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Studies of Wings Supporting Non-unique Solutions in Transonic Flows

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Non-unique numerical solutions of transonic flows over airfoils have been found, first for potential flow equation, and later for Euler and RANS equations. The studies have also been extended to unsteady flow simulations, and non-unique numerical solutions continue to be demonstrated. The question of whether three-dimensional effect can have a further influence on the uniqueness of the transonic flow solution remain an important one. Hitherto wings supporting non-unique solutions in transonic solutions have not been studied. Research in this direction will further our understanding of the behavior of non-unique transonic flows. This paper studied a set of four wings based on recently designed symmetrical airfoils that have been found to support non-unique transonic solutions in both steady and unsteady flows in a narrow band of transonic Mach numbers. The aspect ratios of the wings have been varied as a way to control the extend of the three-dimensional effect. For certain of these airfoils, the non-unique solutions cease to exist when extended to a full wing, while for others, non-unique solutions continue to exist depending on the choice of the aspect ratios of the wings. The flow conditions that support non-unique solutions also tend to change when the airfoils are extended to wings of different aspect ratios. The scope of the study is, at present, limited to Euler solutions.

I. Background

Non-unique solutions of the transonic potential flow equation were discovered by Steinhoff and Jameson[1] (1981), who obtained lifting solutions for a symmetric Joukowski airfoil at zero angle of attack in a narrow range of Mach numbers in the neighborhood of Mach 0.85. This non-uniqueness could not be duplicated with the Euler equations and it was conjectured by Salas et al[2] (1983) that the non-uniqueness was a consequence of the isentropic flow approximation. Subsequently, however, Jameson[3] (1991) discovered several airfoils which supported non-unique solutions of the Euler equations in a narrow Mach band. These airfoils were lifting.

The question of non-unique transonic flows was re-examined by Hafez and Guo[4–6] (1999) who formed both lifting and non-lifting solutions for a 12 percent thick symmetric airfoil with parallel sides from 25 to 75 percent chord in a Mach range from 0.825 to 0.843. The question was further pursued in detail in a series of studies by Kuz'min and Ivanova[7–11] (2004,2006) who confirmed the results of Hafez and Guo, and also showed that airfoils with positive curvature everywhere could support non-unique solutions.

Recently, a set of four symmetrical airfoils was designed by Jameson, Vassberg, and Ou[17] (2011) which were found to support non-unique transonic solutions. The NU4 airfoil was the result of aggressive shape optimization to minimize drag of a 12 percent thick symmetrical airfoil. The JF1 airfoil is an extremely simple parallel sided airfoil. The JB1 airfoil is also parallel sided but has continuous curvature over the entire profile. The JC6 airfoil is convex and C_∞ continuous. It was found that in non-lifting transonic flow these

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airfoils exhibit a transition from a solution with two supersonic zones on each surface below a certain critical Mach number to a situation with one supersonic zone on each surface above the critical Mach number. In the region of instability solutions with positive lift are found in which there is a single supersonic zone on the top surface and two supersonic zones on the lower surface. Solutions with negative lift are the mirror images of the solutions with positive lift. The $CL - \alpha$ plots of these airfoils exhibit three branches at zero angle of attack corresponding to a P-branch with positive lift, a Z-branch with zero lift, and a N-branch with negative lift. In the most recent work by the same authors,[18] unsteady RANS solutions have been performed for the same set of airfoils in the transonic regime. Non-unique transonic solutions continue to be demonstrated for all four airfoils under very similar flow conditions as the Euler cases.

II. Introduction

Given that the equations governing steady inviscid compressible flow are nonlinear, one can anticipate the possibility of non-unique solutions. A familiar example is the case of supersonic flow past a wedge at an angle θ, where there are two solutions with different shock angles β corresponding to the strong and weak branches of the $\beta - \theta$ diagram. Supersonic flows past a three-dimensional cone are qualitatively similar in that for a given cone angle and Mach number, there are two possible oblique shock waves corresponding to the strong- and weak-shock solutions. In going from the wedge to the cone, the maximum angle beyond which the shock becomes detached changes, as a result of three-dimensional relieving effect. The same three-dimensional effect that influences the supersonic flows over the cone can influence transonic flows over the wing. As a result, the conditions supporting non-unique flow solutions might change in going from a two-dimensional airfoil to a three-dimensional wing; alternatively, the non-unique solutions might cease to exist altogether. The purpose of the present study is to investigate the three-dimensional effect on the non-unique transonic flows.

III. Overview of Airfoils Supporting Non-unique Transonic Flows

A set of four airfoils of very different characteristics were designed and found to share the property that in non-lifting transonic flow they exhibit a transition from a solution with two supersonic zones on each surface below a certain critical Mach number to a situation with one supersonic zone on each surface above the critical Mach number. In the region of instability solutions with positive lift are found in which there is a single supersonic zone on the top surface and two supersonic zones on the lower surface, and also solutions with negative lift which are the mirror images of the solutions with positive lift. For further details related to the Euler solutions, please refer to the work by Jameson, Vassberg, and Ou[17] (2011); for details of the unsteady RANS simulations, please refer to the work by Ou, Jameson, and Vassberg[18] (2014); for details of the numerical schemes, please refer to the work by Jameson.[13–15]

The geometric definitions of the airfoils are described below. The non-unique transonic solutions computed using steady Euler simulations are discussed briefly below for completeness.

III.A. NU4 Airfoil

III.A.1. NU4 Airfoil Geometry

The NU4 airfoil was a consequence of a shape optimization study for symmetric airfoils in transonic flow,[12] in which an attempt was made to find a 12 percent thick airfoil with a shock free solution at Mach 0.84. The resulting NU4 airfoil has an almost shock free solution at its design Mach number, but also allows a lifting and non-lifting solution at zero angle of attack.

