Application of High-Order Energy Stable Flux Reconstruction Schemes to the Euler Equations

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The authors recently identified an infinite range of high-order energy stable flux reconstruction (FR) schemes in 1D and on triangular elements in 2D. The new flux reconstruction schemes are linearly stable for all orders of accuracy in a norm of Sobolev type. They are parameterized by a single scalar quantity, which if chosen judiciously leads to the recovery of various well known high-order methods (such as a collocation based nodal discontinuous Galerkin method and a spectral difference method). Identification of such schemes represents a significant advance in terms of understanding why certain FR schemes are stable, whereas others are not. However, to date there have been no studies into how these schemes perform when applied to real world non-linear problems. In this paper, stability and accuracy properties of these new schemes are studied for various two-dimensional inviscid flow problems. The results offer significant insight into the performance of energy stable FR schemes for non-linear problems. It is envisaged the results will aid scheme selection for a given problem, based on its stability and accuracy requirements.

I. Introduction

High-order numerical methods potentially offer better accuracy and lower computational cost than low-order schemes for a variety of problems. In recent years, high-order methods have been used successfully in a wide variety of fields such as computational aero-acoustics and turbulence research. Schemes for structured grids such as compact finite-difference methods have received the most attention due to their simple formulation. However, these high-order methods have limited scope because of their inability to handle complex geometries. Recently, locally discontinuous high-order methods such as the discontinuous Galerkin (DG) and Spectral Difference (SD) methods have emerged as promising and competitive tools for solving problems with high accuracy over complicated geometries.

The discontinuous Galerkin method was developed in 1973 by Reed and Hill¹ to solve the neutron transport equation and numerous variants have been developed for solving the weak form of both hyperbolic² and elliptic systems.³ The basic principle of DG schemes is to decompose the approximate numerical solution both spatially, by tessellating a given computational domain with separate elements, and also spectrally, via a summation of piecewise discontinuous polynomial basis functions within each element. A particularly simple and efficient range of DG schemes utilize high-order Lagrange polynomial basis functions inside each element, defined by solution values at a set of distinct nodal points. Such schemes have become known as nodal DG methods, an exposition of which can be found in the recent textbook by Hesthaven and Warburton.⁴ Similar to nodal DG methods are spectral difference methods, which are based on the governing system in its differential form. The foundation for such schemes was first put forward by Kopriva and Kolias⁵ in 1996 under the name of “staggered grid Chebyshev multidomain” methods. However, several years later in 2006 Liu, Vinokur and Wang⁶ presented a more general formulation for both triangular and quadrilateral

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elements, which they termed the spectral difference (SD) method. In recent years, the SD method has been successfully used to solve a wide variety of problems.⁷⁻⁹

In 2007, Huynh presented the flux reconstruction (FR) approach,¹⁰ which is capable of unifying several high-order methods, including the collocation based nodal DG method and the SD method (at least for linear advection). Furthermore, the FR approach allowed Huynh to create a variety of new high-order schemes with various stability and accuracy properties. Recently, Vincent, Castonguay and Jameson^{11,12} identified an infinite range of linearly stable FR schemes for both quadrilateral and triangular elements, henceforth referred to as Vincent-Castonguay-Jameson (VCJ) schemes. The methodology for developing such schemes is an extension of the approach adopted by Jameson,¹³ who utilized a FR formulation to prove that the SD method is stable in a norm of Sobolev type. The VCJ schemes are parameterized by a single scalar quantity, which if chosen judiciously can lead to the recovery of several well known numerical methods such as the nodal DG method and a particular SD method for quadrilateral elements. Furthermore, as shown in Ref. 14 and Ref. 12, the scalar that defines the VCJ schemes can be selected to provide an increased CFL number while yielding the optimal order of accuracy on linear advection problems.

Although the properties of VCJ schemes have been investigated for linear problems in 1D and 2D, their non-linear properties have not been studied thoroughly. The present article aims to investigate the properties of the VCJ schemes for inviscid flows. Accuracy and stability analysis are presented for the Euler equations on linear and curved meshes. Three test cases are considered: the propagation of an isentropic vortex, the inviscid flow over a cylinder at zero angle of attack and the inviscid flow over an airfoil at an angle of attack. The article begins with a brief review of the FR approach. Following this review, an overview of the VCJ schemes is presented. Numerical experiments are then performed in order to elucidate various properties of VCJ schemes and finally, conclusions are drawn.

II. The Flux Reconstruction Approach to High-Order Methods

This section begins with a brief overview of the flux reconstruction approach in 1D and an extension of the flux reconstruction is presented for triangular elements.

A. Review of the Flux Reconstruction Approach in 1D

Consider solving the following 1D scalar conservation law

$$\frac{\partial u}{\partial t} + \frac{\partial f}{\partial x} = 0 \quad (1)$$

within an arbitrary periodic domain Ω , where x is a spatial coordinate, t is time, $u = u(x, t)$ is a conserved scalar quantity and $f = f(u)$ is the flux of u in the x direction. Furthermore, consider partitioning Ω into N non-overlapping, distinct elements each denoted $\Omega_n = \{x | x_n < x < x_{n+1}\}$. Finally, having partitioned Ω into separate elements, consider representing the exact solution u within each Ω_n by a polynomial of degree p denoted $u_n^\delta = u_n^\delta(x, t)$ (which is in general piecewise discontinuous between elements), and the exact flux f within each Ω_n by a polynomial of degree $p + 1$ denoted $f_n^\delta = f_n^\delta(x, t)$ (which is piecewise continuous between elements).

In order to simplify the implementation, it is advantageous to transform each Ω_n to a reference element $\Omega_S = \{r | -1 \leq r \leq 1\}$ via the mapping

$$x = \Theta_n(r) = \left(\frac{1-r}{2}\right)x_n + \left(\frac{1+r}{2}\right)x_{n+1}, \quad (2)$$

which has the inverse

$$r = \Theta_n^{-1}(x) = 2 \left(\frac{x - x_n}{x_{n+1} - x_n} \right) - 1. \quad (3)$$

Having performed such a transformation, the evolution of u_n^δ within any individual Ω_n (and thus the evolution of u^δ within Ω) can be determined by solving the following transformed governing equation within the reference element Ω_S

$$\frac{\partial \hat{u}^\delta}{\partial t} + \frac{\partial \hat{f}^\delta}{\partial r} = 0, \quad (4)$$

where

$$\hat{u}^\delta = \hat{u}^\delta(r, t) = u_n^\delta(\Theta_n(r), t) \quad (5)$$

is a polynomial of degree p ,

$$\hat{f}^\delta = \hat{f}^\delta(r, t) = \frac{f_n^\delta(\Theta_n(r), t)}{J_n}, \quad (6)$$

is a polynomial of degree $p + 1$, and $J_n = (x_{n+1} - x_n)/2$.

The FR approach to solving Eq. 4 within the standard element Ω_S consists of five stages. The first stage is to define a specific form for $\hat{u}^\delta$. To this end, it is assumed that values of $\hat{u}^\delta$ are known at a set of $p + 1$ solution points inside Ω_S , with each point located at a distinct position r_i ($i = 0$ to p). Lagrange polynomials $l_i = l_i(r)$ defined as

$$l_i = \prod_{j=0, j \neq i}^p \left(\frac{r - r_j}{r_i - r_j} \right) \quad (7)$$

can then be used to construct the following expression for $\hat{u}^\delta$

$$\hat{u}^\delta = \sum_{i=0}^p \hat{u}_i^\delta l_i, \quad (8)$$

where $\hat{u}_i^\delta = \hat{u}_i^\delta(t)$ are the known values of $\hat{u}^\delta$ at the solution points r_i .

