally identified by Huynh to be particularly stable.³⁴ Because VCJH schemes are a subset of FR schemes, 1D VCJH schemes can be extended to quadrilateral elements via the construction of tensor product bases, following the procedure presented in Ref. 34 for FR schemes.

Thus far, the discussion of VCJH schemes has been limited to first order systems. It turns out that the stability of VCJH schemes can be extended to second order systems with diffusive fluxes of the form $f = f(u, \nabla u)$ where f is linear. Details of the stability proof will be presented in a subsequent publication. Here only the result of the proof is presented.

For the diffusion case, the VCJH schemes are linearly stable via a condition involving two broken Sobolev type norms. Here, one norm applies to the solution and the other applies to the auxiliary variable. VCJH correction functions ensure that

$$\frac{1}{2} \frac{d}{dt} \|u^{\delta D}\|_{p,2}^2 + \|q^{\delta D}\|_{p,2}^2 \leq 0 \quad (3.11)$$

where $\|u^{\delta D}\|_{p,2}$ is as before and $\|q^{\delta D}\|_{p,2}$ is given by

$$\|q^{\delta D}\|_{p,2} = \left[\sum_{n=1}^N \int_{x_n}^{x_{n+1}} (q_n^{\delta D})^2 + \frac{c}{2} (J_n)^{2p} \left(\frac{\partial^p q_n^{\delta D}}{\partial x^p} \right)^2 dx \right]^{1/2}. \quad (3.12)$$

B. VCJH schemes on triangular elements

This section presents an energy-stable VCJH approach for selecting the correction fields $\phi_{f,j}$ and $\psi_{f,j}$ on triangular elements. This approach is an extension of the VCJH approach on triangles discussed in Ref. 36.

1. Specifying the correction field $\phi_{f,j}$

Unlike for the 1D VCJH schemes, a unique closed form expression for the correction functions $\mathbf{h}_{f,j}$ in terms of the parameter κ has not been found.³⁶ However, unique values for the coefficients of the polynomials defining each correction field $\phi_{f,j}$ can be obtained from the solution of a system of equations.³⁶ Before, the aforementioned system of equations is presented, some notation must be introduced. First, let the operator $D^{(v,w)}$ be defined as

$$D^{(v,w)} = \frac{\partial^w}{\partial r^{(w-v+1)} \partial s^{(v-1)}} \quad (3.13)$$

where v and w are integers such that $1 \leq v \leq w + 1$. For example,

$$D^{(1,2)} = \frac{\partial^2}{\partial r^2}, \quad D^{(2,2)} = \frac{\partial^2}{\partial r \partial s}, \quad D^{(3,2)} = \frac{\partial^2}{\partial s^2}. \quad (3.14)$$

Also, consider the 2D orthonormal Dubiner basis⁴⁹ given by

$$L_k(\mathbf{r}) = \frac{4}{3^{1/4}} Q_v(a) Q_w^{(2v+1,0)}(b) (1-b)^v \quad (3.15)$$

$$k = w + (p+1)v + 1 - \frac{v}{2}(v-1), \quad (v, w) \geq 0; \quad v + w \leq p$$

where

$$a = \frac{3r}{2 - \sqrt{3}s}, \quad b = \frac{1}{3}(2\sqrt{3}s - 1) \quad (3.16)$$

and $Q_n^{(\alpha,\beta)}$ is the normalized n^{th} order Jacobi polynomial. The Dubiner basis given by Eq. (3.15) is orthonormal on the reference triangle Ω_s . Next, let the binomial coefficients of degree w be defined by

$$\binom{v}{w} = \frac{v!}{w!(v-w)!}. \quad (3.17)$$

Following the presentations in references 36 and 37, define the VCJH scheme coefficients (c_m 's) as

$$c_m = \kappa \binom{p}{m-1} \quad (3.18)$$

where κ is a free parameter which will be specified later. The correction fields $\phi_{f,j}$ are polynomials of degree p and therefore can be expressed as

$$\phi_{f,j} = \sum_{k=1}^{N_p} \sigma_k L_k(\mathbf{r}) \quad (3.19)$$

where σ_k are the expansion coefficients. In a VCJH scheme, the expansion coefficient σ_k of each correction field $\phi_{f,j}$ can be obtained by solving the following system of equations for the unknowns σ_k (for $k = 1$ to N_p)

$$\sum_{k=1}^{N_p} \sigma_k \sum_{m=1}^{p+1} c_m \left(D^{(m,p)} L_i \right) \left(D^{(m,p)} L_k \right) = -\sigma_i + \int_{\Gamma_S} (\mathbf{h}_{f,j} \cdot \hat{\mathbf{n}}) L_i d\Gamma, \quad \text{for } 1 \leq i \leq N_p \quad (3.20)$$

where κ is a free parameter that must lie within the range $0 \leq \kappa \leq \infty$. The last term on the right hand side of Eq. (3.20) can be evaluated exactly using Eq. (2.61) and the fact that $\mathbf{h}_{f,j} \cdot \hat{\mathbf{n}}$ is a polynomial of order p on each edge of Ω_s .

The correction fields $\phi_{f,j}$ obtained from the solution of Eq. (3.20) result in an energy stable flux reconstruction scheme on triangles parameterized by a single scalar parameter κ . Stability is guaranteed for linear fluxes of the form $f = f(u)$ in a norm of the transformed solution $\|\hat{u}^{\delta D}\|_{p,2}$. From Ref. 36, the stability condition is

$$\frac{d}{dt} \|\hat{u}^{\delta D}\|_{p,2}^2 \leq 0, \quad (3.21)$$

where

$$\|\hat{u}^{\delta D}\|_{p,2} = \left[\sum_{n=1}^N \int_{\Omega_n} \left((\hat{u}_n^{\delta D})^2 + \frac{1}{A_S} \sum_{m=1}^{p+1} c_m \left(D^{(m,p)} \hat{u}_n^{\delta D} \right)^2 \right) d\Omega_n \right]^{1/2}. \quad (3.22)$$

Here A_S is the area of the reference element Ω_s . $\|\hat{u}^{\delta D}\|_{p,2}$ is a norm provided that $0 \leq \kappa < \infty$. If $\kappa = \kappa_{dg} = 0$, the VCJH scheme on triangular elements recovers a collocation based nodal DG scheme.

2. Specifying the correction field $\psi_{f,j}$

In section II, part B, the correction field $\psi_{f,j}$ was used to form $\hat{\nabla} \hat{u}^{\delta C}$ (the correction to the gradient). It turns out that choosing $\psi_{f,j}$ to be a VCJH correction field (of the same form as $\phi_{f,j}$) results in an energy-stable treatment for diffusion problems, and for a particular choice of κ recovers the nodal DG approach. The remainder of this section will motivate the form for $\hat{\nabla} \hat{u}^{\delta C}$ that arises when $\psi_{f,j} = \phi_{f,j}$, i.e.

$$\begin{aligned} \hat{\nabla} \hat{u}^{\delta C} &= \sum_{f=1}^3 \sum_{j=1}^{N_{fp}} \Pi_{f,j} \psi_{f,j} \hat{\mathbf{n}}_{f,j} \\ &= \sum_{f=1}^3 \sum_{j=1}^{N_{fp}} \Pi_{f,j} \phi_{f,j} \hat{\mathbf{n}}_{f,j}. \end{aligned} \quad (3.23)$$

First, the authors will show how this form for $\hat{\nabla} \hat{u}^{\delta C}$ allows VCJH schemes to recover the nodal DG approach. A brief review of the nodal DG approach will provide some necessary context. For nodal DG, consider the first order system given by

$$\frac{\partial \hat{u}^{\delta D}}{\partial t} = \hat{\nabla} \cdot \hat{\mathbf{f}}^{\delta D}, \quad (3.24)$$

$$\hat{\mathbf{q}}^{\delta D} = \hat{\nabla} \hat{u}^{\delta D}. \quad (3.25)$$

This discussion will only be concerned with recovering DG within Eq. (3.25). (Castonguay et al.³⁶ has already shown that VCJH schemes recover nodal DG for conservation laws in the form of Eq. (3.24)). In following the standard DG approach, Eq. (3.25) is dotted with an arbitrary vector test function τ and integrated over the element domain Ω_s ,

$$\int_{\Omega_s} \hat{\mathbf{q}}^{\delta D} \cdot \tau d\Omega_s = \int_{\Omega_s} \hat{\nabla} \hat{u}^{\delta D} \cdot \tau d\Omega_s. \quad (3.26)$$

