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# ACCELERATED ITERATION SCHEMES FOR TRANSONIC FLOW CALCULATIONS USING FAST POISSON SOLVERS

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## Abstract

Reliable but slow methods for calculating transonic flows have been developed in recent years. These use central difference formulas in the subsonic (elliptic) zone, and upwind difference formulas in the supersonic (hyperbolic) zone. This report describes an improved iterative method for solving the resulting difference equations. Each iteration consists of two stages: in the first stage a direct method is used to solve Poisson's equation with the nonlinear terms treated as a forcing function; in the second stage the relaxation method is used to sweep out the errors in the supersonic zone. The combined method gives fast convergence, whereas the Poisson method by itself diverges when there is a region of supersonic flow, and relaxation by itself is very slow.

## 1. Introduction.

This report describes an improved iterative method for solving nonlinear partial differential equations of mixed type which appear in the calculation of transonic flows. In particular the equations to be considered are the transonic potential flow equation and the transonic small disturbance equation.

The potential flow equation can be derived from the Euler equations for compressible fluid flow by introducing the assumption that the flow is irrotational, so that we can define a potential $\phi$. Then we find that in smooth regions of flow $\phi$ satisfies the quasilinear equation

$$\phi_{xx} + \phi_{yy} = \frac{1}{a^2}\left(u^2\phi_{xx} + 2uv\phi_{xy} + v^2\phi_{yy}\right) \tag{1.1}$$

In this equation u and v are the velocity components

$$u = \phi_x , \quad v = \phi_y \tag{1.2}$$

and $a$ is the local speed of sound. Given the ratio of specific heats $\gamma$, and the stagnation speed of sound $a_0$, $a$ can be determined from the energy equation

$$a^2 = a_0^2 - \frac{\gamma-1}{2} q^2 \tag{1.3}$$

in which $q$ is the speed

$$q = (u^2 + v^2)^{1/2}$$

Equation (1.1) is elliptic when the local Mach number $q/a < 1$, and hyperbolic when $M > 1$. The boundary conditions are that the normal velocity vanishes at the body,

$$\frac{\partial \phi}{\partial n} = 0 \quad \text{on a given curve} \tag{1.4}$$

and that the flow is uniform with a prescribed speed at infinity.

Smooth transonic solutions are known to exist only in special cases [1]. We must therefore admit weak solutions with appropriate discontinuities [2]. Since an irrotational flow is isentropic the discontinuities are not true shock waves but isentropic jumps. If the normal component of mass flow and the tangential component of velocity are conserved across the discontinuities, and we also exclude discontinuous expansions, then they will be fairly good approximations to shock waves of moderate strength. The momentum deficiency across the discontinuities then provides an approximation to the wave drag [3].

We can ensure satisfaction of these jump conditions by treating the equation in the conservation form

$$\frac{\partial}{\partial x} (\rho\phi_x) + \frac{\partial}{\partial y} (\rho\phi_y) = 0 \tag{1.5}$$

where $\rho$ is the density. If $M_\infty$ is the Mach number of the free stream at infinity, then $\rho$ can be determined from the local speed of sound by the relation

$$\rho^{\gamma-1} = M_\infty^2 a^2 \tag{1.6}$$

The form (1.5) also corresponds to the Bateman variational principle that

$$I = - \iint p \, dx \, dy$$

is stationary, where $p$ is the pressure

$$p = \frac{\rho^\gamma}{\gamma M_\infty^2}$$

If the profile is given in the form

$$y = \tau f(x)$$

where $\tau$ is a sufficiently small parameter, we can expect the disturbances to be small. If we expand the solution in powers of $\tau$ under the assumption that $1 - M_\infty^2 \sim \tau^{2/3}$, and retain only the leading term, we can then obtain the transonic small disturbance equation [4]. A typical form is

$$\phi_{xx} + \phi_{yy} = A \phi_{xx} \tag{1.7}$$

where $A$ is an approximation to the square of the local Mach number given by

$$A = M_\infty^2 (1 + (\gamma+1)\phi_x) \tag{1.8}$$

In this equation $\phi$ is the disturbance potential, so that the velocity components are $1+\phi_x$ and $\phi_y$. The boundary condition at the body is transferred to the x axis, becoming

$$\phi_y = \frac{df}{dx} \quad \text{at} \quad y = 0 \ . \tag{1.9}$$

Thus we have the double simplification that the upwind direction is known to be the x direction, and that the Neumann boundary condition no longer has to be satisfied on a curved profile.