The second stage of the FR approach involves constructing a degree p polynomial $\hat{f}^{\delta D} = \hat{f}^{\delta D}(r, t)$, defined as the approximate transformed discontinuous flux within Ω_S . A collocation projection at the $p + 1$ solution points is employed to obtain $\hat{f}^{\delta D}$, which can hence be expressed as

$$\hat{f}^{\delta D} = \sum_{i=0}^p \hat{f}_i^{\delta D} l_i \quad (9)$$

where the coefficients $\hat{f}_i^{\delta D} = \hat{f}_i^{\delta D}(t)$ are simply values of the transformed flux at each solution point r_i evaluated directly from the approximate solution. The flux $\hat{f}^{\delta D}$ is termed discontinuous since it is calculated directly from the approximate solution, which is in general piecewise discontinuous between elements.

The third stage of the FR approach involves calculating numerical interface fluxes at either end of the standard element Ω_S (at $r = \pm 1$). In order to calculate these fluxes, one must first obtain values for the approximate solution at either end of the standard element via Eq. 8. Once these values have been obtained they can be used in conjunction with analogous information from adjoining elements to calculate numerical interface fluxes. The exact methodology for calculating such numerical interface fluxes will depend on the nature of the equations being solved. For example, when solving the Euler equations one may use a Roe¹⁵ or Rusanov¹⁶ approximate Riemann solver. In what follows the common numerical interface fluxes associated with the left and right hand ends of Ω_n (and transformed appropriately for use in Ω_S) will be denoted $\hat{f}_L^{\delta I}$ and $\hat{f}_R^{\delta I}$ respectively.

The penultimate stage of the FR approach involves adding a degree $p + 1$ transformed correction flux $\hat{f}^{\delta C} = \hat{f}^{\delta C}(r, t)$ to the approximate transformed discontinuous flux $\hat{f}^{\delta D}$, such that their sum equals the transformed numerical interface flux at $r = \pm 1$, yet follows (in some sense) the approximate discontinuous flux within the interior of Ω_S . In order to define $\hat{f}^{\delta C}$ such that it satisfies the above requirements, consider first defining degree $p + 1$ correction functions $h_L = h_L(r)$ and $h_R = h_R(r)$ that approximate zero (in some sense) within Ω_S , as well as satisfying

$$h_L(-1) = 1, \quad h_L(1) = 0, \quad (10)$$

$$h_R(-1) = 0, \quad h_R(1) = 1, \quad (11)$$

and, based on symmetry considerations

$$h_L(r) = h_R(-r). \quad (12)$$

A suitable expression for $\hat{f}^{\delta C}$ can now be written in terms of h_L and h_R as

$$\hat{f}^{\delta C} = (\hat{f}_L^{\delta I} - \hat{f}_L^{\delta D})h_L + (\hat{f}_R^{\delta I} - \hat{f}_R^{\delta D})h_R, \quad (13)$$

where $\hat{f}_L^{\delta D} = \hat{f}^{\delta D}(-1, t)$ and $\hat{f}_R^{\delta D} = \hat{f}^{\delta D}(1, t)$. Using this expression, a degree $p + 1$ approximate total transformed flux $\hat{f}^\delta = \hat{f}^\delta(r, t)$ within Ω_S can be constructed from the discontinuous and correction fluxes as follows

$$\hat{f}^\delta = \hat{f}^{\delta D} + \hat{f}^{\delta C} = \hat{f}^{\delta D} + (\hat{f}_L^{\delta I} - \hat{f}_L^{\delta D})h_L + (\hat{f}_R^{\delta I} - \hat{f}_R^{\delta D})h_R. \quad (14)$$

The final stage of the FR approach involves calculating the divergence of $\hat{f}^\delta$ at each solution point r_i using the expression

$$\frac{\partial \hat{f}^\delta}{\partial r}(r_i) = \sum_{j=0}^k \hat{f}_j^{\delta D} \frac{dl_j}{dr}(r_i) + (\hat{f}_L^{\delta I} - \hat{f}_L^{\delta D}) \frac{dh_L}{dr}(r_i) + (\hat{f}_R^{\delta I} - \hat{f}_R^{\delta D}) \frac{dh_R}{dr}(r_i). \quad (15)$$

These values can then be used to advance the approximate transformed solution $\hat{u}^\delta$ in time via a suitable temporal discretization of the following semi-discrete expression

$$\frac{d\hat{u}_i^\delta}{dt} = -\frac{\partial \hat{f}^\delta}{\partial r}(r_i). \quad (16)$$

The nature of a particular FR scheme depends solely on three factors, namely the location of the solution collocation points r_i , the methodology for calculating the transformed numerical interface fluxes $\hat{f}_L^{\delta I}$ and $\hat{f}_R^{\delta I}$, and finally the form of the flux correction functions h_L (and thus h_R). Huynh has shown that a collocation based (under integrated) nodal DG scheme is recovered in 1D if the corrections functions h_L and h_R are the right and left Radau polynomials respectively.¹⁰ Huynh has also shown that SD type methods can be recovered (at least for a linear flux function) if the corrections h_L and h_R are set to zero at a set of p points within Ω_S (located symmetrically about the origin). Huynh also suggested several additional forms of h_L (and thus h_R), leading to the development of new schemes, with various stability and accuracy properties. For further details of these new schemes, refer to the article by Huynh.¹⁰

The 1D flux reconstruction approach can easily be extended to quadrilateral elements in 2D via the construction of tensor product bases, as described in Ref. 10. With such a construction, the transformed approximate solution within the reference element is represented by a set of p^2 solution points, as shown in figure (1). The location of the solution points along 1D lines in each element automatically fixes the location of the flux points along each edge.

Figure 1: Solution points (circles) and flux points (squares) in the reference quadrilateral element for $p = 2$

B. Extension of the Flux Reconstruction Approach to Triangles

Now, consider the 2D scalar conservation law

$$\frac{\partial u}{\partial t} + \nabla_{xy} \cdot \mathbf{f} = 0 \quad (17)$$

within an arbitrary domain Ω , where x and y are spatial coordinates, t is time, $u = u(x, t)$ is a conserved scalar, $\mathbf{f} = (f, g)$ where $f = f(u)$ and $g = g(u)$ are the fluxes of u in the x and y directions respectively. Consider partitioning the domain Ω into N non-overlapping, conforming triangular elements Ω_n . To facilitate the implementation, each element Ω_n in physical space is mapped to a reference equilateral triangle Ω_S , using a mapping Θ_n , as shown in figure (2).

Figure 2: Mapping between the physical space (x, y) and the computational space (r, s)

For a linear triangular element, the mapping is

$$\mathbf{x} = \boldsymbol{\Theta}_n(\mathbf{r}) = \frac{(-3r + 2 - \sqrt{3}s)}{6}\mathbf{x}_1 + \frac{(2 + 3r - \sqrt{3}s)}{6}\mathbf{x}_2 + \frac{(2 + 2\sqrt{3}s)}{6}\mathbf{x}_3 \quad (18)$$

where $\mathbf{x}_1, \mathbf{x}_2$ and $\mathbf{x}_3$ are the coordinates of the vertices of the triangular element $\boldsymbol{\Omega}_n$ in physical space. Using this mapping, the governing equation (17) in the physical domain can be transformed to an equivalent governing equation in the computational domain which takes the form

$$\frac{\partial \hat{u}}{\partial t} + \nabla_{rs} \cdot \hat{\mathbf{f}} = 0 \quad (19)$$

where

$$\hat{u} = Ju, \quad \hat{\mathbf{f}} = (\hat{f}, \hat{g}) = \left(\frac{\partial y}{\partial s}f - \frac{\partial x}{\partial s}g, -\frac{\partial y}{\partial r}f + \frac{\partial x}{\partial r}g \right), \quad \text{and} \quad J = \frac{\partial x}{\partial r} \frac{\partial y}{\partial s} - \frac{\partial x}{\partial s} \frac{\partial y}{\partial r}. \quad (20)$$

The approximate solution $\hat{u}^\delta$ within the reference element $\boldsymbol{\Omega}_S$ is represented by a multi-dimensional polynomial of degree p , defined by its values at a set of $N_p = \frac{(p+1)(p+2)}{2}$ solution points (represented by hollow circles in figure (3)).