III.A.2. Non-unique Solutions for NU4 Airfoil at M.840

The non-unique solutions for NU4 airfoils at M.840 are shown in Figure 1 and 2. In Figure 1, the CL versus α curve shows the presence of three distinctive solutions at zero angle of attack. In Figure 2, the corresponding Cp distribution and pressure contour at zero angle of attack are plotted. Figure 2(b) corresponds to the solution in the Z-branch. Figure 2(a) corresponds to the solution in the P-branch. The solution in the N-branch is the mirror image of the solution in the P-branch, hence is not plotted.

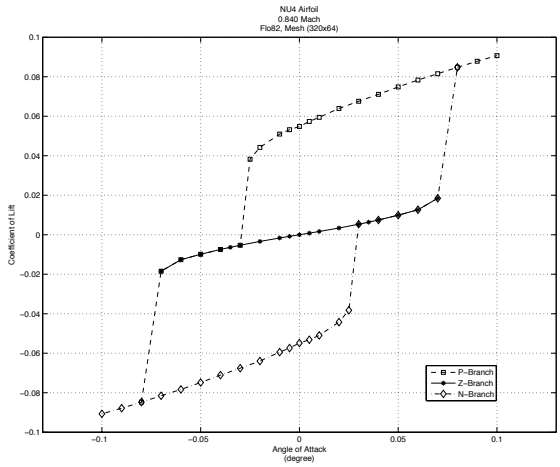
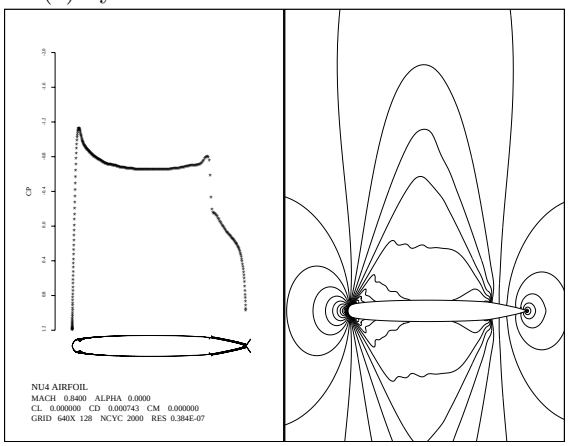


Figure 1. Plot of CL-α sweep for flows over the NU4 airfoil at M.840, showing the presence of the positive, negative, and zero branches.

(a) Symmetrical solution on the Z-branch



(b) Asymmetrical solution on the P-branch

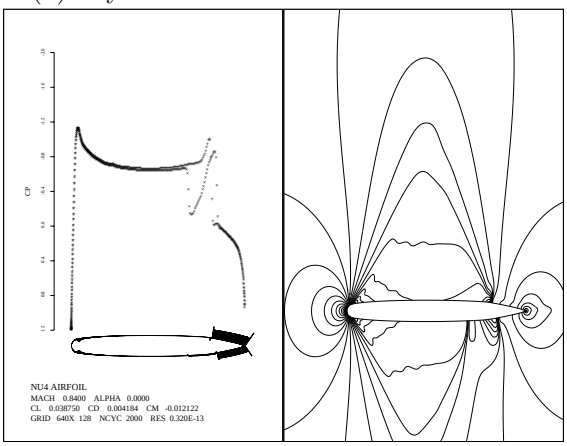


Figure 2. Pressure contours showing non-unique solutions of NU4 airfoil at $\alpha = 0^\circ$, M=.840

III.B. JF1 Airfoil

III.B.1. JF1 Airfoil Geometry

The JF1 airfoil has a shape that is very simple, consisting of a parallel sided slab closed by a semi-circular nose and two parabolic arcs at the rear. Depending on the extent of the parabolic arcs a Mach range exists in which lifting solutions can be found at zero angle of attack.

III.B.2. Non-unique Solutions for JF1 Airfoil at M.835

The CL versus α curve at M.835 is plotted in Figure 3. The corresponding Cp distributions and pressure contours at zero angle of attack are shown in Figure 4(a) for the symmetrical Z-branch solution, and in Figure 4(b) for the asymmetrical P-branch solution.

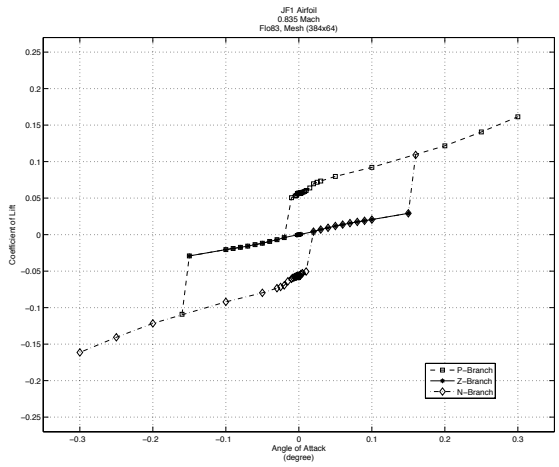
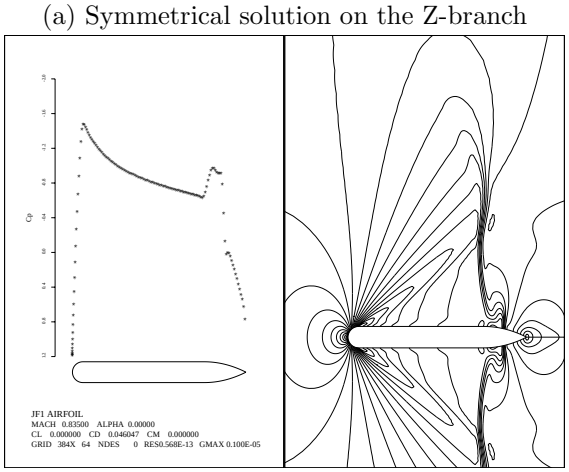


Figure 3. Plots of CL-α sweep for flows over the JF1 airfoil at M.840, showing the presence of the positive, negative, and zero branches.