In order to ensure satisfaction of the proper jump conditions we can also write the small disturbance equation in conservation form as

$$\frac{\partial}{\partial x}\{(1-M_\infty^2)\phi_x - \frac{\gamma+1}{2} M_\infty^2\phi_x^2\} + \phi_{yy} = 0 \tag{1.10}$$

Reliable but slow methods have been developed in the last few years for solving both the transonic small disturbance equation and the transonic potential flow equation. These methods are based on the idea, first introduced by Murman and Cole [5], of using central difference formulas in the subsonic zone and upwind difference formulas in the supersonic zone. As an illustration consider the small disturbance equation in conservation form (1.10). Let $p_{ij}$ be a central difference approximation to the terms containing the $x$ derivatives:

$$\begin{aligned} p_{ij} &= \frac{1-M_\infty^2}{\Delta x^2}\{\phi_{i+1,j} - 2\phi_{ij} + \phi_{i-1,j}\} \\ &\quad - \frac{(\gamma+1)M_\infty^2}{2\Delta x^3}\{(\phi_{i+1,j} - \phi_{ij})^2 - (\phi_{ij} - \phi_{i-1,j})^2\} \\ &= \frac{1-A_{ij}}{\Delta x^2}\{\phi_{i+1,j} - 2\phi_{ij} + \phi_{i-1,j}\} \end{aligned}$$

where

$$A_{ij} = M_\infty^2\{1 + \frac{\gamma+1}{2\Delta x}(\phi_{i+1,j} - \phi_{i-1,j})\}$$

Also let $q_{ij}$ be a central difference approximation to $\phi_{yy}$

$$q_{ij} = \frac{1}{\Delta y^2}\left\{\phi_{i,j+1} - 2\phi_{ij} + \phi_{i,j-1}\right\}.$$

Define a switching function $\mu_{ij}$ with the value unity at supersonic points and zero at subsonic points

$$\mu_{ij} = \begin{cases} 0 & \text{if } A_{ij} < 1 \\ 1 & \text{if } A_{ij} \geq 1 \end{cases}$$

Then we approximate (1.10) by

$$(1.11) \qquad p_{ij} + q_{ij} - \mu_{ij} p_{ij} + \mu_{i-1,j}\, p_{i-1,j} = 0$$

This gives the standard central difference approximation for an elliptic equation at subsonic points. At supersonic points $p_{ij}$ is replaced by the upwind formula $p_{i-1,j}$, giving a scheme similar to the usual implicit scheme for the wave equation. Supersonic points immediately after entering the supersonic zone are treated as 'parabolic' points, since $p_{ij}$ is cancelled but $\mu_{i-1,j} = 0$. Subsonic points immediately after leaving the supersonic zone are treated as shock points [6] at which $p_{ij}$ and $p_{i-1,j}$ are summed.

The added terms $\mu_{ij} p_{ij} - \mu_{i-1,j} p_{i-1,j}$ are an approximation to $\partial P/\partial x$ where

$$P = \mu\, \Delta x \frac{\partial}{\partial x} \{(1-M_\infty^2)\phi_x - \frac{\gamma+1}{2} M_\infty^2 \phi_x^2\} .$$

Thus the conserved quantity is modified by a term of order $\Delta x$. The scheme satisfies Lax's conditions for a difference scheme in conservation form [2], and in the limit as $\Delta x \to 0$ it can be shown that the correct conservation law is satisfied across discontinuities [6]. We can regard $\partial P/\partial x$ as an artificial viscosity introduced by the numerical scheme to give the correct directional property that the region of dependence is upwind, and to enforce satisfaction of a proper entropy condition by excluding the appearance of expansion shocks.

The usual method for solving the nonlinear difference equation (1.11) is a generalization of the relaxation method for elliptic equations. At each point we first calculate $A_{ij}$ using values of $\phi_{ij}$ from the previous iteration, and hence determine the appropriate form of the difference equations. Then with the coefficients $A_{ij}$ frozen we solve a set of linear equations for the correction to the potential on each successive vertical line, marching downstream. In forming these equations we add in the correction at adjacent upstream lines to make use of the latest available information, and at elliptic points we use an over-relaxation factor to increase the magnitude of the correction.

Let $R_{ij}$ be the residual multiplied by $\Delta x^2$, evaluated at each point using values of $\phi_{ij}$ from the last cycle,

$$R_{ij} = \Delta x^2 \ (p_{ij} + q_{ij} - \mu_{ij} p_{ij} + \mu_{i-1,j}\ p_{i-1,j})$$

Then the equations to be solved for the correction $C_{ij}$ which must be added to $\phi_{ij}$ are

$$\left(\frac{\Delta x}{\Delta y}\right)^2 (C_{i,j+1} - 2C_{ij} + C_{i,j-1}) + (1-\mu_{ij}) A_{ij} \left(-\frac{2}{\omega} C_{ij} + C_{i-1,j}\right)$$

(1.12)

$$+ \mu_{i-1,j}\ A_{i-1,j}\ (C_{ij} - 2C_{i-1,j} + C_{i-2,j}) = -R_{ij}$$

where $\omega$ is the overrelaxation factor for subsonic points.