Here τ is vector whose components are polynomials of degree p (i.e. $\tau \in P_p^2(\Omega_s)$). Integration by parts is then applied to the right hand side of Eq. (3.26)

$$\int_{\Omega_s} \hat{\mathbf{q}}^{\delta D} \cdot \tau d\Omega_s = \int_{\Gamma_s} \hat{u}^{\delta D} (\tau \cdot \hat{\mathbf{n}}) d\Gamma_s - \int_{\Omega_s} \hat{u}^{\delta D} (\hat{\nabla} \cdot \tau) d\Omega_s. \quad (3.27)$$

The quantity $\hat{u}^{\delta D}$ is not well-defined along the element boundary Γ_s . To remedy this (and to ensure communication between elements), DG introduces a numerical flux $\hat{u}^{\delta I}$ to the boundary term

$$\int_{\Omega_s} \hat{\mathbf{q}}^{\delta D} \cdot \tau d\Omega_s = \int_{\Gamma_s} \hat{u}^{\delta I} (\tau \cdot \hat{\mathbf{n}}) d\Gamma_s - \int_{\Omega_s} \hat{u}^{\delta D} (\hat{\nabla} \cdot \tau) d\Omega_s. \quad (3.28)$$

If integration by parts is applied to the second term on the right hand side of Eq. (3.28), we obtain

$$\begin{aligned} \int_{\Omega_s} \hat{\mathbf{q}}^{\delta D} \cdot \tau d\Omega_s &= \int_{\Gamma_s} \hat{u}^{\delta I} (\tau \cdot \hat{\mathbf{n}}) d\Gamma_s - \int_{\Gamma_s} \hat{u}^{\delta D} (\tau \cdot \hat{\mathbf{n}}) d\Gamma_s + \int_{\Omega_s} \hat{\nabla} \hat{u}^{\delta D} \cdot \tau d\Omega_s, \\ &= \int_{\Gamma_s} (\hat{u}^{\delta I} - \hat{u}^{\delta D}) (\tau \cdot \hat{\mathbf{n}}) d\Gamma_s + \int_{\Omega_s} \hat{\nabla} \hat{u}^{\delta D} \cdot \tau d\Omega_s. \end{aligned} \quad (3.29)$$

Here $\hat{u}^{\delta C} = \hat{u}^{\delta I} - \hat{u}^{\delta D}$, and thus

$$\int_{\Omega_s} \hat{\mathbf{q}}^{\delta D} \cdot \tau d\Omega_s = \int_{\Gamma_s} \hat{u}^{\delta C} (\tau \cdot \hat{\mathbf{n}}) d\Gamma_s + \int_{\Omega_s} \hat{\nabla} \hat{u}^{\delta D} \cdot \tau d\Omega_s. \quad (3.30)$$

Eq. (3.30) can be written in differential form if the integral over Γ_s on the right hand side can be transformed into an integral over Ω_s . In nodal DG, the necessary transformation is

$$\boxed{\int_{\Gamma_s} \hat{u}^{\delta C} (\tau \cdot \hat{\mathbf{n}}) d\Gamma_s = \int_{\Omega_s} \hat{\nabla} \hat{u}^{\delta C} \cdot \tau d\Omega_s.} \quad (3.31)$$

This transformation is called a ‘lifting’ transformation. Substituting this transformation into Eq. (3.30) yields

$$\int_{\Omega_s} \hat{\mathbf{q}}^{\delta D} \cdot \tau d\Omega_s = \int_{\Omega_s} \hat{\nabla} \hat{u}^{\delta C} \cdot \tau d\Omega_s + \int_{\Omega_s} \hat{\nabla} \hat{u}^{\delta D} \cdot \tau d\Omega_s, \quad (3.32)$$

and

$$\hat{\mathbf{q}}^{\delta D} = \hat{\nabla} \hat{u}^{\delta D} + \hat{\nabla} \hat{u}^{\delta C}. \quad (3.33)$$

The nodal DG differential form for $\hat{\mathbf{q}}^{\delta D}$ (given by Eq. (3.33)) is the same form proposed by flux reconstruction in section II (see Eq. (2.64)). However, a FR approach will not recover a nodal DG approach unless it produces the proper form for $\hat{\nabla} \hat{u}^{\delta C}$ (the correction to the gradient). In particular, the FR approach must produce a form for $\hat{\nabla} \hat{u}^{\delta C}$ that satisfies the lifting transformation (Eq. (3.31)).

The VCJH form for $\hat{\nabla} \hat{u}^{\delta C}$ (Eq. (3.23)) recovers nodal DG by satisfying Eq. (3.31). This can be demonstrated by an examination of the properties of VCJH correction functions. If $\kappa = \kappa_{dg} = 0$, the VCJH correction functions ($\mathbf{h}_{f,j}$ ’s) satisfy the following ‘lifting identities’, which allow transformations between volume and surface integrals (or in this particular 2D case, between integrations over the element area and the element boundary). The first identity (due to Ref. 36) is

$$\int_{\Gamma_s} (\mathbf{h}_{f,j} \cdot \hat{\mathbf{n}}) L_i d\Gamma_s = \int_{\Omega_s} \phi_{f,j} L_i d\Omega_s. \quad (3.34)$$

This identity can be rewritten as

$$\int_{\Gamma_f} (\mathbf{h}_{f,j} \cdot \hat{\mathbf{n}}_{f,j}) L_i d\Gamma_f = \int_{\Omega_s} \phi_{f,j} L_i d\Omega_s. \quad (3.35)$$

Here the properties of $\mathbf{h}_{f,j}$ in Eq. (2.61) have been utilized. Eq. (2.61) requires the correction function $\mathbf{h}_{f,j}$ to vanish when dotted with normals on adjacent faces. Thus, the boundary integral of the quantity $\mathbf{h}_{f,j} \cdot \hat{\mathbf{n}}$ only remains non-trivial on face f . As a result, the integral over Γ_s in Eq. (3.34) becomes an integral over Γ_f in Eq. (3.35).

The second lifting identity is simply a generalization of Eq. (3.35), where the functions L_i (orthonormal degree p polynomials) are replaced by vector dot products of the form $(\tau \cdot \hat{\mathbf{n}}_{f,j})$. The expression for the second identity is

$$\int_{\Gamma_f} (\mathbf{h}_{f,j} \cdot \hat{\mathbf{n}}_{f,j}) (\tau \cdot \hat{\mathbf{n}}_{f,j}) d\Gamma_f = \int_{\Omega_s} \phi_{f,j} (\tau \cdot \hat{\mathbf{n}}_{f,j}) d\Omega_s. \quad (3.36)$$

Once again, τ is an arbitrary test vector whose components are polynomials of degree p . Consider multiplying Eq. (3.36) by $\Pi_{f,j}$ and summing over f and j to get

$$\sum_{f=1}^3 \left[\int_{\Gamma_f} \sum_{j=1}^{N_{fp}} \Pi_{f,j} (\mathbf{h}_{f,j} \cdot \hat{\mathbf{n}}_{f,j}) (\tau \cdot \hat{\mathbf{n}}_{f,j}) d\Gamma_f \right] = \int_{\Omega_s} \left(\sum_{f=1}^3 \sum_{j=1}^{N_{fp}} \Pi_{f,j} \phi_{f,j} \hat{\mathbf{n}}_{f,j} \right) \cdot \tau d\Omega_s. \quad (3.37)$$

Using (2.62) and (2.65) this is equivalent to

$$\sum_{f=1}^3 \left[\int_{\Gamma_f} \hat{u}^{\delta C} (\tau \cdot \hat{\mathbf{n}}_{f,j}) d\Gamma_f \right] = \int_{\Omega_s} \hat{\nabla} \hat{u}^{\delta C} \cdot \tau d\Omega_s, \quad (3.38)$$

and simplifying the LHS yields

$$\int_{\Gamma_s} \hat{u}^{\delta C} (\tau \cdot \hat{\mathbf{n}}) d\Gamma_s = \int_{\Omega_s} \hat{\nabla} \hat{u}^{\delta C} \cdot \tau d\Omega_s. \quad (3.39)$$

Here the lifting transformation given by Eq. (3.31) is recovered. In Eq. (3.37), the correction field multiplied by the face normal acts as the nodal DG ‘lifting’ operator (here the term $\phi_{f,j} \hat{\mathbf{n}}_{f,j}$ is traditionally referred to as the lifting operator in the DG context).