(a) Symmetrical solution on the Z-branch



(b) Asymmetrical solution on the P-branch

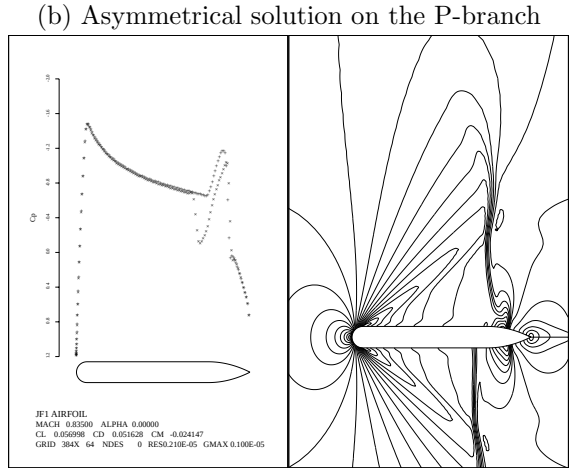


Figure 4. Pressure contours showing non-unique solutions of NU4 airfoil at $\alpha = 0°$, M=.840

III.C. JB1 Airfoil

III.C.1. JB1 Airfoil Geometry

The JB1 airfoil also has a parallel center section but the nose and tail are closed by higher order curves which maintain continuity of the curvature at the junction points. The nose section is defined by a Bezier curve with the control points

$$x_1 = 0, y_1 = 0$$

$$x_2 = 0, y_2 = 1$$
$$x_3 = 1, y_3 = 1$$
$$x_4 = 1, y_4 = 1$$

scaled to a length of .125 and a height of .0625. The upper surface curve is defined by

$$\begin{array}{c} x = .125(3t^2 - 2t^3) \\ y = .0625(3t - 3t^2 + t^3) \end{array}, \quad 0 \le t \le 1$$

The trailing curve from $x = .625$ to 1 is

$$y = .0625\left[1 - \left(1 - \left(\frac{1-x}{.375}\right)\right)^3\right]$$

III.C.2. Non-unique Solutions for JB1 Airfoil at M.827

The CL versus α curve at M.827 is plotted in Figure 5. The corresponding Cp distributions and pressure contours at zero angle of attack are shown in Figure 6(a) for the symmetrical Z-branch solution, and in Figure 6(b) for the asymmetrical P-branch solution.

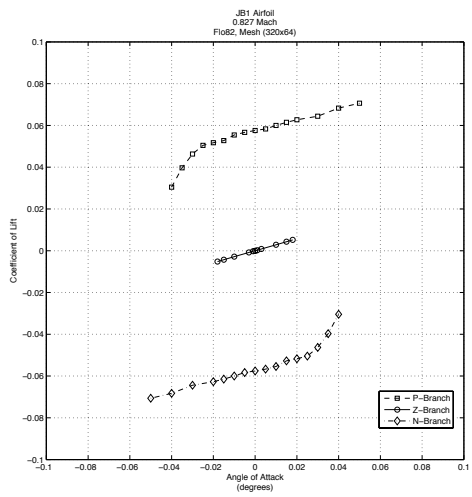
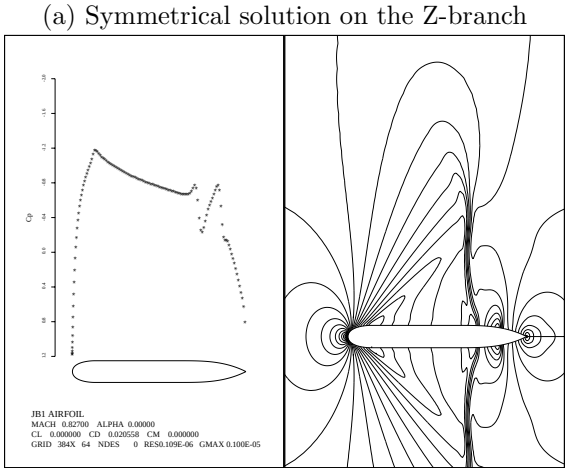


Figure 5. Plots of CL-α sweep for flows over the JF1 airfoil at M.827, showing the presence of the positive, negative, and zero branches.

(a) Symmetrical solution on the Z-branch



(b) Asymmetrical solution on the P-branch

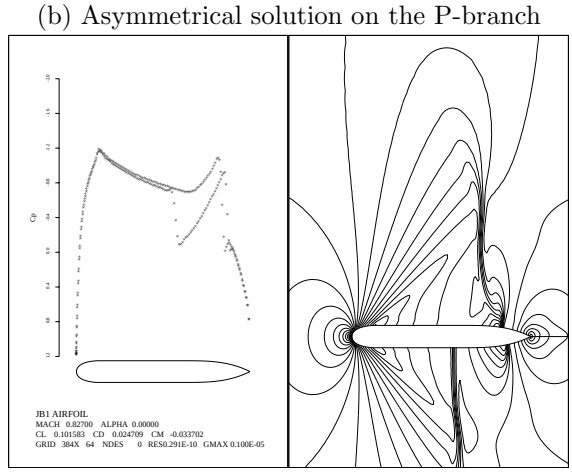


Figure 6. Pressure contours showing non-unique solutions of JB1 airfoil at $\alpha = 0^\circ$, M=.827

III.D. JC6 Airfoil

III.D.1. JC6 Airfoil Geometry

The JC6 airfoil is a fully convex airfoil defined by a simple algebraic formula

$$y(x) = Cx^{\frac{1}{n}}(1 - x^n), 0 \leq x \leq 1$$

where the constant C, with a value of 0.06817, is adjusted to give the specified maximum thickness, 12 percent of the chord. The choice n=6 results in a very blunt-nosed airfoil with maximum thickness at about 55 percent of the chord, which has positive curvature everywhere and is C_∞ continuous.