Figure 3: Solution points (circles) and flux points (squares) in the reference element for $p = 2$

The approximate solution in the reference element takes the form

$$\hat{u}^\delta(\mathbf{r}, t) = \sum_{i=1}^{N_p} \hat{u}_i^\delta l_i(\mathbf{r}) \quad (21)$$

where $\hat{u}_i^\delta = J_n \cdot u^\delta(\boldsymbol{\Theta}_n^{-1}(\mathbf{r}_i), t)$ is the value of $\hat{u}$ at the solution point i of element n , and $l_i(\mathbf{r})$ is the multi-dimensional Lagrange polynomial associated with the solution point i in the reference equilateral triangle $\boldsymbol{\Omega}_S$. The approximate solution $\hat{u}^\delta$ lies in the space $P_p(\boldsymbol{\Omega}_S)$, defined as the space of polynomials of degree $\leq p$

on Ω_S . Following the 1D flux reconstruction approach, the total transformed approximate flux $\hat{\mathbf{f}}^\delta = (\hat{f}^\delta, \hat{g}^\delta)$ is written as the sum of a discontinuous component $\hat{\mathbf{f}}^{\delta D}$ and a correction component $\hat{\mathbf{f}}^{\delta C}$,

$$\hat{\mathbf{f}}^\delta = \hat{\mathbf{f}}^{\delta D} + \hat{\mathbf{f}}^{\delta C}. \quad (22)$$

The approximate discontinuous flux $\hat{\mathbf{f}}^{\delta D} = (\hat{f}^{\delta D}, \hat{g}^{\delta D})$ is computed by constructing a degree p polynomial for each of its components as

$$\hat{f}^{\delta D} = \sum_{i=1}^{N_p} \hat{f}_i^{\delta D} l_i, \quad \hat{g}^{\delta D} = \sum_{i=1}^{N_p} \hat{g}_i^{\delta D} l_i \quad (23)$$

where the coefficients $\hat{f}_i^{\delta D}$ and $\hat{g}_i^{\delta D}$ are the values of the transformed flux at the solution point i evaluated directly from the approximate solution $\hat{u}_i$ ($\hat{f}_i^{\delta D} = \hat{f}(\hat{u}_i)$ and $\hat{g}_i^{\delta D} = \hat{g}(\hat{u}_i)$). The divergence of the discontinuous flux is therefore

$$\nabla_{rs} \cdot \hat{\mathbf{f}}^{\delta D} = \sum_{i=1}^{N_p} \hat{f}_i^{\delta D} \frac{\partial l_i}{\partial r} + \sum_{i=1}^{N_p} \hat{g}_i^{\delta D} \frac{\partial l_i}{\partial s}. \quad (24)$$

On each edge of the element, a set of $N_{fp} = (p+1)$ flux points (illustrated by squares in figure (3)) are defined and used to couple the solution between neighbouring elements. The correction flux $\hat{\mathbf{f}}^{\delta C}$ is constructed as follows

$$\hat{\mathbf{f}}^{\delta C}(\mathbf{r}) = \sum_{f=1}^3 \sum_{j=1}^{N_{fp}} \left[(\hat{\mathbf{f}} \cdot \hat{\mathbf{n}})_{f,j}^{\delta I} - (\hat{\mathbf{f}}^{\delta D} \cdot \hat{\mathbf{n}})_{f,j} \right] \mathbf{h}_{f,j}(\mathbf{r}) \quad (25)$$

$$= \sum_{f=1}^3 \sum_{j=1}^{N_{fp}} \Delta_{f,j} \mathbf{h}_{f,j}(\mathbf{r}). \quad (26)$$

Equations (25) and (26) deserve explanation. First, expressions subscripted by the indices f, j correspond to a quantity at the flux point j of face f , where $1 \leq f \leq 3$ and $1 \leq j \leq N_{fp}$ and the convention used to number the faces and flux points is illustrated in figure (4).

Figure 4: Numbering convention for the faces and flux points on the reference triangular element. Example shown corresponds to $p = 2$

For example, $(\hat{\mathbf{f}}^{\delta D} \cdot \hat{\mathbf{n}})_{f,j}$ is the normal component of the discontinuous transformed flux $\hat{\mathbf{f}}^{\delta D}$ at the flux point f, j . In equation (25), $(\hat{\mathbf{f}} \cdot \hat{\mathbf{n}})_{f,j}^{\delta I}$ is a normal transformed numerical flux computed at flux point f, j . As in the 1D flux reconstruction method, it is computed by first evaluating the multiply defined values of u^δ at each flux point using equation (21). At each flux point, we define u_-^δ to be the value of u^δ computed using the information local to the current element and u_+^δ to be the value of u^δ computed using information from the neighbouring element sharing the same flux point. Once both approximate solution values (u_-^δ and u_+^δ) are evaluated at each flux point, a solver specific to the equations being solved is used to compute a common interface flux based on u_+^δ , u_-^δ and the local normal vector $\mathbf{n}_{f,j}$. From this common numerical flux, the transformed normal numerical flux denoted by $(\hat{\mathbf{f}} \cdot \hat{\mathbf{n}})_{f,j}^{\delta I}$ can be obtained. In equation (26), $\Delta_{f,j}$ is defined

as the difference between the normal transformed numerical flux and the normal transformed discontinuous flux at the flux point f, j . Finally, $\mathbf{h}_{f,j}(\mathbf{r})$ is a vector correction function associated with flux point f, j . Each vector correction function $\mathbf{h}_{f,j}(\mathbf{r})$ is restricted to lie in the Raviart-Thomas space¹⁷ of order p , denoted by $RT_p(\Omega_S)$. Because of this property,

$$\begin{aligned} \nabla_{rs} \cdot \mathbf{h}_{f,j} &\in P_p(\Omega_S) \\ \mathbf{h}_{f,j} \cdot \mathbf{n} \Big|_{\Gamma_S} &\in R_p(\Gamma_S) \end{aligned} \quad (27)$$

where $P_p(\Omega_S)$ is the space of polynomials of degree $\leq p$ on the reference element Ω_S and $R_p(\Gamma_S)$ is the space of polynomials of degree $\leq p$ on the boundary of Γ_S . In other words, the divergence of each correction function ($\nabla_{rs} \cdot \mathbf{h}_{f,j}$) is a polynomial of degree p and the normal trace $\mathbf{h}_{f,j} \cdot \hat{\mathbf{n}}$ is also a polynomial of degree p along each edge. Furthermore, it is required that the correction functions $\mathbf{h}_{f,j}$ satisfy

$$\mathbf{h}_{f,j}(\mathbf{r}_{f_2,j_2}) \cdot \mathbf{n}_{f_2,j_2} = \begin{cases} 1 & \text{if } f = f_2 \text{ and } j = j_2 \\ 0 & \text{otherwise} \end{cases} \quad (28)$$

Hence, the following holds at each flux point f, j

$$\hat{\mathbf{f}}^{\delta C}(\mathbf{r}_{f,j}) \cdot \hat{\mathbf{n}}_{f,j} = \left[(\hat{\mathbf{f}} \cdot \hat{\mathbf{n}})_{f,j}^{\delta I} - (\hat{\mathbf{f}}^{\delta D} \cdot \hat{\mathbf{n}})_{f,j} \right] = \Delta_{f,j}. \quad (29)$$

An example of a vector correction function $\mathbf{h}_{f,j}$ is shown in figure (5) for the case $p = 2$.