Each cycle is equivalent, after the calculation of the coefficients $A_{ij}$, to a line relaxation scheme in the subsonic zone, and a marching scheme in the supersonic zone. Since the supersonic difference scheme is implicit there is no limit on the step length $\Delta x$. With an explicit scheme $\Delta x$ would have to be reduced at points near the sonic line where $A_{ij}$ approaches zero, in order to satisfy the Courant Friedrichs Lewy stability condition. On every line we have to solve a tridiagonal set of equations. The tridiagonal matrix is diagonally dominant and can be safely factorized as the product of upper and lower bidiagonal matrices. Thus the total number of operations in a cycle is directly proportional to the number of points.

Generalizations of this approach have been used for the transonic potential flow equation [7,8,9]. In this case the upwind direction is no longer necessarily aligned with the x coordinate, and it may be necessary to use a 'rotated' upwind difference scheme in the supersonic zone [9]. The use of one-sided differencing corresponds to an essential property of transonic flow, that the solution for a profile with fore and aft symmetry is not symmetric. Instead there is a smooth acceleration over the front half of the body followed by a discontinuous recompression through a shock wave. If a completely symmetric numerical scheme were used, any solution that could be computed for such a profile would be symmetric, and must therefore exhibit improper discontinuities in the flow.

The effectiveness of these methods is by now well established, at least for two dimensional and axisymmetric calculations [10, 11,12]. It is a common practice for aircraft to cruise at a high subsonic speed just below the point at which the appearance of strong shock waves causes the onset of drag rise. In this regime the approximation of potential flow does not lead to serious error. In fact, if a correction is made for the displacement effect of the viscous boundary layer, the agreement with experimental data is often excellent [12,13]. Their main shortcoming is the slow rate of convergence of the iterative method, requiring hundreds or even thousands of cycles. As a result it is a common practice to terminate calculations at a point where it is not clear that the result has fully converged. A faster rate of convergence would alleviate this difficulty. It is also extremely desirable if calculations are to be attempted for more complex geometric configurations.

## 2. Construction of a Fast Iterative Scheme for the Small Disturbance Equation

Denoting by L the finite difference operator which approximates the partial differential operator, let the discrete form of (1.7) be written as

$$L\phi = 0 \tag{2.1}$$

where $\phi$ is now regarded as a vector. To set up an iterative procedure let $\phi^{(k)}$ denote the potential before the start of the $k^{th}$ iteration. Then consider a scheme of the form

$$N\phi^{(k+1)} = (N-L)\phi^{(k)} \tag{2.2}$$

where N is a finite difference operator which should be as close to L as possible. If we denote the correction by

$$C = \phi^{(k+1)} - \phi^{(k)}$$

and the residual by

$$R = L\phi^{(k)}$$

it is useful to allow generalizations of the form

$$N\ C = -\ \omega\ R \tag{2.3}$$

where $\omega$ is a relaxation factor.

As a guide to the convergence of such a scheme we can consider the linearized equation which is obtained by freezing the nonlinear coefficients $A_{ij}$. Let $\hat{L}$ be the corresponding linearized operator and let $\phi$ be the solution of

$$\hat{L}\,\phi = 0$$

Also let $e^{(k)}$ be the error at the $k^{th}$ step

$$e^{(k)} = \phi^{(k)} - \phi$$

Then

$$N(e^{(k+1)} - e^{(k)}) = -\,\omega\,\hat{L}\,e^{(k)}$$

or

$$e^{(k)} = M\,e^{(k)}$$

where M is the iteration operator

$$M = N^{-1}(N - \omega\,\hat{L}) \tag{2.4}$$

A necessary condition for convergence is

$$\lim_{k\to\infty} \|M^k\| = 0\ .$$

If this condition is satisfied, but for some k, $\|M^k\| >> 1$, then the round off error introduced k steps previously will be amplified. Thus it is safer to use a scheme for which $\|M\| < 1$.

The relaxation method is derived by first linearizing L at each iteration, and then choosing N as the lower triangular part of the corresponding matrix operator $\hat{L}^{(k)}$. By choosing for N a closer approximation to $\hat{L}^{(k)}$ we can expect to obtain a faster rate of convergence. We are constrained, however, by the need to find an economical method of performing the computations for each iteration.