The VCJH framework, which recovers nodal DG for $\kappa = \kappa_{dg} = 0$, can be expanded to recover alternative schemes for $\kappa \neq 0$. In this case, VCJH correction functions satisfy a more general version of the second lifting identity. The general form of Eq. (3.36) becomes

$$\begin{aligned} \int_{\Gamma_f} (\mathbf{h}_{f,j} \cdot \hat{\mathbf{n}}_{f,j}) (\tau \cdot \hat{\mathbf{n}}_{f,j}) d\Gamma_f &= \int_{\Omega_s} \phi_{f,j} (\tau \cdot \hat{\mathbf{n}}_{f,j}) d\Omega_s \\ &+ \sum_{m=1}^{p+1} c_m \left(D^{(m,p)} (\tau \cdot \hat{\mathbf{n}}_{f,j}) \right) \left(D^{(m,p)} \phi_{f,j} \right). \end{aligned} \quad (3.40)$$

Following the previous development for nodal DG, multiply Eq. (3.40) by $\Pi_{f,j}$ and sum over f and j to produce

$$\begin{aligned} \sum_{f=1}^3 \left[\int_{\Gamma_f} \sum_{j=1}^{N_{fp}} \Pi_{f,j} (\mathbf{h}_{f,j} \cdot \hat{\mathbf{n}}_{f,j}) (\tau \cdot \hat{\mathbf{n}}_{f,j}) d\Gamma_f \right] &= \int_{\Omega_s} \left(\sum_{f=1}^3 \sum_{j=1}^{N_{fp}} \Pi_{f,j} \phi_{f,j} \hat{\mathbf{n}}_{f,j} \right) \cdot \tau d\Omega_s \\ &+ \sum_{m=1}^{p+1} c_m D^{(m,p)}(\tau) \cdot D^{(m,p)} \left(\sum_{f=1}^3 \sum_{j=1}^{N_{fp}} \Pi_{f,j} \phi_{f,j} \hat{\mathbf{n}}_{f,j} \right) \end{aligned} \quad (3.41)$$

where (by analogy with DG) this is equivalent to

$$\sum_{f=1}^3 \left[\int_{\Gamma_f} \hat{u}^{\delta C} (\tau \cdot \hat{\mathbf{n}}_{f,j}) d\Gamma_f \right] = \int_{\Omega_s} \hat{\nabla} \hat{u}^{\delta C} \cdot \tau d\Omega_s + \sum_{m=1}^{p+1} c_m D^{(m,p)}(\tau) \cdot D^{(m,p)}(\hat{\nabla} \hat{u}^{\delta C}). \quad (3.42)$$

Simplifying the LHS yields

$$\int_{\Gamma_s} \hat{u}^{\delta C} (\tau \cdot \hat{\mathbf{n}}) d\Gamma_s = \int_{\Omega_s} \hat{\nabla} \hat{u}^{\delta C} \cdot \tau d\Omega_s + \sum_{m=1}^{p+1} c_m D^{(m,p)}(\tau) \cdot D^{(m,p)}(\hat{\nabla} \hat{u}^{\delta C}). \quad (3.43)$$

If κ is nonzero then c_m is nonzero and these expressions define an alternative family of VCJH lifting operators. The resulting VCJH schemes have some desirable properties for diffusion problems. It can be shown that the stability of these schemes is guaranteed for diffusive fluxes of the form $f = f(u, \nabla u)$ where f is linear. The stability proof for VCJH schemes on triangles will be presented in a subsequent publication. Here only the result of the proof is presented.

On triangles, VCJH schemes are stable for linear diffusion problems in a condition involving the norm of the transformed solution $\|\hat{u}^{\delta D}\|_{p,2}$ and a similar norm on the transformed auxiliary variable $\|\hat{\mathbf{q}}^{\delta D}\|_{p,2}$. The criterion for stability is

$$\frac{1}{2} \frac{d}{dt} \|\hat{u}^{\delta D}\|_{p,2}^2 + \|\hat{\mathbf{q}}^{\delta D}\|_{p,2}^2 \leq 0 \quad (3.44)$$

where the norm on the transformed auxiliary variable has the form

$$\|\hat{\mathbf{q}}^{\delta D}\|_{p,2} = \left[\sum_{n=1}^N \int_{\Omega_n} \left(\hat{\mathbf{q}}_n^{\delta D} \cdot \hat{\mathbf{q}}_n^{\delta D} + \frac{1}{A_S} \sum_{m=1}^{p+1} c_m \left(D^{(m,p)}(\hat{\mathbf{q}}_n^{\delta D}) \cdot D^{(m,p)}(\hat{\mathbf{q}}_n^{\delta D}) \right) \right) d\Omega_n \right]^{1/2}. \quad (3.45)$$

C. Reference values of c and κ

The previous sections demonstrate that VCJH schemes in 1D (and their tensor-product extensions to quadrilateral elements) are parameterized by a single scalar c , and that VCJH schemes on triangles are parameterized by a single scalar κ . A particular choice of these constants allows the VCJH framework to recover common schemes (DG and SD type methods) as well as new methods (Huynh type methods).³⁵ For this paper, three different VCJH schemes are of particular interest.

1. The DG scheme. This scheme arises from $c = c_{dg} = 0$ and $\kappa = \kappa_{dg} = 0$. This scheme serves as a reference with which to compare the following two schemes.
2. The SD scheme. This scheme arises from $c = c_{sd}$ and $\kappa = \kappa_{sd}$. On quadrilaterals this scheme is equivalent to a provably stable form of the spectral difference method for linear fluxes of the form $f = f(u)$ and $f = f(u, \nabla u)$.²⁹ On triangles this scheme, (while not formally equivalent to the traditional SD method of Ref. 28), has similar accuracy and stability properties as the 1D SD scheme.³⁶
3. The HU scheme. This scheme arises from $c = c_{hu}$ and $\kappa = \kappa_{hu}$. On quadrilaterals, this scheme (referred to by Huynh as the g_2 scheme) was shown to have favorable stability and accuracy properties.³⁴ This scheme is also provably stable for linear fluxes of the form $f = f(u)$ and $f = f(u, \nabla u)$.³⁵ On triangles this scheme, (while not formally equivalent to the original g_2 approach), has similar accuracy and stability properties as the 1D g_2 scheme.³⁶

For linear advection, numerical experiments indicate that the DG scheme possesses the most restrictive CFL limit. The SD and HU schemes have significantly larger CFL limits (see references 35 - 37 for details). Thus, the SD, and HU schemes are of interest for applications in which explicit time-stepping is desirable.

For the SD and HU approaches, appropriate values for c and κ depend on the degree of the polynomial basis p . Reference values for these quantities appear in table (1) below.

Table 1: Reference values of c and κ

	p=2	p=3	p=4
c_{dg}	0	0	0
c_{sd}	$2.96 \cdot 10^{-2}$	$9.52 \cdot 10^{-4}$	$1.61 \cdot 10^{-5}$
c_{hu}	$6.67 \cdot 10^{-2}$	$1.69 \cdot 10^{-3}$	$2.52 \cdot 10^{-5}$
κ_{dg}	0	0	0
κ_{sd}	$6.18 \cdot 10^{-3}$	$1.60 \cdot 10^{-4}$	$1.83 \cdot 10^{-6}$
κ_{hu}	$1.39 \cdot 10^{-2}$	$2.85 \cdot 10^{-4}$	$2.86 \cdot 10^{-6}$

Consult references 36 and 37 for details regarding how these values were determined.