Figure 5: Correction function associated with flux point $f = 2, j = 1$ for $p = 2$

To simplify the notation in the following sections, the correction field $\phi_{f,j}(\mathbf{r})$ defined as the divergence of the correction function $\mathbf{h}_{f,j}(\mathbf{r})$ is introduced, i.e.

$$\phi_{f,j}(\mathbf{r}) = \nabla_{rs} \cdot \mathbf{h}_{f,j}(\mathbf{r}). \quad (30)$$

Finally, combining equations (19), (22), (24) and (26), the approximate solution values at the solution points can be updated from

$$\begin{aligned} \frac{d\hat{u}_i^\delta}{dt} &= - \left(\nabla_{rs} \cdot \hat{\mathbf{f}}^\delta \right) \Big|_{\mathbf{r}_i} \\ &= - \left(\nabla_{rs} \cdot \hat{\mathbf{f}}^{\delta D} \right) \Big|_{\mathbf{r}_i} - \left(\nabla_{rs} \cdot \hat{\mathbf{f}}^{\delta C} \right) \Big|_{\mathbf{r}_i} \\ &= - \sum_{k=1}^{N_p} \hat{f}_k^{\delta D} \frac{\partial l_k}{\partial r} \Big|_{\mathbf{r}_i} - \sum_{k=1}^{N_p} \hat{g}_k^{\delta D} \frac{\partial l_k}{\partial s} \Big|_{\mathbf{r}_i} - \sum_{f=1}^3 \sum_{j=1}^{N_{fp}} \Delta_{f,j} \phi_{f,j}(\mathbf{r}_i). \end{aligned} \quad (31)$$

On triangular elements, the nature of a particular FR scheme on triangular elements depends on the location of the solution points, the location of the flux points, the methodology for calculating the transformed numerical interface flux $(\hat{\mathbf{f}} \cdot \hat{\mathbf{n}})_{f,j}^{\delta I}$, and finally, the form of the divergence $\phi_{f,j}$ of the correction functions $\mathbf{h}_{f,j}$.

It has been shown in Ref. 12 that a collocation-based (under integrated) nodal DG scheme is recovered on triangular elements if every correction function $\mathbf{h}_{f,j}$ is chosen such that

$$\int_{\Omega_S} \mathbf{h}_{f,j} \varphi d\Omega_S = 0 \quad (32)$$

for any polynomial $\varphi \in P_{p-1}(\Omega_S)$, i.e. the space of polynomials of degree $p-1$. For further details, see the article by Castonguay et al.¹²

III. Vincent-Castonguay-Jameson Schemes

This section begins with a presentation of the methodology to generate 1D VCJ schemes. Then, the procedure to generate VCJ schemes on triangular elements is presented.

A. VCJ schemes in 1D

VCJ schemes in 1D can be recovered if the left and right corrections functions h_L and h_R respectively are defined as

$$h_L = \frac{(-1)^p}{2} \left[L_p - \left(\frac{\eta_p L_{p-1} + L_{p+1}}{1 + \eta_p} \right) \right], \quad (33)$$

and

$$h_R = \frac{1}{2} \left[L_p + \left(\frac{\eta_p L_{p-1} + L_{p+1}}{1 + \eta_p} \right) \right], \quad (34)$$

where

$$\eta_p = \frac{c(2p+1)(a_p p!)^2}{2}, \quad a_p = \frac{(2p)!}{2^p (p!)^2}, \quad (35)$$

L_p is a Legendre polynomial of degree p , and c is a free scalar parameter that must lie within the range

$$\frac{-2}{(2p+1)(a_p p!)^2} < c < \infty. \quad (36)$$

Such correction functions satisfy

$$\int_{-1}^1 h_L \frac{\partial \hat{u}^\delta}{\partial r} dr - c \left(\frac{\partial^p \hat{u}^\delta}{\partial r^p} \right) \left(\frac{d^{p+1} h_L}{dr^{p+1}} \right) = 0, \quad (37)$$

and

$$\int_{-1}^1 h_R \frac{\partial \hat{u}^\delta}{\partial r} dr - c \left(\frac{\partial^p \hat{u}^\delta}{\partial r^p} \right) \left(\frac{d^{p+1} h_R}{dr^{p+1}} \right) = 0, \quad (38)$$

within the standard element Ω_S for any transformed solution $\hat{u}^\delta$, and ensure that the resulting VCJ scheme will be linearly stable in the broken Sobolev type norm $\|u^\delta\|_{p,2}$, defined as

$$\|u^\delta\|_{p,2} = \left[\sum_{n=1}^N \int_{x_n}^{x_{n+1}} (u_n^\delta)^2 + \frac{c}{2} (J_n)^{2p} \left(\frac{\partial^p u_n^\delta}{\partial x^p} \right)^2 dx \right]^{1/2}. \quad (39)$$

It can be noted that several existing methods are encompassed by the new class of VCJ schemes. In particular if $c = c_{dg} = 0$ then a collocation based nodal DG scheme is recovered. Alternatively, if

$$c = c_{sd} = \frac{2p}{(2p+1)(p+1)(a_p p!)^2}, \quad (40)$$

an SD method is recovered (at least for a linear flux function). It is in fact the only SD type scheme that can be recovered from the range of VCJ schemes. Further, it is identical to the SD scheme that Jameson¹³ proved to be linearly stable, which is the same as the only SD scheme that Huynh found to be devoid of weak instabilities.¹⁰ Finally, if

$$c = c_{hu} = \frac{2(p+1)}{(2p+1)p(a_p p!)^2}, \quad (41)$$

then a so called g_2 FR method is recovered, which was originally identified by Huynh¹⁰ to be particularly stable. The 1D VCJ schemes can be extended to quadrilateral elements via the construction of tensor product bases, following the procedure presented in Ref. 10.

B. VCJ schemes on triangular elements

Unlike for the 1D VCJ schemes, a closed form expression of the correction functions $\mathbf{h}_{f,j}$ and their divergence $\phi_{f,j}$ in terms of the parameter κ has not been found. The coefficients of the polynomials defining each correction field $\phi_{f,j}$ can be obtained from the solution of a system of equation. Before, the aforementioned system of equations is presented, some notation must be introduced. First, let the operator $D^{(v,w)}$ be defined as

$$D^{(v,w)} = \frac{\partial^w}{\partial r^{(w-v+1)} \partial s^{(v-1)}} \quad (42)$$

where v and w are integers such that $1 \leq v \leq w + 1$. For example,

$$D^{(1,2)} = \frac{\partial^2}{\partial r^2}, \quad D^{(2,2)} = \frac{\partial^2}{\partial r \partial s}, \quad D^{(3,2)} = \frac{\partial^2}{\partial s^2}. \quad (43)$$