III.D.2. Non-unique Solutions of JC6 Airfoil at M.847

The CL versus α curve at M.847 is plotted in Figure 7. The corresponding Cp distributions and pressure contours at zero angle of attack are shown in Figure 8(a) for the symmetrical Z-branch solution, and in Figure 8(b) for the asymmetrical P-branch solution. In Figure 7, results of OVERFLOW[16] simulations for the same flow conditions are also plotted and compared with the FLO82 results. Very similar results were obtained using both methods.

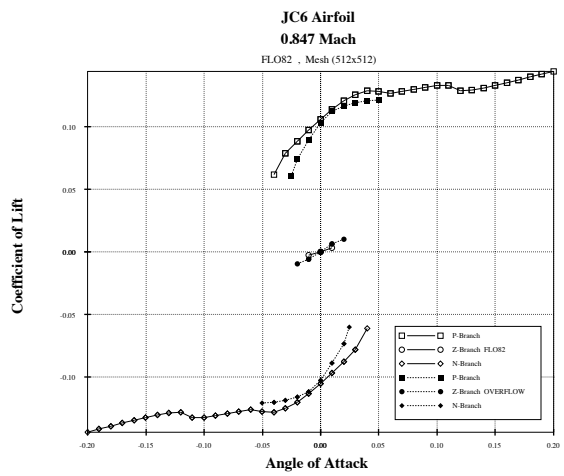
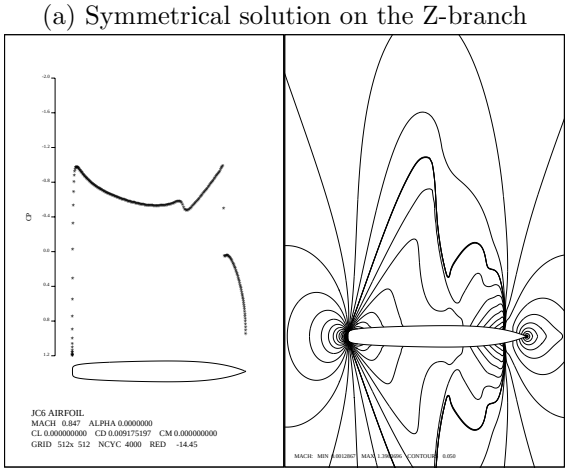


Figure 7. Plots of CL-α sweep for flows over the JC6 airfoil at M.847, showing the presence of the positive, negative, and zero branches.

(a) Symmetrical solution on the Z-branch



(b) Asymmetrical solution on the P-branch

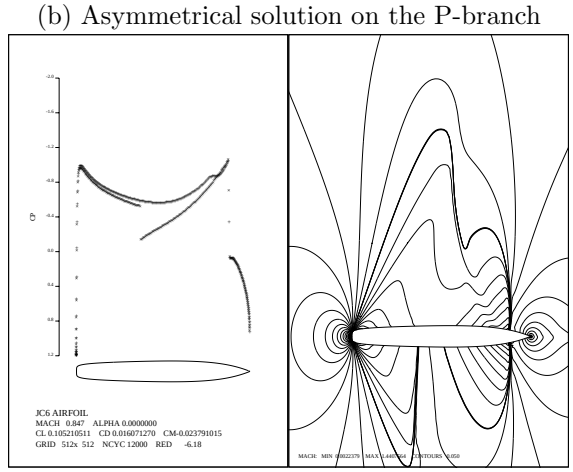
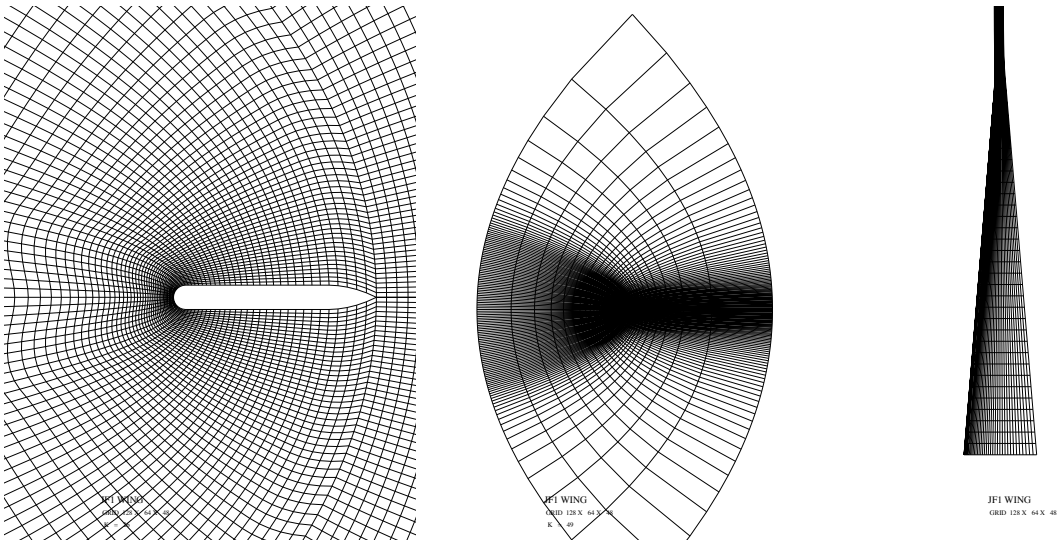


Figure 8. Pressure contours showing non-unique solutions of JC6 airfoil at $\alpha = 0^\circ$, M=.847

IV. Three-Dimensional Results

A set of four wings were designed based on the above airfoils that were found to exhibit non-unique transonic solutions. Flow simulations based on Euler equations (FLO88)[13] were performed on meshes that contain 128 cells in the clockwise direction, 64 cells in the normal direction, and 48 cells in the spanwise direction. An example of the mesh used for JF1 wing is shown in Figure 9. The aspect ratios of the wings are varied as a way to control the extend of the three-dimensional effect.



