A note on terminology in multigrid methods

Philipp Birken[1,*], **Jonathan Bull**[2], and **Antony Jameson**[3]

[1] Lund University, Centre for the Mathematical Sciences, Numerical Analysis, Box 118, 22100 Lund
[2] Uppsala University, Division of Scientific Computing, Dept. of Information Technology, Box 337, 75105 Uppsala
[3] Stanford University, Department of Aeronautics & Astronautics, Stanford, CA 94305

We compare terminology used in the literature on multigrid methods for compressible computational fluid dynamics to that used in linear multigrid theory. Several popular iterative and direct smoothers are presented side-by-side using the same terminology. We argue for greater analysis of these methods in order to place them into a more rigorous framework and to identify the most promising candidates for future development.

1 Multigrid Methods

Multigrid methods were developed in the 60s and 70s for the solution of linear elliptic PDEs. Since then, textbook multigrid efficiency has been both obtained and proven for large classes of linear elliptic PDEs [1,2]. In the middle of the 80s, multigrid methods were applied to compressible fluid dynamics problems with text book multigrid efficiency being demonstrated for the steady Euler equations [3]. The development of the former was driven mainly by mathematicians, whereas the latter, being mathematically much harder, was mainly driven in the application oriented engineering community. This has led to a difference in language between the two communities. Our aim in this article is to translate the concepts from the different communities to each other and show some relations that, although simple, are not widely known.

1.1 Linear Multigrid

Consider the linear equation system

$$Ax = b, \quad A \in R^{n\times n},\, x, b \in R^n. \tag{1}$$

arising from the discretization of a PDE on a grid. A multigrid method for this problem is a linear iterative method, obtained from the composition of two other linear iterative methods, a smoother and a coarse grid correction.

1.2 Nonlinear Multigrid

The class of methods most used in compressible CFD are Full Approximation Schemes (FAS), which are multigrid methods for the solution of nonlinear equation systems

$$F(u) = 0, \quad u \in R^n, F: R^n \to R^n. \tag{2}$$

Again, these are the composition of two iterations, but both of these will be nonlinear. In the mathematical literature, these will still be called a smoother and a coarse grid correction. In the engineering nomenclature, there will be a base scheme, typically an explicit time integration scheme on the finest level only and the addition of a multigrid scheme to this is called *convergence acceleration*.

2 Specific smoothers in both communities

An important class of iterative solvers in CFD are explicit time iteration schemes. To this end, a time derivative is added to (2) and a cheap, explicit time integration method is employed to advance a solution in time. In case this asymptotically approaches a steady state, the time derivative will be small, meaning that we in this way recover the solution. For the explicit Euler method, this would look like

$$u^{k+1} = u^k + \Delta t^* F(u^k), \tag{3}$$

where Δt^* is a time step size. We denote this with a star, because it is actually a pseudo time and only there for the purpose of an iteration. In the engineering community, this will be called the physical time, if (2) describes a steady state equation and a dual time, if (2) results from the implicit time integration of an unsteady process.

* Corresponding author: e-mail philipp.birken@na.lu.se, phone +46 462 223 165

Table 1: Smoother terminology in CFD and numerical linear algebra

CFD	Linear formula	Num. linear Algebra
Exp. E	$x^{k+1} = x^k + \Delta t^* r^k$	Richardson
ERK	$x^{k+1} = P_s(-\Delta t^* A)x^k + S_{ERK}(\Delta t^* A)b$	Polynomial sm.
ARK	$x^{k+1} = P_s(\Delta t^*, -A_{conv}, -A_{visc})x^k + S_{ARK}(\Delta t^*, A_{conv}, A_{visc})b$	Not analyzed
IRK	$x^{k+1} = R_s(-\Delta t^* A)x^k + S_{IRK}(-\Delta t^* A)b$	Not analyzed
LU-SGS	$x_{k+1} = (I - (D+U)^{-1}D(L+D)^{-1}A)x^k + (D+U)^{-1}D(L+D)^{-1}b$	SGS
GMRES-s	$x^{k+1} = P_s(A, r^0)x^k + S_{GMRES}(A, r^0)b$	GMRES-s
IPS	Small blocks from PDE system	Block-Jacobi
ILS	Order unknowns by lines	Block-Jacobi

When looking at the linear equation (1), the same approach can be used when we consider the equation $b - Ax = 0$ instead. We then see that (3) is nothing but the Richardson iteration

$$x^{k+1} = x^k + \omega(b - Ax^k)$$

with the damping parameter ω taking the role of the pseudo time.

Extending this to general explicit Runge-Kutta (ERK) methods, it can be seen that these correspond to polynomial smoothers in the linear case. This allows to relate it to using s steps of a nonlinear iterative method like GMRES as a smoother. There, because a Krylov subspace is used, the iteration matrix will again be a polynomial of degree s in A, but it will depend on the initial residual as well [4].

However, the methods used in practice are additive Runge-Kutta (ARK) methods. These use different coefficients for different parts of the PDE, meaning the convective or the diffusive part. Thus, for a linear problem these correspond to multivariate polynomials. There is no analysis of their properties in the numerical linear algebra community. In the case of implicit Runge-Kutta (IRK) smoothers the polynomial becomes a rational expression; there has been little or no analysis of this class as yet.

A second important class are symmetric Gauß-Seidel methods (SGS). In the linear case, these are given by

$$x^{k+1} = (I - (D+U)^{-1}D(L+D)^{-1}A)x^k + (D+U)^{-1}D(L+D)^{-1}b, \tag{4}$$

where we decompose the matrix A into a strictly lower triangular, strictly upper triangular and its diagonal part: $A = L + D + U$.

An extension to the nonlinear case would be a so called nonlinear symmetric Gauß-Seidel process. This method uses a forward and a backward sweep over the nonlinear equations. Each single one is solved in some way, e.g. using a local Newton iteration. However, the method in wide use is to linearize (2) around the current iterate and then use one step of (4) to solve the linearized equation system. A particular implementation of this is known in CFD as the LU-SGS method.

The final class in wide use are so called point or line smoothers. These group unknowns together, either by point or by a line of l points, and then solve the nonlinear equation systems with respect to these unknowns simultaneously. Thus, in the numerical linear algebra world, this is the block Jacobi method for specific choices of blocks. When solving d-dimensional systems of equations the blocks have dimension d in the point case and $d \times l$ in the line case. The block systems are usually solved directly via LU decomposition or SGS.

Aside from the basic iterative and direct smoothing techniques, convergence may be further accelerated by the use of preconditioning, for example local timestepping, which is a special variant of Jacobi preconditioning. In the case of systems of equations, more complex techniques are employed, see for example [5].

We summarize these thoughts in Table 1. We hope that this reference will stimulate numerical analysis of commonly-used smoothers to put them on a firmer footing.

References

[1] W. Hackbusch, Multi-Grid Methods and Applications (Springer, 1985)

[2] U. Trottenberg, C. W. Oosterlee, and S. Schüller, Multigrid (Elsevier Academic Press, 2001)

[3] A. Jameson, 33rd AIAA Applied Aerodynamics Conference Dallas, TX, 2015.

[4] P. Birken, J. Bull, and A. Jameson, ECCOMAS Congress 2016, VII European Congress on Computational Methods in Applied Sciences and Engineering, accepted

[5] S. Langer, J. Comp. Phys. **277** (2014).