Also, consider the 2D orthonormal Dubiner basis given by

$$\psi_k(\mathbf{r}) = \frac{4}{3^{1/4}} Q_v(a) Q_w^{(2v+1,0)}(b) (1-b)^v \quad (44)$$

$$k = w + (p+1)v + 1 - \frac{v}{2}(v-1), \quad (v, w) \geq 0; \quad v + w \leq p$$

where

$$a = \frac{3r}{2 - \sqrt{3}s}, \quad b = \frac{1}{3}(2\sqrt{3}s - 1) \quad (45)$$

and $Q_n^{(\alpha,\beta)}$ is the normalized n^{th} order Jacobi polynomial. The Dubiner basis given by equation (44) is orthonormal on the reference triangle Ω_S . Finally, let the binomial coefficients of degree w be defined by

$$\binom{v}{w} = \frac{v!}{w!(v-w)!}. \quad (46)$$

The correction fields $\phi_{f,j}$ are polynomials of degree p and therefore can be expressed as

$$\phi_{f,j} = \sum_{k=1}^{N_p} \sigma_k \psi_k(\mathbf{r}) \quad (47)$$

where σ_k are the expansion coefficients. In a VCJ scheme, the expansion coefficient σ_k of each correction field $\phi_{f,j}$ can be obtained by solving the following system of equations for the unknowns σ_k (for $k = 1$ to N_p)

$$\kappa \sum_{k=1}^{N_p} \sigma_k \sum_{m=1}^{p+1} \binom{p}{m-1} \left(D^{(m,p)} \psi_i \right) \left(D^{(m,p)} \psi_k \right) = -\sigma_i + \int_{\Gamma_S} (\mathbf{h}_{f,j} \cdot \hat{\mathbf{n}}) \psi_i d\Gamma, \quad \text{for } 1 \leq i \leq N_p \quad (48)$$

where κ is a free parameter that must lie within the range $0 \leq \kappa \leq \infty$. The last term on the right hand side of (48) can be evaluated exactly using equation (28) and the fact that $\mathbf{h}_{f,j} \cdot \hat{\mathbf{n}}$ is a polynomial of order p on each edge of Ω_S .

The correction fields $\phi_{f,j}$ obtained from the solution of equation (48) result in an energy stable flux reconstruction scheme on triangles parameterized by a single scalar parameter κ . Stability is guaranteed (for linear advection) in the norm

$$\left[\sum_{n=1}^N \int_{\Omega_n} \left(\frac{u^2}{2} + \frac{\kappa}{2A_S} \sum_{m=1}^{p+1} \binom{p}{m-1} \left(D^{(m,p)} u \right)^2 \right) d\Omega_n \right]^{1/2} \quad (49)$$

where A_S is the area of the reference element Ω_S , provided that $0 \leq \kappa < \infty$. If $\kappa = \kappa_{dg} = 0$, the VCJ scheme on triangular elements recovers a collocation based nodal DG scheme.

IV. Reference values of c and κ

In the previous section, it was shown that the 1D VCJ schemes (and their tensor-product extensions to quadrilateral elements) are parameterized by the single scalar c and the VCJ schemes on triangles by the single scalar κ . In 1D, it was shown in Ref. 11 that one can recover a collocation based DG scheme by setting $c = c_{dg} = 0$, a spectral difference scheme with $c = c_{sd}$ and a Huynh type scheme with $c = c_{hu}$. On triangular elements, one can recover a collocation-based nodal DG scheme by setting $\kappa = \kappa_{dg} = 0$.

Changing the parameters c and κ leads to an infinite number of flux reconstruction schemes, guaranteed to be stable for linear problems. In order to identify schemes that have desirable characteristics, accuracy and stability analysis of the VCJ schemes were conducted for the linear advection problem in Ref. 14 and 12. These studies gave insight into the effect of changing the parameters c and κ on the properties of the VCJ schemes. For example, by conducting a 2D von Neumann stability analysis of the third order VCJ schemes on triangles for the linear advection equation, figure (6) was obtained. The plot shows how the maximum allowable non-dimensional time step varies with the parameter κ and indicates that certain values of κ lead to an increased maximum time step. The von Neumann stability analysis was performed on a grid obtained by repeating the generating pattern shown in figure (7), with τ set to 60° . An upwind numerical flux was used and the solution polynomial order was set to $p = 2$. More details on the analysis can be found in Ref. 12.

Figure 6: Maximum allowable time step versus the parameter κ for linear advection on triangular grid with different explicit time advancement schemes

Figure 7: Mesh generating pattern used for von Neumann stability analysis on triangles

From similar plots obtained for different solution polynomial orders p in 1D and 2D, the values of c and κ that lead to a maximum allowable time step were identified. Such values will be denoted by c_+ and κ_+ and they depend on both the explicit time advancement scheme used and the order of accuracy of the VCJ scheme. For the low-storage five stage, 4th order Runge-Kutta scheme,¹⁸ the values of c_+ and k_+ are listed in table (1) for polynomial orders $p = 2$ to 4.

Table 1: Values of c_+ and κ_+ for the low-storage RK4 time-stepping scheme

Polynomial degree p	c_+	κ_+
2	0.206	4.30e-2
3	3.80e-3	6.40e-4
4	4.67e-5	5.30e-6

Because the class of VCJ schemes on triangular elements only has two values of κ as reference points (κ_{dg} and κ_+), while the family of 1D VCJ schemes has four (c_{dg} , c_{sd} , c_{hu} and c_+), two intermediate reference

values of κ are defined as follows

$$\kappa_{sd} = \frac{c_{sd}}{c_+} \kappa_+ \quad (50)$$

$$\kappa_{hu} = \frac{c_{hu}}{c_+} \kappa_+. \quad (51)$$

Furthermore, two additional reference values of c and κ are defined as

$$c_{++} = 2 \cdot c_+ \quad (52)$$

$$\kappa_{++} = 2 \cdot \kappa_+. \quad (53)$$

Table (2) shows the numerical values of the five reference values of c and κ defined in this section (namely $c_{dg}, c_{sd}, c_{hu}, c_+, c_{++}$ and $\kappa_{dg}, \kappa_{sd}, \kappa_{hu}, \kappa_+, \kappa_{++}$), along with the equation or table used to compute them.

Table 2: Reference values of c and κ

	Equation/Table	p=2	p=3	p=4
c_{dg}	0	0	0	0
c_{sd}	Eq. (40)	2.96e-2	9.52e-4	1.61e-5
c_{hu}	Eq. (41)	6.67e-2	1.69e-3	2.52e-5
c_+	Table (1)	2.06e-1	3.80e-3	4.67e-5
c_{++}	Eq. (52)	4.12e-1	7.60e-3	9.34e-5
κ_{dg}	0	0	0	0
κ_{sd}	Eq. (50)	6.18e-3	1.60e-4	1.83e-6
κ_{hu}	Eq. (51)	1.39e-2	2.85e-4	2.86e-6
κ_+	Table (1)	4.30e-2	6.40e-4	5.30e-6
κ_{++}	Eq. (53)	8.60e-2	1.28e-3	1.06e-5

In the following section, numerical experiments will be performed using these values of c and κ in order to compare the properties of their corresponding VCJ scheme.

V. Numerical Experiments

In this section, numerical experiments are performed on the Euler equations to elucidate various properties of VCJ schemes on quadrilateral and triangular elements. The two-dimensional Euler equations are

$$\frac{\partial q}{\partial t} + \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = 0 \quad (54)$$

where the state q and the inviscid fluxes f and g are

$$q = \begin{pmatrix} \rho \\ \rho u \\ \rho v \\ \rho e \end{pmatrix}, \quad f = \begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ \rho eu + p \end{pmatrix}, \quad g = \begin{pmatrix} \rho v \\ \rho uv \\ \rho v^2 + p \\ \rho ev + p \end{pmatrix}. \quad (55)$$

In equation (55), ρ is the density, u is the velocity component in the x direction, v is the velocity component in the y direction and e is the total energy per unit mass. In all the numerical experiments presented in this section, the Rusanov flux¹⁶ is used to compute the common interface flux $(\mathbf{f} \cdot \mathbf{n})^{\delta I} = (fn_x + gn_y)^{\delta I}$.