(a) Nearfield (b) Farfield (c) Surface

Figure 9. Mesh for JF1 wing

The flow simulations were obtained by first perturbing the wing to a finite angle of attack from zero. Once the flow has sufficiently converged, the perturbation was removed so that the wing attitude returned to zero degree, and the flow simulation was reconverged starting from the previously converged state. A non-unique solution will be obtained if the reconverged flow is asymmetrical. The other solution is the symmetrical zero lift solution, as is expected for flow over a symmetrical wing at zero angle of attack.

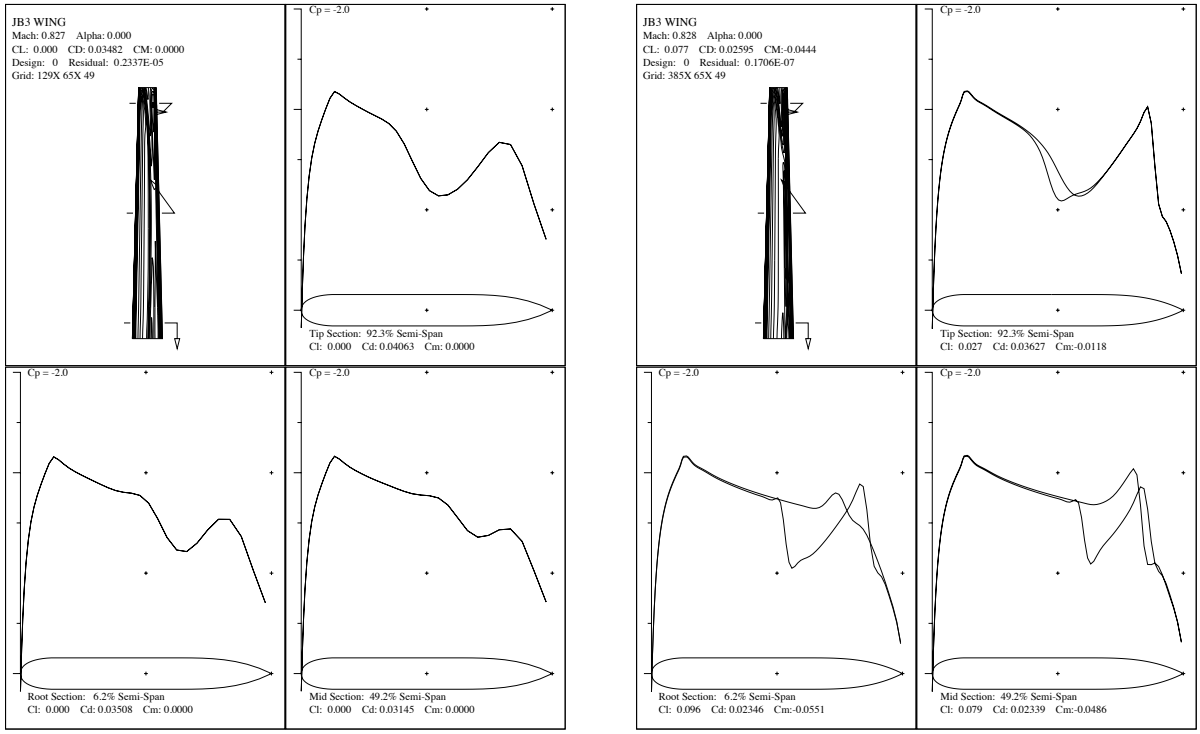
For wings with very high aspect ratio, the three-dimensional effect is small, and one expects the non-unique behaviors of the airfoils to carry over to the case of the wings. If non-unique solution can be replicated for high aspect ratio wings, the aspect ratio can be gradually decreased to investigate the non-unique flow characteristics. Because the three-dimensional effect can affect the conditions for certain flow characteristics, the Mach numbers at which the airfoils exhibit non-unique solutions can be different for wings of different aspect ratios. As a result, as the aspect ratio changes, the Mach number might need to be varied to check the existence of non-unique solutions. This is the methodology adopted for the following study.

IV.A. Wings based on JB1 airfoil

A slender wing based on JB1 airfoil was constructed. The wing has a high aspect ratio of 16 and an aspect ratio of 0.6. For the two-dimensional JB1 airfoil, non-unique solutions were found at M=.827. When the three-dimensional simulations were performed for the high aspect ratio wing in the neighborhood this Mach number, non-unique solutions were again found to exist at M=.827. For the same flow conditions, the wing assumes a lifting solution and a zero-lift solution. The exact non-unique solution depends on the initial condition of the flow simulation. The Cp distributions showing the symmetrical and asymmetrical flow solutions at zero angle of attack are plotted in Figure 10.

The CL versus α sweep of the simulation solutions shows the presence of the P-branch and N-branch, corresponding to the positive lift and negative lift solutions. The P- and N-branches can be extended past zero degree of angle of attack, with a narrow overlapping region. The Z-branch cannot be obtained other than the single point corresponding to the zero lift solution at zero angle of attack. The result is plotted in Figure 11.