IV. Numerical Results

This section presents results from numerical tests of the VCJH schemes. Here, the DG, SD, and HU versions of the VCJH schemes are examined. Each scheme is coupled with a LDG numerical flux with $\gamma = 1$. Test cases on model problems and on the compressible Navier-Stokes equations are presented.

A. Time-dependent Diffusion Equation

A standard procedure for confirming the validity of a viscous discretization involves finding solutions to the time-dependent diffusion equation. The time-dependent diffusion equation (also known as the heat equation) is a useful tool for analysis: it isolates the diffusive mechanisms of a scheme and prevents convective terms from interfering with analysis of the viscous discretization. It also serves as a model for the viscous phenomena encountered in the solutions of the compressible Navier-Stokes equations. The time-dependent diffusion equation is a parabolic partial differential equation which in 2D (cartesian form) becomes

$$\frac{\partial u}{\partial t} = \alpha \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right). \quad (4.1)$$

The exact solution to Eq. (4.1) is a function of the form $u = u(x, y, t) = u(\mathbf{x}, t)$. The temporal evolution of the solution depends on the initial condition $u = u(x, y, 0)$ and the boundary conditions on the 2D domain Ω . Note that the domain boundary is denoted Γ .

Define a 2D domain Ω which is a unit square $[0, 1] \times [0, 1]$. Consider the case where Dirichlet boundary conditions are applied on all boundaries ($\Gamma = \Gamma_D$). Define an initial condition at $t = 0$ as

$$u = u(\mathbf{x}, 0) = \sin(\gamma\pi x) \sin(\gamma\pi y). \quad (4.2)$$

This standard combination⁵⁰ of initial and boundary conditions produces an analytical solution of the form

$$u = u(\mathbf{x}, t) = e^{(-\beta\pi^2 t)} \sin(\gamma\pi x) \sin(\gamma\pi y) \quad \beta = 2\alpha\gamma^2. \quad (4.3)$$

For this test case, the parameters were set as $\alpha = 0.25$, $\beta = 0.5$, and $\gamma = 1.0$. Upon setting the parameters, the problem definition was complete. Thereafter, the time-dependent diffusion problem was solved using the DG, SD, and HU approaches. Each of the three schemes was marched forward in time using a low-storage 4th order Runge-Kutta scheme.⁵¹ The time step was chosen small enough to ensure that temporal errors were negligible. Each scheme was evaluated on a series of meshes to determine the order of accuracy. Tests were performed on three different sets of meshes.

1. Structured quadrilateral meshes. These meshes were created by splitting the unit square domain into $n \times n$ Cartesian grids of n^2 rectangular elements. Structured meshes with $n = 3, 6, 12, 24, 48$, and 96 were obtained (see figure (6)).
2. Structured triangular meshes. These meshes were created by splitting $n \times n$ Cartesian grids to form meshes with $2n^2$ triangular elements. Structured meshes with $n = 3, 6, 12, 24$, and 48 were obtained (see figure (7)).
3. Unstructured mixed meshes. These meshes were created by splitting the unit square domain into an unstructured grid of triangular elements. The triangular elements were evenly divided into two groups. Each triangle in the first group was subdivided into four triangles and each triangle in the second group was subdivided into three quadrilaterals. The result was an unstructured mesh of 91 elements. The grid was further subdivided to create finer meshes. A total of four meshes with 91, 364, 1456, and 5824 elements were obtained (see figure (8)).

On each mesh, polynomials of degree $p = 2, 3$, and 4 were employed. The theoretical expected order of accuracy in the L2 norm was $p + 1$ (see Ref. 20), corresponding to 3rd, 4th, and 5th order respectively. The theoretical expected order of accuracy in the H1 norm was p (see Ref. 20). Let E_k correspond to the error on the k th grid where k is an integer. The numerical order of accuracy was computed using

$$\frac{\ln\left(\frac{E_k}{E_{k-1}}\right)}{\ln(1/2)}, \quad k \geq 1 \quad (4.4)$$

where the grids become successively refined (by factors of 2) as k increases. In this context, errors in the L2 and H1 norms were evaluated. The error was evaluated after $t = 0.25$ s of simulation time. The error in the L2 norm ($E^{(L2)}$) was evaluated using

$$E^{(L2)} = \sqrt{\int_{\Omega} (u^e(\mathbf{x}) - u^\delta(\mathbf{x}))^2 d\Omega} = \sqrt{\sum_{n=1}^N \int_{\Omega_n} (u^e(\mathbf{x}) - u^\delta(\mathbf{x}))^2 d\Omega_n} \quad (4.5)$$

and the error in the H1 norm ($E^{(H1)}$) was evaluated using

$$\begin{aligned} E^{(H1)} &= \sqrt{\int_{\Omega} \left[(u^e(\mathbf{x}) - u^\delta(\mathbf{x}))^2 + \left(\frac{\partial u^e(\mathbf{x})}{\partial x} - \frac{\partial u^\delta(\mathbf{x})}{\partial x} \right)^2 + \left(\frac{\partial u^e(\mathbf{x})}{\partial y} - \frac{\partial u^\delta(\mathbf{x})}{\partial y} \right)^2 \right] d\Omega} \\ &= \sqrt{\sum_{n=1}^N \int_{\Omega_n} \left[(u^e(\mathbf{x}) - u^\delta(\mathbf{x}))^2 + \left(\frac{\partial u^e(\mathbf{x})}{\partial x} - \frac{\partial u^\delta(\mathbf{x})}{\partial x} \right)^2 + \left(\frac{\partial u^e(\mathbf{x})}{\partial y} - \frac{\partial u^\delta(\mathbf{x})}{\partial y} \right)^2 \right] d\Omega_n}. \end{aligned} \quad (4.6)$$

In particular, the H1 norm is important because (relative to the L2 norm) it yields a more complete description of the scheme's accuracy. The error in the H1 norm encapsulates the accuracy of both the solution and the solution gradient.

The test cases show that the solution and its gradient converge with the expected order of accuracy. Figures (9), (10), and (11) show that the VCJH schemes produce the expected order of accuracy in the L2 norm for all three grid types (namely the structured quadrilateral, structured triangular, and unstructured mixed grids). Similarly, figures (12), (13), and (14) show that the VCJH schemes produce the expected order of accuracy in the H1 norm for all three grid types. In each figure, the error in the appropriate norm is plotted vs. the grid spacing. The slopes of the plots correspond to the orders of accuracy for the schemes. For each of the plots, the DG, SD, and HU schemes all appear to have the same slopes (and thus the same orders of accuracy). These qualitative results are supported by quantitative results. The quantified numerical results in the form of values for the order accuracy, the L2 norm, and the H1 norm appear in tables (2) - (7). In general, the quantitative results from the tables confirm the qualitative results from the figures. The results in the tables also provide some additional insights.

While the results on all grids converge towards the expected order of accuracy, the tables reveal that the sharpest convergence occurs on the structured grids. This is expected, as the greater regularity of the structured grids tends to encourage faster convergence towards the smooth exact solution. In addition, the tables and the figures both reveal that (in the majority of cases) the DG scheme produces the least error, followed by the SD scheme, and finally the HU scheme. This is expected, as the the DG scheme has $\kappa_{dg} = 0$, which (in this context) corresponds to a minimum of amount of dissipation added to the solution. The SD and HU schemes have progressively larger values of κ and c , which means that each is progressively more dissipative. This effectively provides them with a larger time step limit, but also makes them slightly less accurate. Nevertheless, the errors of the DG, SD, and HU schemes are very close together in magnitude (virtually the same order of magnitude in all cases). Overall, the DG, SD, and HU schemes all achieve acceptable error values and rates of convergence.

B. Couette Flow

Couette flow comprises one of the few known solutions to the compressible Navier-Stokes equations. This allows it to serve as an acceptable metric for determining the accuracy of viscous discretizations when applied to the Navier-Stokes equations. The Couette flow test case provides a substantially more rigorous evaluation of the VCJH schemes as it requires them to accurately solve a coupled system of non-linear partial differential equations. However, any errors that arise during Couette flow testing are the combined result of inviscid and viscous discretization errors. Thus some exclusivity is lost (the viscous discretization is no longer evaluated in isolation). Nevertheless, this test case is a necessary step towards the demonstration of VCJH schemes within the context of practical problems, as it simulates the boundary layers that arise in many of these problems.