A. Isentropic Vortex Test Case

The first test case considered is the Euler vortex test case used by Shu¹⁹, for which an exact solution is known. The initial condition consists of a uniform flow with $\{\rho, u, v, p\} = \{1, 1, 1, 1\}$ onto which an isentropic vortex is added. The isentropic vortex has perturbations in u, v and temperature $T = p/\rho$ and no perturbation in entropy $S = p/\rho^\gamma$. The perturbations in u, v and T are given by

$$\Delta u = -\frac{\epsilon}{2\pi} e^{0.5(1-R^2)y} \quad (56)$$

$$\Delta v = \frac{\epsilon}{2\pi} e^{0.5(1-R^2)x} \quad (57)$$

$$\Delta T = \frac{(\gamma - 1)\epsilon^2}{8\gamma\pi^2} e^{1-R^2} \quad (58)$$

where $R = \sqrt{x^2 + y^2}$ and the vortex strength $\epsilon = 5$. The exact solution for the conditions given above is the propagation of the vortex with a velocity of (1,1), assuming that the domain is infinitely big. In the numerical simulations, the domain is taken to be $-10 \leq x \leq 10$, $-10 \leq y \leq 10$ with periodic boundary conditions imposed on the boundaries. Furthermore, all simulations are advanced in time using a low-storage 4th order Runge-Kutta scheme.¹⁸

1. Preliminary study on the effect of solution points and flux points location

The nature of VCJ schemes depends on the location of the solution points $\mathbf{r}_i$, the numerical flux function used to compute the common interface fluxes ($f^{\delta I}$ in 1D and $(\mathbf{f} \cdot \mathbf{n})^{\delta I}$ in 2D) and the value of the parameters c and κ used to define the correction functions. On triangular elements, the nature of the schemes also depends on the location of the flux points along each edge, while for quadrilateral elements, the edge flux point locations are fixed by the position of the solution points. Recently, it was shown by Jameson, Vincent and Castonguay²⁰ that non-linearity stability of the 1D VCJ schemes is not guaranteed since they utilize a collocation projection in order to construct an approximation of the flux. Using such a projection leads to aliasing errors, and consequently aliasing driven instabilities (of the same form as those which afflict collocation based DG schemes if the solution is under-resolved). It was also noted that the location of the solution points (at which the collocation projection is performed) is critical in terms of non-linear stability. These findings suggest that the solution points within the reference elements and the flux points along edges should be located at good integration points to mitigate aliasing errors. To illustrate the effect of the solution and flux point locations on non-linearity stability on triangular elements, two numerical simulations were performed on the grid shown in figure (8) for the isentropic vortex problem with $p = 4$. The initial solution described in the previous subsection was used and is shown in figure (9). The two simulations used a different set of solution points and flux points. In the first simulation, the location of the solution and flux points was chosen as in figure (10). Along each edge, the flux points are located at Legendre-Gauss-Lobatto points, while the interior solution points are located at α -optimized points.⁴ In the second simulation, the flux points are located at Gauss points along each edge, and the solution points at 15 numerical quadrature points that exactly integrate polynomials of degree 7, as shown in figure (11). The locations of the solution points were taken from Ref. 21. Note that because flux points and solution points overlap in the first set of points, the resulting scheme is computationally more efficient than the second.

Figure 8: Mesh used to investigate effect of solution and flux points location

Figure 9: Initial solution for the isentropic vortex test case

Figure 10: First set of flux and solution points for $p = 4$. Flux points are located at Gauss-Lobatto points along each edge. Solution points are located at α -optimized points (Ref. 4)

Figure 11: Second set of flux and solution points for $p = 4$. Flux points are located at Gauss points along each edge. Solution points are located at good numerical quadrature points given in Ref. 21

For both simulations, the parameter κ was set to 0, thus recovering a collocation-based nodal DG scheme. The time step was taken sufficiently small so that time discretization errors are negligible and the simulation was run until $t = 358$. The final solutions obtained with the two set of points are illustrated in figures (12) and (13). The effect of aliasing driven instabilities is clearly seen in the first simulation (the one with the set of points shown in figure (10)) and they eventually cause the solution to blow up at $t = 360$. The second simulation was not afflicted by aliasing instabilities and remained stable until it was stopped at $t=4000$. Based on these preliminary numerical experiments in 2D and the 1D theoretical analysis conducted in Ref. 20, the simulations in the remainder of this article are performed with the flux points located at Gauss points along each edge, and for triangular elements, the solution points are placed at the numerical quadrature points given in Ref. 21. For quadrilateral elements, the solution points are located at Gauss points in each coordinate direction. Future work will further investigate the effect of the solution and flux points location on the accuracy and stability of VCJ schemes.

Figure 12: Contours of density at $t=358$, obtained using the solution and flux points shown in figure (10)

Figure 13: Contours of density at $t=358$, obtained using the solution and flux points shown in figure (11)

2. Stability and accuracy analysis of VCJ schemes

In this section, a comprehensive set of numerical experiments are performed for the isentropic vortex problem in order to investigate how the accuracy and stability of the VCJ schemes are affected by the choice of the parameters c and κ . The numerical experiments are performed on three different grid types: an unstructured triangular grid, a structured quadrilateral grid and finally, a grid containing a mix of quadrilateral and triangular elements. The number of elements in each grid is $N_x \times N_x \times 2$ for the triangular and mixed grids and $N_x \times N_x$ for the quadrilateral grids, where N_x is varied from 10 to 160. Each grid is refined recursively by splitting each of its elements into four. The grid spacing Δx is defined as $\Delta x = \frac{N_x}{20}$. Figures (14), (15) and (16) show the three sets of grids used.

First, the order of accuracy of the VCJ schemes is investigated. For each grid, the simulation is run until $t = 2$ and the approximate solution is compared with the exact solution. The L2 norm of the error in density is used as a measure of accuracy, which can be computed from

$$\text{L2 error} = \sqrt{\int_{\Omega} (\rho^e(x, y) - \rho^\delta(x, y))^2 d\Omega} = \sqrt{\sum_{n=1}^N \int_{\Omega_n} (\rho^e(x, y) - \rho^\delta(x, y))^2 d\Omega_n} \quad (59)$$

where N is the total number of elements in the mesh, ρ^e is the exact density and ρ^δ is the approximate density obtained using the VCJ schemes.

A total of 14 different values of c and κ are used, among which are the values shown in table (2). For the mixed grid, the notation (c, κ) is used to indicate the value of c used for the VCJ scheme on quadrilateral elements and the value of κ used on triangular elements. On mixed grid, the value of κ is set to be proportional to the value of c as follows

$$\kappa = \frac{c}{c_+} \kappa_+. \quad (60)$$

Figures (17(a)) through (19(c)) show plots of the L2 error versus the mesh spacing Δx for the three different grid types, three different polynomial orders p , and 5 different values of c , κ or a pair (c, κ) . The figures indicate that on the three grid types and for polynomial orders $p = 2, 3$ and 4, the VCJ schemes corresponding to $c = c_{dg}$, $\kappa = \kappa_{dg}$ and $(c, \kappa) = (c_{dg}, \kappa_{dg})$ give the most accurate solutions, and increasing the value of c or κ leads to an increase in the magnitude of the error. For each line in figures (17(a)) through (19(c)), the average slope was computed in order to obtain a measure of the order of accuracy of the VCJ schemes. The order of accuracy calculated is shown as a function of the parameters c and/or κ for the three different grid types in figures (20), (21) and (22). For a solution obtained with a solution polynomial of order p , the optimal rate of convergence is $O(h^{p+1/2})$ when the Rusanov flux is used, as discussed in section 5.5

of Ref. 4. For the values of c and κ listed in table (2), the optimal convergence rate is obtained for all cases except for $p = 4$ on the mixed grid. Furthermore, for almost all cases, increasing the value of c and κ past c_{++} and κ_{++} respectively, leads to a significant decrease in the order of accuracy.