However, as the aspect ratio of the wing is further decreased, the non-unique solutions cannot be found for the wing, even after different Mach numbers were tried in the neighborhood of the original condition. For



(a) Z-branch solution, M=.827, $\alpha = 0°$ (b) P-branch solution, M=.827, $\alpha = 0°$

Figure 10. Euler solutions for wing based on JB1 airfoil at M.827, and $\alpha = 0°$. Wing aspect ration=16. Taper ratio=0.6.

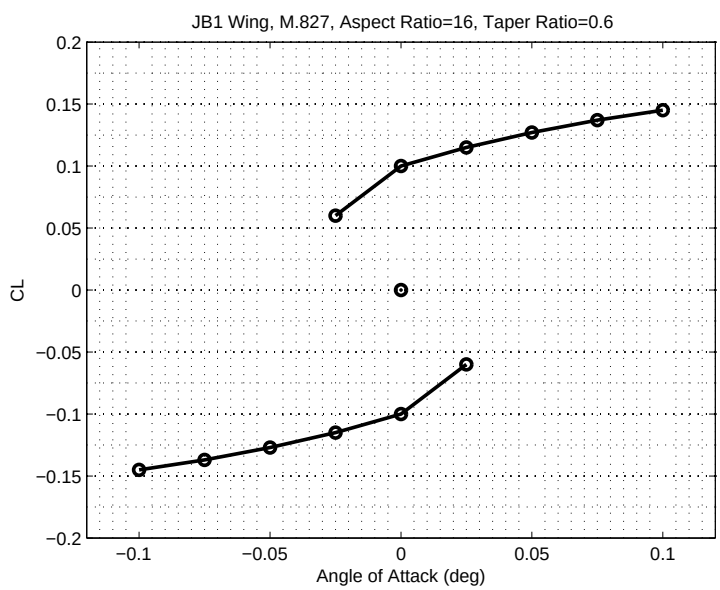


Figure 11. C_L versus α sweep for wing based on JB1 airfoil at M.827.

the wing based on JB1 airfoil, non-unique solutions are only supported when the aspect ratio is sufficiently large.

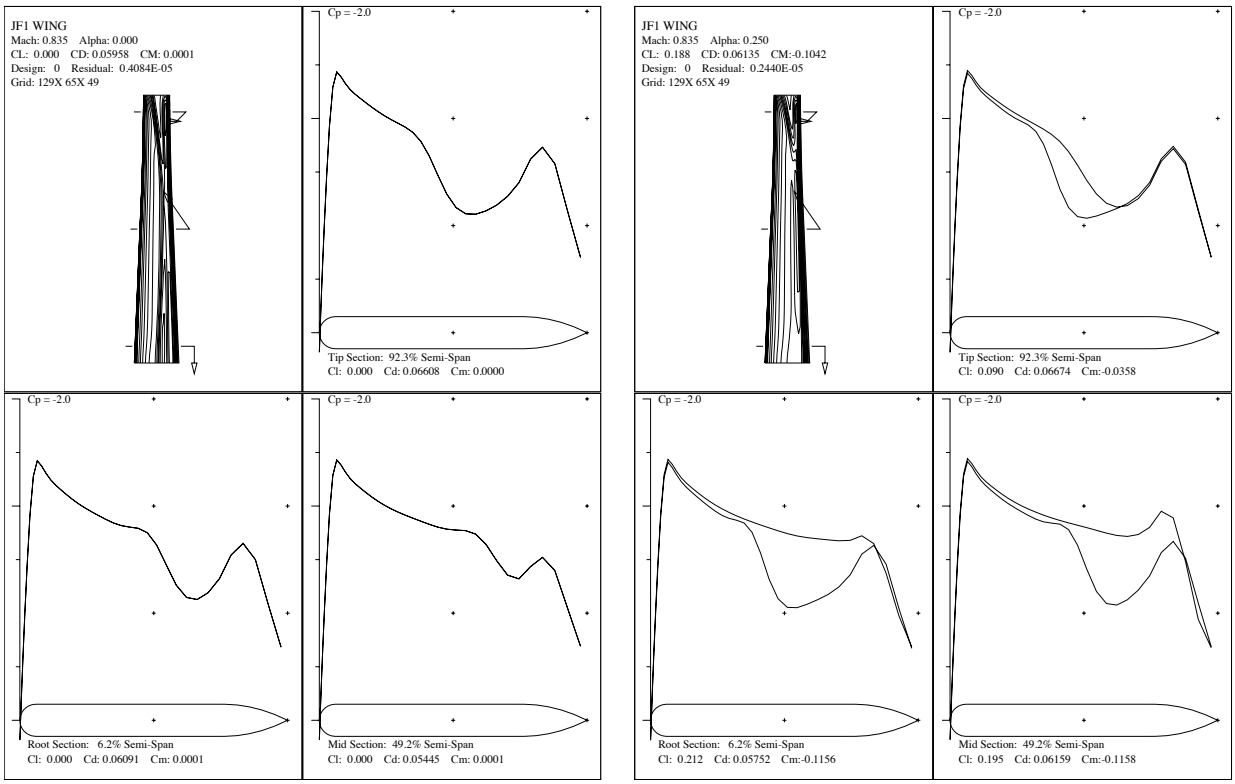
IV.B. Wings based on NU4 airfoil and JC6 Airfoil

When similar analyses were carried out for wings based on the NU4 and JC6 airfoils, non-unique solutions cannot be found even when very high aspect ratio wings were used. The results indicate that the non-unique

behaviors over these airfoil shapes are very sensitive to the influence of three-dimensional effects.

IV.C. Wings based on JF1 airfoil

Wings based on JF1 airfoil, on the other hand, were found to support non-unique solutions over a much wider range of aspect ratios. Non-unique solutions have been found for aspect ratios of 16, 14, and 12 at M=.835. The corresponding Cp distributions for the wing with aspect ratio of 12 are plotted in Figure 12.