Having established the motivation for this test case, consider the governing equations and their exact solution. The governing equations for Couette flow are the compressible Navier-Stokes equations with constant viscosity. In 2D Cartesian form, these equations become

$$\frac{\partial U}{\partial t} + \nabla \cdot \mathbf{F}_e(U) = \nabla \cdot \mathbf{F}_v(U, \nabla U), \quad (4.7)$$

where the conservative variables U and the x and y components $f_e(U)$ and $g_e(U)$ of the inviscid flux vector $\mathbf{F}_e(U)$ are given by

$$U = \begin{Bmatrix} \rho \\ \rho u \\ \rho v \\ E \end{Bmatrix}, \quad f_e(U) = \begin{Bmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ u(E + p) \end{Bmatrix}, \quad g_e(U) = \begin{Bmatrix} \rho v \\ \rho uv \\ \rho v^2 + p \\ v(E + p) \end{Bmatrix}. \quad (4.8)$$

In the equations above, ρ is the density, u and v are the velocity components in x and y directions, p is the pressure and E is the total energy (internal and kinetic). The pressure is related to the total energy by

$$E = \frac{p}{\gamma - 1} + \frac{1}{2}\rho(u^2 + v^2) \quad (4.9)$$

where γ is a constant ratio of specific heats determined by the molecular properties of the fluid. In all subsequent test cases, $\gamma = 1.4$ which is the standard value for air. The x and y components $f_v(U, \nabla U)$ and $g_v(U, \nabla U)$ of the viscous flux $\mathbf{F}_v(U, \nabla U)$ are given as

$$f_v(U, \nabla U) = \mu \begin{Bmatrix} 0 \\ 2u_x + \lambda(u_x + v_y) \\ v_x + u_y \\ u[2u_x + \lambda(u_x + v_y)] + v(v_x + u_y) + \frac{C_p}{Pr} T_x \end{Bmatrix},$$

$$g_v(U, \nabla U) = \mu \begin{Bmatrix} 0 \\ v_x + u_y \\ 2v_y + \lambda(u_x + v_y) \\ v[2v_y + \lambda(u_x + v_y)] + u(v_x + u_y) + \frac{C_p}{Pr} T_y \end{Bmatrix}. \quad (4.10)$$

Here μ is the dynamic viscosity, C_p is the specific heat, and P_r is the Prandtl number. T is the temperature which can be derived from the ideal gas assumption. Finally, λ is -2/3 by the Stokes hypothesis.

These equations can be solved to give a closed-form solution for Couette flow. Consider the case where there is airflow between two flat plates with $\mu = \text{const.}$ These plates are of infinite extent in the x and z directions. The plates are separated by distance H , with the top plate moving with velocity U_t , while the bottom plate is stationary. The temperature of the bottom plate is T_b and the temperature of the top plate is T_t . The analytical solution for the velocity of the flow is

$$u = U_t \left(\frac{y}{H} \right), \quad v = 0. \quad (4.11)$$

The thermodynamic quantities are

$$T = T_b + \left(\frac{y}{H} \right) (T_t - T_b) + \frac{P_r U_t^2}{2C_p} \left(\frac{y}{H} \right) \left[1 - \left(\frac{y}{H} \right) \right], \quad (4.12)$$

$$\rho = \frac{p}{RT}, \quad p = \text{const.} \quad (4.13)$$

Numerically, Couette flow was reproduced in a rectangular computational domain with $-1 \leq x \leq 1$ and $0 \leq y \leq 1$. The left and right boundaries of the domain were periodic. The top plate was a moving isothermal no-slip wall and the bottom plate was a stationary isothermal no-slip wall. The pressure at the top wall was set constant (the pressure boundary condition on the top wall was exact). As an initial condition, the flow everywhere was given the velocity of the top plate $u = U_t$ and $v = 0$. The Mach number of the top plate was $M = 0.2$ and the flow Reynolds number was $Re = 200$. The temperatures of the top and bottom plates were 300K and 315K respectively. As in the previous test case, the flow was marched forward in time using a 4th order Runge Kutta scheme. The simulation was terminated once an approximate steady state was achieved (when the magnitude of the residual dropped below 10^{-15}).

For this test case, the three VCJH schemes (DG, SD, and HU) were evaluated on a set of mixed grids. These unstructured grids of quadrilaterals and triangles were formed using the methodology outlined in the description for the time-dependent diffusion test case. Grids with 3, 12, 48, and 192 elements were created. The topology of the grids is illustrated in figure (15). On each grid, the schemes were tested with $p = 2, 3$, and 4 degree polynomials. The expected order of accuracy is obtained for each of the schemes. Figure (16) demonstrates that the desired order of accuracy is obtained in the L2 norm. Figure (17) demonstrates the same favorable results for the H1 norm. Note that each figure shows (respectively) the L2 and H1 norms of the total energy E vs. the typical mesh spacing. The full numerical results with precise values for the order of accuracy, the L2 norm, and the H1 norm appear in tables (8) and (9).

Overall, all schemes perform as expected. The DG, SD, and HU schemes produce very similar results. All schemes converge with the expected order of accuracy. As in the previous test case, the error magnitude (in both the L2 and H1 norms) is similar across schemes. This demonstrates that all three schemes are suitable for application to the viscous fluxes in the compressible Navier Stokes equations.

C. Viscous flow past Naca 0012 airfoil

Consider the case of laminar subsonic flow over a NACA 0012 airfoil. The free-stream (far-field) Mach number is $M = 0.5$, and the free-stream Reynolds number is set as $Re = 5000$. The airfoil is oriented so that its angle of attack $\alpha = 0^\circ$. For this configuration, estimates for the pressure and friction drag coefficients have been generated by a number of authors (see references 22, 52, and 53). In fact, this test case is a popular standard for evaluating viscous discretizations. Its popularity is due to its complexity, as the Reynolds number places the flow in the latter portion of the laminar regime. Under these conditions, the flow near the trailing edge of the airfoil separates, creating a small recirculation bubble in the wake.²² This recirculation bubble contributes to unsteadiness in the wake, and the development of a vortex sheet several chord lengths down stream from the trailing edge.¹³

With this as motivation, efforts were made to simulate the NACA 0012 test case with each of the three VCJH schemes. Each scheme was tested on a mixed grid of 4,650 elements (composed from 3,242 triangular elements and 1,408 quadrilateral elements). An illustration of the grid near the airfoil appears in figure (18). Within this grid, the NACA 0012 airfoil geometry was defined using the formula

$$y = \pm\beta \left(ax^{1/2} + bx + cx^2 + dx^3 + ex^4 \right) \quad (4.14)$$

where

$$\begin{aligned} \beta &= 0.594689181 & a &= 0.298222773 & b &= -0.127125232 \\ c &= -0.357907906 & d &= 0.291984971 & e &= -0.105174696 \end{aligned} \quad (4.15)$$

This equation for the airfoil thickness corresponds to a NACA 0012 airfoil with a sharp trailing edge. The airfoil surface was represented using quadratic polynomials. The boundary conditions on the airfoil were those of a no-slip adiabatic wall. The far-field boundary conditions were subsonic characteristic boundary conditions. The compressible Navier-Stokes equations with variable dynamic viscosity were solved under these conditions. The dynamic viscosity was assumed to vary according to Sutherland's Law

$$\left(\frac{\mu}{\mu_*} \right) = \left(\frac{T}{T_*} \right)^{3/2} \frac{T_* + 110}{T + 110}, \quad (4.16)$$

where reference quantities are subscripted with $*$. Here, the units for temperature are Kelvin, and the units for viscosity are $kg/(m \cdot s)$. For air at STP, $\mu_* = 1.7894 \times 10^{-5} \text{ kg}/(m \cdot s)$ and $T_* = 288.16 \text{ K}$ (see Ref. 54 for details).