As another metric to evaluate the characteristics of VCJ schemes, the maximum allowable time step is computed experimentally for orders $p = 2, 3$ and 4 for the grids in figures (14(c)), (15(c)) and (16(c)). The maximum allowable time step was determined by gradually increasing the time step by increments of $1e-5$, until the solution went unstable before reaching a time $t = 100$. The results are shown in figures (23(a)), (23(b)) and (23(c)). The linear stability analysis conducted in Ref. 14 and 12 showed that the VCJ schemes corresponding to values of c_+ and κ_+ allowed a significant increase in the maximum allowable time step, when compared to the collocation-based nodal DG schemes (which correspond to $c = 0$ and $\kappa = 0$). A similar trend is observed for the isentropic vortex test case. On the three grid types, increasing the value of the parameters c and κ leads to an increase in the maximum allowable time step compared to a collocation-based nodal DG.

Finally, combining the results obtained from the accuracy and stability analysis, figures (24(a)) through (25(c)) were generated. In figure (24(a)), (24(b)) and (24(c)), the maximum time step is plotted versus the order of accuracy for VCJ schemes of orders $p = 2, 3$ and 4, respectively. The figures indicate that for all combinations of p and the grid type (except for $p = 4$ and the mixed grid), increasing the values of c and κ from 0 to c_+ and κ_+ respectively, leads to a significant increase in the allowable time step, without a significant change in the order of accuracy of the scheme. Figures (25(a)), (25(b)) and (25(c)) on the other hand show how the maximum time step varies as a function of the magnitude of the error on the grids of dimension $N_x = 40$, for orders $p = 2, 3$ and 4, respectively. These results clearly aid identification of VCJ schemes with desirable properties. For example, consider the case $p = 2$ on the quadrilateral mesh, shown by the red line in figure (25(a)). The VCJ scheme corresponding to c_+ is certainly superior to the one corresponding to c_{++} , since both provide the same maximum time step, while the value of c_+ leads to a smaller error magnitude. The results also indicate that by choosing values of c and κ in the range $0 \leq c \leq c_+$ and $0 \leq \kappa \leq \kappa_+$, respectively, one can trade-off accuracy for stability, and vice versa, while maintaining the same order of accuracy. For example, for the case $p = 2$ on the quadrilateral grid, if one wishes to increase the maximum allowable time step, the VCJ scheme corresponding to c_{hu} would provide an approximately two-fold increase in the CFL number compared to a collocation-based DG scheme, at the expense of an increase in the magnitude of the error by a factor of approximately 2.5. Similar trends are observed for the different grid types and solution polynomial orders p . The results presented in this section offer a quantitative measure of the trade-offs between accuracy and maximum time step for different VCJ schemes. These quantitative results can be utilized to choose the appropriate VCJ scheme to use based on the requirements of a simulation.

B. Inviscid flow over a cylinder

The numerical experiments performed on the isentropic vortex test case illustrate how the values of c and κ can be selected to obtain a scheme tailored to specific needs in terms of order of accuracy, allowable time step and absolute error magnitude. In order to investigate if those trends are also observed for steady state simulations with curved boundaries, numerical experiments are performed for the inviscid flow over a half-cylinder at a Mach number of 0.2. For all cases presented, quadratic elements are used to represent the geometry, and the grids were generated with the mesh generating software Gambit. For time advancement, the explicit low-storage 4th order Runge-Kutta¹⁸ scheme is used, and for all cases, the residual is converged to machine zero. Characteristics boundary conditions are used at the inflow and outflow boundaries.

Numerical experiments are performed on three different grids: a structured quadrilateral mesh, an unstructured triangular mesh and a low-quality mixed mesh, shown in figures (32(a)), (33(a)) and (34(a)), respectively. It should be noted that the mixed grids purposefully contains highly skewed elements and a highly non-uniform cell size distribution. For each grid, steady state solutions are obtained with the polynomial order p set to 2. For the quadrilateral grid and triangular grids, the value of c and κ from table (2) are used. For the mixed grids, the VCJ schemes corresponding to the following pairs (c, κ) are used: (c_{dg}, κ_{dg}) , (c_{sd}, κ_{sd}) , (c_{hu}, κ_{hu}) , (c_+, κ_+) and (c_{++}, κ_{++}) .

First, for every combination of a grid and a VCJ scheme, the time step leading to the fastest convergence rate to machine zero is identified. This is done by gradually increasing the value of Δt by increments of $1e-4$ and recording the number of time steps required to reach a value of 10^{-13} in the L2 norm of the density residual. For each value of c , κ and (c, κ) , the time step leading to the fastest convergence rate is

identified, and the corresponding residual convergence history is plotted in figures (26), (28) and (30) for the quadrilateral, triangular and mixed grids, respectively. Results are similar to the ones obtained for the isentropic vortex test case. The values of c_{hu} , c_+ , c_{++} , κ_{hu} , κ_+ and κ_{++} allow a significant increase in the maximum allowable time step, and therefore, a significant reduction in the number of iterations required to converge to machine zero. On average, those VCJ schemes converge in half the number of iterations required by the collocation-based nodal DG schemes. Obviously, using an explicit time-stepping scheme to converge to a steady state solution is not efficient. However, in the context of a multi-grid algorithm that uses an explicit smoother, the aforementioned VCJ schemes would allow a significant improvement in terms of number of iterations compared to the collocation-based nodal DG approach. It should be noted that for the low-quality mixed grid, the VCJ scheme corresponding to (c_{++}, κ_{++}) failed to converge, highlighting the fact that increasing the values of c and κ past c_+ and κ_+ can lead to a significant decrease in accuracy.

In order to investigate the effect of changing the parameters c and κ on the accuracy, the entropy error at the wall is computed. Figures (27), (29) and (31) show the entropy error of the steady state solution for the different VCJ schemes, on the quadrilateral, triangular and mixed grids respectively. On the quadrilateral grid, the different VCJ schemes give very similar entropy error levels and on the triangular grids, all schemes converge to the exact same entropy error. On the mixed grid, the solutions obtained with (c_{dg}, κ_{dg}) and (c_{sd}, κ_{sd}) have the lowest entropy error, but overall, all schemes yield very similar error levels.

The steady state density contours obtained for the various VCJ schemes on different grids are plotted in figures (32) through (34). The results obtained on the quadrilateral grid are nearly identical for all schemes. On the triangular grid, as discussed earlier, the converged solution is the same for all schemes, hence, only one contour plot is shown. On the low-quality mixed grid, the VCJ schemes corresponding to (c_{dg}, κ_{dg}) , (c_{sd}, κ_{sd}) , (c_{hu}, κ_{hu}) and (c_+, κ_+) give smooth solution contours, even if the mesh contains highly skewed elements. The collocation-based nodal DG scheme gives the smoothest density contours.

The results obtained on the half cylinder seems to confirm the findings of the previous subsection, which showed that increasing the values of c from 0 to c_+ and κ_+ leads to an increase in the maximum allowable time step, and therefore an increase in the convergence rate to steady state. This improvement is obtained at the expense of a decrease in the accuracy of the solution.