(a) Z-branch solution, M=.835, $\alpha = 0°$ (b) P-branch solution, M=.835, $\alpha = 0°$

Figure 12. Euler solutions for wing based on JF1 airfoil at M.835, and $\alpha = 0°$. Wing aspect ration=12. Taper ratio=0.6.

It is also found that, for the same aspect ratio, changing the taper ratio does not significantly change the non-unique behavior of the flow, as is shown in Figure 13. In this case, the aspect ratio is increased from 0.6 to 1.

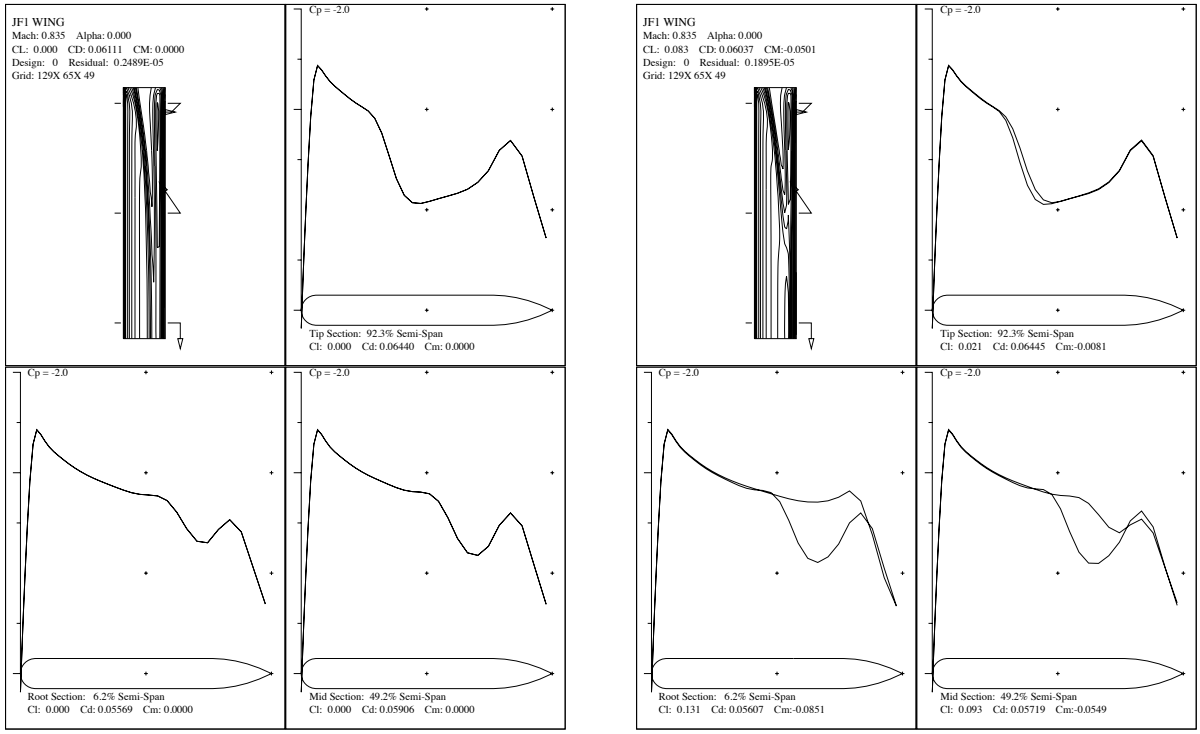
The CL versus α sweep of the simulation solutions also shows the presence of the P-branch and N-branch. The P- and N-branches can extend past zero degree of angle of attack resulting in a narrow overlapping region. The Z-branch cannot be obtained other than the single point corresponding to the zero lift solution at zero angle of attack. The result is plotted in Figure 14.

The non-unique solution disappeared when the aspect ratio is further reduced to 10. As the three-dimensional effect becomes more significant when the aspect ratio decreases, the flow conditions that support non-unique flows also change. When the simulations were performed at M=.840, the non-unique solutions were rediscovered for the wing with an aspect ratio of 10. The flow solution is plotted in Figure 15. The corresponding plot of CL versus α sweep is shown in Figure 16.

When the aspect ratio is further decreased, the non-unique solutions disappeared. Varying the Mach number did not yield further non-unique solutions.

V. Conclusion

In this work, two-dimensional airfoils that were previously found to exhibit non-unique transonic solutions were extended to three-dimensional wings to investigate the influence of the three-dimensional effect on non-unique flow behavior. It is found that certain airfoil shapes are more sensitive to the three-dimensional effect



(a) Z-branch solution, M=.835, $\alpha = 0°$ (b) P-branch solution, M=.835, $\alpha = 0°$

Figure 13. Euler solutions for wing based on JF1 airfoil at M.835, and $\alpha = 0°$. Wing aspect ration=12. Taper ratio=1.