For this test case, the DG, SD, and HU schemes were executed with $p = 3$ degree polynomials. This corresponded to 4th order convergence of the solution. Figures (19) and (20) illustrate the predicted pressure and skin friction distributions. A visual comparison of the results reveals that they are very similar. Figures (21), (22), and (23) show the Mach contours produced by each of the schemes. Again, note the similarity in results between the three schemes. Finally, the schemes are compared to an independent reference. In table (10) the pressure-based and skin friction-based drag coefficients for the three VCJH schemes are compared to results predicted by an assortment of other methods. The three schemes all produce results which fall within the range of the data produced by the alternative methods.

V. Conclusion

This paper has presented details of the FR approach, and its application to linear and non-linear advection-diffusion problems on mixed grids. In particular, it has shown that VCJH schemes are capable of correctly characterizing second-order systems of partial differential equations ranging from the heat equation to the compressible Navier-Stokes equations. The application of VCJH schemes to these problems has been successful in the following ways: 1) The optimal order of accuracy is recovered on quadrilaterals and triangles in L2 and H1 norms of the solution. 2) The numerical values for engineering quantities (such as the drag coefficient) agree with well-established numerical results. Future research will involve applying the VCJH schemes to flows around geometries of greater complexity and to flows at higher (turbulent) Reynolds numbers. In addition, the performance of VCJH schemes in 3D will be explored on meshes of hexahedral, tetrahedral, and prismatic elements.

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Figure 6: An example of three structured quadrilateral grids used for the order of accuracy study on the time-dependent model diffusion problem

Figure 7: An example of three structured triangular grids used for the order of accuracy study on the time-dependent model diffusion problem

Figure 8: An example of three unstructured mixed grids used for the order of accuracy study on the time-dependent model diffusion problem

Figure 9: L2 error vs. mesh spacing for polynomials of degree $p = 2, 3$, and 4 on structured quadrilateral grids

Figure 10: L2 error vs. mesh spacing for polynomials of degree $p = 2, 3$, and 4 on structured triangular grids

Figure 11: L2 error vs. mesh spacing for polynomials of degree $p = 2, 3$, and 4 on unstructured mixed grids

Figure 12: H1 error vs. mesh spacing for polynomials of degree $p = 2, 3$, and 4 on structured quadrilateral grids

Figure 13: H1 error vs. mesh spacing for polynomials of degree $p = 2, 3$, and 4 on structured triangular grids

Figure 14: H1 error vs. mesh spacing for polynomials of degree $p = 2, 3$, and 4 on unstructured mixed grids

Figure 15: An example of three unstructured mixed grids used for the order of accuracy study on the Couette flow problem, $M = 0.2$, $Re = 200$

Figure 16: L2 error of E vs. mesh spacing for polynomials of degree $p = 2, 3$, and 4 on unstructured mixed grids for Couette flow, $M = 0.2$, $Re = 200$

Figure 17: H1 error of E vs. mesh spacing for polynomials of degree $p = 2, 3$, and 4 on unstructured mixed grids for Couette flow, $M = 0.2$, $Re = 200$

Figure 18: Close-up of mixed mesh for NACA 0012, $M = 0.5$, $Re = 5000$

Figure 19: Pressure coefficient vs. x for top and bottom surfaces of NACA 0012, $M = 0.5$, $Re = 5000$

Figure 20: Skin Friction coefficient vs. x for top and bottom surfaces of NACA 0012, $M = 0.5$, $Re = 5000$

Figure 21: Mach contours for 4th order DG scheme around Naca 0012, $M = 0.5$, $Re = 5000$

Figure 22: Mach contours for 4th order SD scheme around Naca 0012, $M = 0.5$, $Re = 5000$

Figure 23: Mach contours for 4th order HU scheme around Naca 0012, $M = 0.5$, $Re = 5000$

Table 2: Error measured in the L2 norm for the time-dependent model diffusion problem on structured quadrilateral grids

p	N elements	DG L_2 err.	$O(L_2)$	SD L_2 err.	$O(L_2)$	HU L_2 err.	$O(L_2)$
2	36	$1.247 \cdot 10^{-4}$		$1.720 \cdot 10^{-4}$		$2.554 \cdot 10^{-4}$	
	144	$1.662 \cdot 10^{-5}$	2.91	$2.414 \cdot 10^{-5}$	2.83	$3.651 \cdot 10^{-5}$	2.81
	576	$2.160 \cdot 10^{-6}$	2.94	$3.255 \cdot 10^{-6}$	2.89	$5.028 \cdot 10^{-6}$	2.86
	2304	$2.757 \cdot 10^{-7}$	2.97	$4.240 \cdot 10^{-7}$	2.94	$6.640 \cdot 10^{-7}$	2.92
	9216	$3.483 \cdot 10^{-8}$	2.98	$5.414 \cdot 10^{-8}$	2.97	$8.544 \cdot 10^{-8}$	2.96
3	9	$7.209 \cdot 10^{-5}$		$1.036 \cdot 10^{-4}$		$1.333 \cdot 10^{-4}$	
	36	$4.652 \cdot 10^{-6}$	3.95	$6.959 \cdot 10^{-6}$	3.90	$9.095 \cdot 10^{-6}$	3.87
	144	$2.910 \cdot 10^{-7}$	4.00	$4.485 \cdot 10^{-7}$	3.96	$5.965 \cdot 10^{-7}$	3.93
	576	$1.816 \cdot 10^{-8}$	4.00	$2.857 \cdot 10^{-8}$	3.97	$3.852 \cdot 10^{-8}$	3.95
	2304	$1.134 \cdot 10^{-9}$	4.00	$1.806 \cdot 10^{-9}$	3.98	$2.455 \cdot 10^{-9}$	3.97
4	9	$3.243 \cdot 10^{-6}$		$4.777 \cdot 10^{-6}$		$5.943 \cdot 10^{-6}$	
	36	$1.068 \cdot 10^{-7}$	4.92	$1.587 \cdot 10^{-7}$	4.91	$1.964 \cdot 10^{-7}$	4.92
	144	$3.528 \cdot 10^{-9}$	4.92	$5.471 \cdot 10^{-9}$	4.86	$6.838 \cdot 10^{-9}$	4.84
	576	$1.138 \cdot 10^{-10}$	4.95	$1.812 \cdot 10^{-10}$	4.92	$2.283 \cdot 10^{-10}$	4.90
	2304	$3.619 \cdot 10^{-12}$	4.97	$5.841 \cdot 10^{-12}$	4.96	$7.389 \cdot 10^{-12}$	4.95

Table 3: Error measured in the H1 norm for the time-dependent model diffusion problem on structured quadrilateral grids

p	N elements	DG H_1 err.	$O(H_1)$	SD H_1 err.	$O(H_1)$	HU H_1 err.	$O(H_1)$
2	36	$8.078 \cdot 10^{-3}$		$9.937 \cdot 10^{-3}$		$1.249 \cdot 10^{-2}$	
	144	$2.103 \cdot 10^{-3}$	1.94	$2.669 \cdot 10^{-3}$	1.90	$3.441 \cdot 10^{-3}$	1.86
	576	$5.391 \cdot 10^{-4}$	1.96	$7.008 \cdot 10^{-4}$	1.93	$9.262 \cdot 10^{-4}$	1.89
	2304	$1.366 \cdot 10^{-4}$	1.98	$1.801 \cdot 10^{-4}$	1.96	$2.416 \cdot 10^{-4}$	1.94
	9216	$3.439 \cdot 10^{-5}$	1.99	$4.567 \cdot 10^{-5}$	1.98	$6.178 \cdot 10^{-5}$	1.97
3	9	$3.336 \cdot 10^{-3}$		$4.626 \cdot 10^{-3}$		$5.663 \cdot 10^{-3}$	
	36	$4.353 \cdot 10^{-4}$	2.94	$6.152 \cdot 10^{-4}$	2.91	$7.592 \cdot 10^{-4}$	2.90
	144	$5.491 \cdot 10^{-5}$	2.99	$7.874 \cdot 10^{-5}$	2.97	$9.814 \cdot 10^{-5}$	2.95
	576	$6.886 \cdot 10^{-6}$	3.00	$9.981 \cdot 10^{-6}$	2.98	$1.255 \cdot 10^{-5}$	2.97
	2304	$8.621 \cdot 10^{-7}$	3.00	$1.258 \cdot 10^{-6}$	2.99	$1.591 \cdot 10^{-6}$	2.98
4	9	$2.233 \cdot 10^{-4}$		$3.226 \cdot 10^{-4}$		$3.813 \cdot 10^{-4}$	
	36	$1.453 \cdot 10^{-5}$	3.94	$2.112 \cdot 10^{-5}$	3.93	$2.493 \cdot 10^{-5}$	3.93
	144	$9.511 \cdot 10^{-7}$	3.93	$1.425 \cdot 10^{-6}$	3.89	$1.702 \cdot 10^{-6}$	3.87
	576	$6.106 \cdot 10^{-8}$	3.96	$9.325 \cdot 10^{-8}$	3.93	$1.123 \cdot 10^{-7}$	3.92
	2304	$3.870 \cdot 10^{-9}$	3.98	$5.972 \cdot 10^{-9}$	3.96	$7.227 \cdot 10^{-9}$	3.96