C. Inviscid flow over an airfoil

As a more realistic test case, the subsonic flow around a NACA0012 is simulated on the mesh shown in figure (35) which contains quadrilateral elements near the surface and triangular elements elsewhere, with a total of 2883 cells. The airfoil surface is represented using quadratic elements. The far field is located at 15 chords from the leading edge of the airfoil and characteristics boundary conditions are used at the inflow and outflow. The incoming Mach number is set to 0.2 and the angle of attack to 5° . The two VCJ schemes corresponding to (c_{dg}, κ_{dg}) and (c_+, κ_+) with $p = 2$ and 3 are used. For the two VCJ schemes, the entropy error of the steady solution at the wall is shown in figure (36) for $p = 2$ and $p = 3$. For $p = 2$, the collocation-based nodal DG schemes provides the most accurate solution, while for $p = 3$, the two schemes yield nearly identical entropy error levels. In Ref. 11, the authors indicate that changing the value of the parameter c affects the damping of the highest degree component of the solution polynomial. For a smooth solution, as the order p is increased, the highest degree component of the solution polynomial becomes less important relative the lowest degree components, and therefore it is expected that all VCJ scheme should produce similar results. The results obtained here support this hypothesis. Density contours from the four simulations are shown in figure (37). Good agreement is observed between the simulations obtained with the two VCJ schemes, especially for $p = 3$. For $p = 2$, the VCJ scheme corresponding to (c_+, κ_+) produces more oscillations in the density contours near the surface of the airfoil.

VI. Conclusions

This article presents a comprehensive study of the stability and accuracy properties of a new set of energy stable flux reconstruction schemes (referred to as VCJ schemes) for two-dimensional inviscid flows. The study shows that the parameters defining the VCJ schemes can be changed to trade-off accuracy for maximum allowable time step, and vice-versa, while maintaining the optimal order of accuracy. The results obtained in this work provide the user with a quantitative measure of those trade-offs, and shows how the appropriate VCJ scheme can be selected based on the accuracy and stability needs of a particular simulation. These

results are also important because until now, a comparison between various flux reconstruction schemes in the context of non-linear equations on arbitrary grid in 2D had not been performed. The findings should motivate further research into the properties of VCJ schemes in the context of viscous flow simulations in three dimensions.

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Figure 14: Set of three unstructured triangular grids used in the order of accuracy analysis for the isentropic vortex problem

Figure 15: Set of three structured quadrilateral grids used in the order of accuracy analysis for the isentropic vortex problem

Figure 16: Set of three unstructured mixed grid used in the order of accuracy analysis for the isentropic vortex problem

(a) $p=2$

(b) $p=3$

(c) $p=4$

Figure 17: L2 error versus mesh spacing Δx for simulations on unstructured triangular grids

(a) $p=2$

(b) $p=3$

(c) $p=4$

Figure 18: L2 error versus mesh spacing Δx for simulations on structured quadrilateral grids

(a) $p=2$

(b) $p=3$

(c) $p=4$

Figure 19: L2 error versus mesh spacing Δx for simulations on mixed grids

Figure 20: Order of accuracy versus the parameter κ for the Euler vortex test case on an unstructured triangular grid [$*$ = κ_{sd} , o = κ_{hu} , $+$ = κ_+ , Δ = κ_{++}]

Figure 21: Order of accuracy versus the parameter c for the Euler vortex test case on a structured quadrilateral grid [$*$ = c_{sd} , o = c_{hu} , $+$ = c_+ , Δ = c_{++}]

Figure 22: Order of accuracy versus the parameter c (and implicitly κ via 60) for the Euler vortex test case on a mixed grid [$*$ = (c_{sd}, κ_{sd}) , o = (c_{hu}, κ_{hu}) , $+$ = (c_+, κ_+) , Δ = (c_{++}, κ_{++})]

(a) $p=2$

(b) $p=3$

(c) $p=4$

Figure 23: Maximum allowable time-step versus the parameters c and κ for polynomials order $p = 2$ (a), $p = 3$ (b) and $p = 4$ (c), for the isentropic vortex problem. $\square = (c_{dg}, \kappa_{dg})$, $*$ $= (c_{sd}, \kappa_{sd})$, $\circ = (c_{hu}, \kappa_{hu})$, $+$ $= (c_+, \kappa_+)$, $\Delta = (c_{++}, \kappa_{++})$]

(a) $p=2$

(b) $p=3$

(c) $p=4$

Figure 24: Plots of maximum allowable time step, versus order of accuracy for $p = 2$ (a), $p = 3$ (b) and $p = 4$ (c) for the isentropic vortex problem. $\square = (c_{dg}, \kappa_{dg})$, $*$ = (c_{sd}, κ_{sd}) , $\circ = (c_{hu}, \kappa_{hu})$, $+$ = (c_+, κ_+) , $\Delta = (c_{++}, \kappa_{++})$]

(a) $p=2$

(b) $p=3$

(c) $p=4$

Figure 25: Plots of maximum allowable time step, versus L2 norm of error in density for $p = 2$ (a), $p = 3$ (b) and $p = 4$ (c) for the isentropic vortex problem. $[\square] = (c_{dg}, \kappa_{dg})$, $[\ast] = (c_{sd}, \kappa_{sd})$, $[\circ] = (c_{hu}, \kappa_{hu})$, $[\ast] = (c_{+}, \kappa_{+})$, $[\Delta] = (c_{++}, \kappa_{++})$

Figure 26: Convergence history of the L2 norm of the density residual for the structured quadrilateral grid over cylinder

Figure 27: Entropy error on surface of the cylinder for the structured quadrilateral grid

Figure 28: Convergence history of the L2 norm of the density residual for the unstructured triangular grid over cylinder

Figure 29: Entropy error on surface of the cylinder for the unstructured triangular grid

Figure 30: Convergence history of the L2 norm of the density residual for the low-quality mixed grid over cylinder

Figure 31: Entropy error on surface of the cylinder for the low-quality mixed grid

(a) Structured quadrilateral mesh around a half-cylinder

(b) c_{dg}

(c) c_{sd}

(d) c_{hu}

(e) c_+

(f) c_{++}

Figure 32: Density contours for the flow over a cylinder at $M = 0.2$. Solution is obtained with $p = 2$ with different VCJ schemes on the quadrilateral mesh shown in (a)

(a) Unstructured triangular mesh over cylinder

(b) c_{dg} , c_{sd} , c_{hu} , c_+ and c_{++}

Figure 33: Density contours for the flow over a cylinder at $M = 0.2$. Solution is obtained with different VCJ schemes with $p = 2$, on the unstructured triangular mesh shown in (a)

(a) Low quality mixed mesh over cylinder

(b) (c_{dg}, κ_{dg})

(c) (c_{sd}, κ_{sd})

(d) (c_{hu}, κ_{hu})

(e) (c_+, κ_+)

(f) (c_{++}, κ_{++})

Figure 34: Density contours for the flow over a cylinder at $M = 0.2$. Solution is obtained with different VCJ schemes with $p = 2$, on the low-quality mesh shown in (a)

Figure 35: Mixed grid over NACA0012 airfoil

Figure 36: Entropy error on NACA 0012 airfoil for VCJ schemes with (c_{dg}, κ_{dg}) and (c_+, κ_+) for $p = 2$ and $p = 3$

(a) $p = 2, (c_{dg}, \kappa_{dg})$

(b) $p = 2, (c_+, \kappa_+)$

(c) $p = 3, (c_{dg}, \kappa_{dg})$

(d) $p = 3, (c_+, \kappa_+)$

Figure 37: Density contours for the flow over the NACA 0012 airfoil at $M = 0.2$ and angle of attack 5° . Solution is obtained with $p = 2$ (a,b) and $p = 3$ (c,d) for two different VCJ schemes on the mesh shown in figure (35)

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