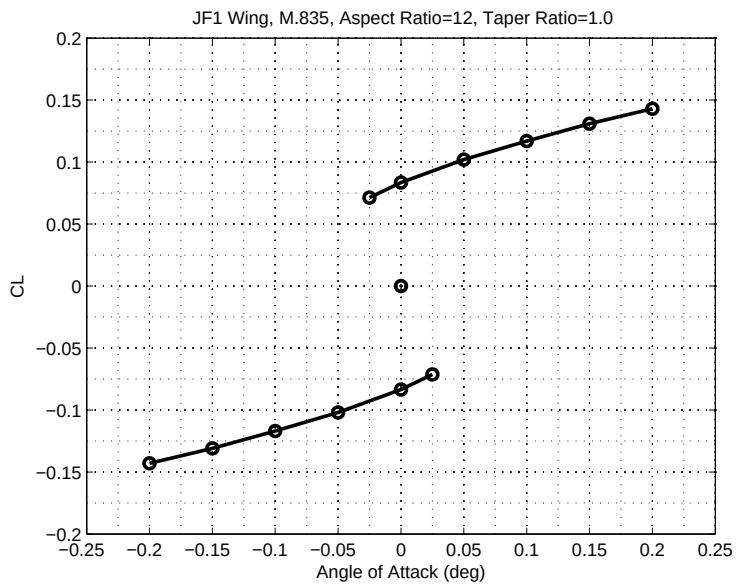
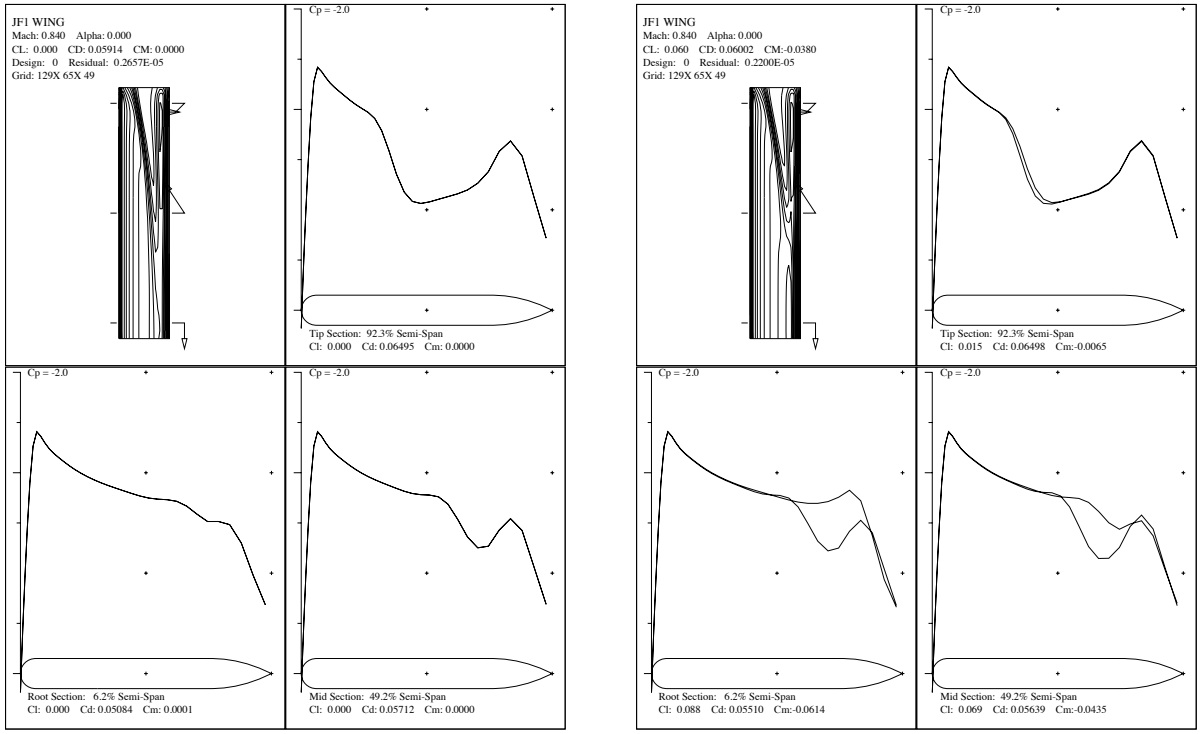


Figure 14. C_L versus α sweep for wing based on JF1 airfoil at M.835.

than other airfoils, with the following cases observed. For wings based on NU4 and JC6 airfoils, non-unique transonic solutions can not be found even when the wings have very high aspect ratio. For wings based on JB1 airfoil, non-unique transonic solutions were found, but only when the wings have sufficiently high aspect ratios. For wings based on JF1 airfoil, non-unique transonic solutions were supported for a wider range of aspect ratios. The lowest aspect ratio that was achieved so far that still supports the non-unique solutions is around 10. As another effect, the three-dimensional flows also change the exact condition for which the



(a) Z-branch solution, M=.840, $\alpha = 0°$ (b) P-branch solution, M=.840, $\alpha = 0°$

Figure 15. Euler solutions for wing based on JF1 airfoil at M.840, and $\alpha = 0°$. Wing aspect ration=10. Taper ratio=1.

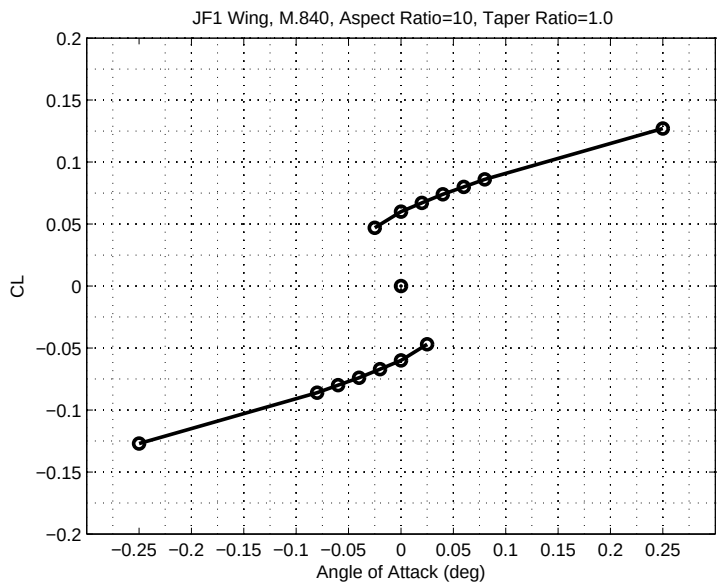


Figure 16. C_L versus α sweep for wing based on JF1 airfoil at M.840.

non-unique solutions are admitted when the aspect ratio decreases.

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