Table 4: Error measured in the L_2 norm for the time-dependent model diffusion problem on structured triangular grids

p	N elements	DG L_2 err.	$O(L_2)$	SD L_2 err.	$O(L_2)$	HU L_2 err.	$O(L_2)$
2	18	$1.917 \cdot 10^{-3}$		$2.029 \cdot 10^{-3}$		$2.488 \cdot 10^{-3}$	
	72	$2.451 \cdot 10^{-4}$	2.97	$2.608 \cdot 10^{-4}$	2.96	$3.042 \cdot 10^{-4}$	3.03
	288	$3.060 \cdot 10^{-5}$	3.00	$3.276 \cdot 10^{-5}$	2.99	$3.775 \cdot 10^{-5}$	3.01
	1152	$3.824 \cdot 10^{-6}$	3.00	$4.100 \cdot 10^{-6}$	3.00	$4.709 \cdot 10^{-6}$	3.00
	4608	$4.783 \cdot 10^{-7}$	3.00	$5.128 \cdot 10^{-7}$	3.00	$5.883 \cdot 10^{-7}$	3.00
3	18	$2.198 \cdot 10^{-4}$		$2.315 \cdot 10^{-4}$		$2.508 \cdot 10^{-4}$	
	72	$1.421 \cdot 10^{-5}$	3.95	$1.490 \cdot 10^{-5}$	3.96	$1.591 \cdot 10^{-5}$	3.98
	288	$8.995 \cdot 10^{-7}$	3.98	$9.420 \cdot 10^{-7}$	3.98	$1.001 \cdot 10^{-6}$	3.99
	1152	$5.656 \cdot 10^{-8}$	3.99	$5.920 \cdot 10^{-8}$	3.99	$6.285 \cdot 10^{-8}$	3.99
	4608	$3.545 \cdot 10^{-9}$	4.00	$3.710 \cdot 10^{-9}$	4.00	$3.937 \cdot 10^{-9}$	4.00
4	18	$2.093 \cdot 10^{-5}$		$2.128 \cdot 10^{-5}$		$2.165 \cdot 10^{-5}$	
	72	$6.998 \cdot 10^{-7}$	4.90	$7.088 \cdot 10^{-7}$	4.91	$7.184 \cdot 10^{-7}$	4.91
	288	$2.251 \cdot 10^{-8}$	4.96	$2.275 \cdot 10^{-8}$	4.96	$2.302 \cdot 10^{-8}$	4.96
	1152	$7.120 \cdot 10^{-10}$	4.98	$7.189 \cdot 10^{-10}$	4.98	$7.268 \cdot 10^{-10}$	4.98
	4608	$2.239 \cdot 10^{-11}$	4.99	$2.259 \cdot 10^{-11}$	4.99	$2.284 \cdot 10^{-11}$	4.99

Table 5: Error measured in the H_1 norm for the time-dependent model diffusion problem on structured triangular grids

p	N elements	DG H_1 err.	$O(H_1)$	SD H_1 err.	$O(H_1)$	HU H_1 err.	$O(H_1)$
2	18	$5.910 \cdot 10^{-2}$		$6.422 \cdot 10^{-2}$		$7.493 \cdot 10^{-2}$	
	72	$1.518 \cdot 10^{-2}$	1.96	$1.620 \cdot 10^{-2}$	1.99	$1.814 \cdot 10^{-2}$	2.05
	288	$3.803 \cdot 10^{-3}$	2.00	$4.034 \cdot 10^{-3}$	2.01	$4.450 \cdot 10^{-3}$	2.03
	1152	$9.498 \cdot 10^{-4}$	2.00	$1.005 \cdot 10^{-3}$	2.01	$1.103 \cdot 10^{-3}$	2.01
	4608	$2.372 \cdot 10^{-4}$	2.00	$2.507 \cdot 10^{-4}$	2.00	$2.745 \cdot 10^{-4}$	2.01
3	18	$9.401 \cdot 10^{-3}$		$1.023 \cdot 10^{-2}$		$1.118 \cdot 10^{-2}$	
	72	$1.205 \cdot 10^{-3}$	2.96	$1.300 \cdot 10^{-3}$	2.98	$1.398 \cdot 10^{-3}$	3.00
	288	$1.512 \cdot 10^{-4}$	2.99	$1.629 \cdot 10^{-4}$	3.00	$1.744 \cdot 10^{-4}$	3.00
	1152	$1.890 \cdot 10^{-5}$	3.00	$2.035 \cdot 10^{-5}$	3.00	$2.175 \cdot 10^{-5}$	3.00
	4608	$2.360 \cdot 10^{-6}$	3.00	$2.542 \cdot 10^{-6}$	3.00	$2.716 \cdot 10^{-6}$	3.00
4	18	$1.183 \cdot 10^{-3}$		$1.234 \cdot 10^{-3}$		$1.269 \cdot 10^{-3}$	
	72	$7.336 \cdot 10^{-5}$	4.01	$7.643 \cdot 10^{-5}$	4.01	$7.852 \cdot 10^{-5}$	4.01
	288	$4.511 \cdot 10^{-6}$	4.02	$4.700 \cdot 10^{-6}$	4.02	$4.829 \cdot 10^{-6}$	4.02
	1152	$2.790 \cdot 10^{-7}$	4.02	$2.907 \cdot 10^{-7}$	4.01	$2.987 \cdot 10^{-7}$	4.01
	4608	$1.734 \cdot 10^{-8}$	4.01	$1.807 \cdot 10^{-8}$	4.01	$1.857 \cdot 10^{-8}$	4.01

Table 6: Error measured in the L_2 norm for the time-dependent model diffusion problem on unstructured mixed grids

p	N elements	DG L_2 err.	$O(L_2)$	SD L_2 err.	$O(L_2)$	HU L_2 err.	$O(L_2)$
2	91	$2.237 \cdot 10^{-4}$		$2.558 \cdot 10^{-4}$		$3.306 \cdot 10^{-4}$	
	364	$2.887 \cdot 10^{-5}$	2.95	$3.448 \cdot 10^{-5}$	2.89	$4.608 \cdot 10^{-5}$	2.84
	1456	$3.720 \cdot 10^{-6}$	2.96	$4.369 \cdot 10^{-6}$	2.98	$5.856 \cdot 10^{-6}$	2.98
	5824	$4.686 \cdot 10^{-7}$	2.99	$5.679 \cdot 10^{-7}$	2.94	$7.685 \cdot 10^{-7}$	2.93
3	91	$7.592 \cdot 10^{-6}$		$8.575 \cdot 10^{-6}$		$1.003 \cdot 10^{-5}$	
	364	$5.092 \cdot 10^{-7}$	3.90	$5.907 \cdot 10^{-7}$	3.86	$7.102 \cdot 10^{-7}$	3.82
	1456	$3.260 \cdot 10^{-8}$	3.97	$3.861 \cdot 10^{-8}$	3.94	$4.763 \cdot 10^{-8}$	3.90
	5824	$2.025 \cdot 10^{-9}$	4.01	$2.478 \cdot 10^{-9}$	3.96	$3.061 \cdot 10^{-9}$	3.96
4	91	$2.818 \cdot 10^{-7}$		$3.140 \cdot 10^{-7}$		$3.408 \cdot 10^{-7}$	
	364	$9.011 \cdot 10^{-9}$	4.97	$1.097 \cdot 10^{-8}$	4.84	$1.263 \cdot 10^{-8}$	4.75
	1456	$2.932 \cdot 10^{-10}$	4.94	$3.492 \cdot 10^{-10}$	4.97	$4.048 \cdot 10^{-10}$	4.96
	5824	$9.298 \cdot 10^{-12}$	4.98	$1.133 \cdot 10^{-11}$	4.95	$1.311 \cdot 10^{-11}$	4.95

Table 7: Error measured in the H_1 norm for the time-dependent model diffusion problem on unstructured mixed grids

p	N elements	DG H_1 err.	$O(H_1)$	SD H_1 err.	$O(H_1)$	HU H_1 err.	$O(H_1)$
2	91	$1.708 \cdot 10^{-2}$		$1.913 \cdot 10^{-2}$		$2.274 \cdot 10^{-2}$	
	364	$4.282 \cdot 10^{-3}$	2.00	$4.983 \cdot 10^{-3}$	1.94	$6.148 \cdot 10^{-3}$	1.89
	1456	$1.071 \cdot 10^{-3}$	2.00	$1.231 \cdot 10^{-3}$	2.02	$1.523 \cdot 10^{-3}$	2.01
	5824	$2.762 \cdot 10^{-4}$	1.96	$3.190 \cdot 10^{-4}$	1.95	$3.886 \cdot 10^{-4}$	1.97
3	91	$8.222 \cdot 10^{-4}$		$9.674 \cdot 10^{-4}$		$1.115 \cdot 10^{-3}$	
	364	$1.089 \cdot 10^{-4}$	2.92	$1.301 \cdot 10^{-4}$	2.89	$1.538 \cdot 10^{-4}$	2.86
	1456	$1.384 \cdot 10^{-5}$	2.98	$1.673 \cdot 10^{-5}$	2.96	$2.022 \cdot 10^{-5}$	2.93
	5824	$1.761 \cdot 10^{-6}$	2.97	$2.166 \cdot 10^{-6}$	2.95	$2.593 \cdot 10^{-6}$	2.96
4	91	$4.103 \cdot 10^{-5}$		$4.779 \cdot 10^{-5}$		$5.197 \cdot 10^{-5}$	
	364	$2.557 \cdot 10^{-6}$	4.00	$3.260 \cdot 10^{-6}$	3.87	$3.752 \cdot 10^{-6}$	3.79
	1456	$1.625 \cdot 10^{-7}$	3.98	$2.028 \cdot 10^{-7}$	4.01	$2.355 \cdot 10^{-7}$	3.99
	5824	$1.064 \cdot 10^{-8}$	3.93	$1.325 \cdot 10^{-8}$	3.94	$1.513 \cdot 10^{-8}$	3.96

Table 8: Error measured in the L2 norm for Couette flow, $M = 0.2$, $Re = 200$, on unstructured mixed grids

p	N elements	DG L_2 err.	$O(L_2)$	SD L_2 err.	$O(L_2)$	HU L_2 err.	$O(L_2)$
2	3	$7.469 \cdot 10^{-5}$		$7.327 \cdot 10^{-5}$		$7.232 \cdot 10^{-5}$	
	12	$9.907 \cdot 10^{-6}$	2.91	$9.802 \cdot 10^{-6}$	2.90	$9.761 \cdot 10^{-6}$	2.89
	48	$1.271 \cdot 10^{-6}$	2.96	$1.248 \cdot 10^{-6}$	2.97	$1.240 \cdot 10^{-6}$	2.98
	192	$1.620 \cdot 10^{-7}$	2.97	$1.580 \cdot 10^{-7}$	2.98	$1.561 \cdot 10^{-7}$	2.99
3	3	$4.581 \cdot 10^{-5}$		$4.632 \cdot 10^{-5}$		$4.670 \cdot 10^{-5}$	
	12	$2.302 \cdot 10^{-6}$	4.31	$2.376 \cdot 10^{-6}$	4.28	$2.426 \cdot 10^{-6}$	4.27
	48	$1.399 \cdot 10^{-7}$	4.04	$1.415 \cdot 10^{-7}$	4.07	$1.426 \cdot 10^{-7}$	4.09
	192	$8.856 \cdot 10^{-9}$	3.98	$8.834 \cdot 10^{-9}$	4.00	$8.838 \cdot 10^{-9}$	4.01
4	3	$7.591 \cdot 10^{-7}$		$7.782 \cdot 10^{-7}$		$7.873 \cdot 10^{-7}$	
	12	$1.951 \cdot 10^{-8}$	5.28	$1.962 \cdot 10^{-8}$	5.31	$1.971 \cdot 10^{-8}$	5.32
	48	$5.631 \cdot 10^{-10}$	5.12	$5.599 \cdot 10^{-10}$	5.13	$5.608 \cdot 10^{-10}$	5.14

Table 9: Error measured in the H1 norm for Couette flow, $M = 0.2$, $Re = 200$, on unstructured mixed grids

p	N elements	DG H_1 err.	$O(H_1)$	SD H_1 err.	$O(H_1)$	HU H_1 err.	$O(H_1)$
2	3	$1.620 \cdot 10^{-3}$		$1.618 \cdot 10^{-3}$		$1.618 \cdot 10^{-3}$	
	12	$4.065 \cdot 10^{-4}$	2.00	$4.065 \cdot 10^{-4}$	1.99	$4.068 \cdot 10^{-4}$	1.99
	48	$1.015 \cdot 10^{-4}$	2.00	$1.016 \cdot 10^{-4}$	2.00	$1.017 \cdot 10^{-4}$	2.00
	192	$2.514 \cdot 10^{-5}$	2.01	$2.516 \cdot 10^{-5}$	2.01	$2.519 \cdot 10^{-5}$	2.01
3	3	$5.270 \cdot 10^{-4}$		$5.287 \cdot 10^{-4}$		$5.307 \cdot 10^{-4}$	
	12	$5.921 \cdot 10^{-5}$	3.15	$5.954 \cdot 10^{-5}$	3.15	$5.963 \cdot 10^{-5}$	3.15
	48	$7.277 \cdot 10^{-6}$	3.02	$7.292 \cdot 10^{-6}$	3.03	$7.298 \cdot 10^{-6}$	3.03
	192	$9.221 \cdot 10^{-7}$	2.98	$9.216 \cdot 10^{-7}$	2.98	$9.217 \cdot 10^{-7}$	2.99
4	3	$1.373 \cdot 10^{-5}$		$1.372 \cdot 10^{-5}$		$1.373 \cdot 10^{-5}$	
	12	$7.405 \cdot 10^{-7}$	4.21	$7.419 \cdot 10^{-7}$	4.21	$7.416 \cdot 10^{-7}$	4.21
	48	$4.555 \cdot 10^{-8}$	4.02	$4.571 \cdot 10^{-8}$	4.02	$4.579 \cdot 10^{-8}$	4.02

Table 10: Comparison of pressure and viscous parts of drag coefficient ($C_{d,p}$ and $C_{d,v}$) with results from assorted structured and unstructured solvers for NACA 0012, $M = 0.5$, $Re = 5000$ test case

Approach	Ref.	Order of accuracy	N elements	$C_{d,p}$	$C_{d,v}$
VCJH DG scheme	-	4	4650	0.02222	0.03348
VCJH SD scheme	-	4	4650	0.02224	0.03345
VCJH HU scheme	-	4	4650	0.02226	0.03343
BR1 DG scheme	[22]	4	1024	0.02208	0.03303
Cell-centered scheme	[52]	2	20480	0.02190	0.03370
Cell-vertex scheme	[53]	2	16384	0.02270	0.03270
Cell-centered scheme	[53]	2	16384	0.02256	0.03301

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