In recent years fast direct methods have been developed for solving finite difference approximations to Poisson's equation on a rectangle

$$Q\,\phi = R \tag{2.5}$$

Here the matrix $Q$ representing the discrete Laplacian operator has the block structure

$$Q = \begin{bmatrix} T & -I & & & \\ -I & T & -I & & \\ & -I & T & -I & \\ & & -I & T & -I \\ & & & -I & T \end{bmatrix} \tag{2.6}$$

where $I$ is the identity matrix, and for equal mesh spacings in the $x$ and $y$ directions $T$ has the form

$$T = \begin{bmatrix} 4 & -1 & & & \\ -1 & 4 & -1 & & \\ & -1 & 4 & -1 & \\ & & -1 & 4 & -1 \\ & & & -1 & 4 \end{bmatrix} \tag{2.7}$$

By taking advantage both of the sparse structure of $Q$, and of its regularity, it is possible to solve the discrete Poisson equation on a square with $n$ cells on each side with a number of operations proportional to $n^2 \log n$. This operation count is obtained by matrix decomposition methods using the fast Fourier transform [14,15] and also by the method of odd-even reduction, a numerically stable variant of which was developed by Buneman [16,17].

The form (1.7) of the small disturbance equation suggests that the discrete Laplacian $Q\phi$ is a fair approximation to $L\phi$, at least for small Mach numbers. Considering the favorable operation count for solving Poisson's equation, a natural choice for the iterative scheme is to take $N = Q$. A scheme of this type was proposed by Martin and Lomax [18]. A similar procedure has also been used by Periaux for subsonic flow calculations using the finite element method [19]. In order to estimate the rate of convergence that might be expected consider the Prandt Glauert equation, which is obtained by replacing A by $M_\infty^2$ in (1.7). Then if H and V are non-negative finite difference operators representing $-\frac{\partial^2}{\partial x^2}$ and $-\frac{\partial^2}{\partial y^2}$ we have

$$(H + V)\phi^{(k+1)} = M_\infty^2 H \phi^{(k)}$$

or

$$H^{1/2} \phi^{(k+1)} = M_\infty^2 K H^{1/2}\phi^{(k)}$$

where

$$K = H^{1/2}(H+V)^{-1}H^{1/2}$$

Thus

$$\| H^{1/2}\phi^{(k+1)} \| \leq M_\infty^2 \| K \| \| H^{1/2}\phi^{(k)} \|$$

and since K is Hermitian, if we use a Euclidean norm

$$\|K\| = \lambda_{max}(K)$$

$$= \max \frac{(x,Kx)}{(x,\ x)}$$

$$= \max \frac{(y,Hy)}{(y,Hy)+(y,Vy)}$$

where

$$H^{1/2}y = K^{1/2}x$$

Thus $\|K\| \leq 1$ and

$$\|H^{1/2}\phi^{(n+1)}\| \leq M_\infty^2 \|H^{1/2}\phi^{(n)}\| \ .$$

This estimate serves to indicate that for subsonic flows the scheme should converge at a rate independent of the mesh size.

The above analysis also suggests that it is doubtful whether such a scheme would converge for a flow with a substantial supersonic zone. To get an idea of what can be expected consider the case of linearized supersonic flow with

$$A = M_\infty^2 > 1$$

Using the Murman difference scheme (1.11), the residual is then

$$R_{ij} = (1-M_\infty^2)(\phi_{ij} - 2\phi_{i-1,j} + \phi_{i-2,j}) + \left(\frac{\Delta x}{\Delta y}\right)^2 (\phi_{i,j+1} - 2\phi_{ij} + \phi_{i,j-1})$$

and the equation for the correction $C_{ij}$ is

$$(2.8) \quad C_{i+1,j} - 2C_{ij} + C_{i-1,j} + \left(\frac{\Delta x}{\Delta y}\right)^2 (C_{i,j+1} - 2C_{ij} + C_{i,j-1}) = -R_{ij}.$$

Taking the case of periodic boundary conditions for simplicity, suppose that

$$\phi_{ij}^{(k)} = G^k e^{imx} e^{iny} \tag{2.9}$$

where G is the amplification factor, and in the exponentials i denotes $\sqrt{-1}$. Let

$$\xi = m\,\Delta x\,, \qquad \eta = n\,\Delta y$$

Then we find that

$$\left(-4\sin^2\frac{\xi}{2} - 4\left(\frac{\Delta x}{\Delta y}\right)^2 \sin^2\frac{\eta}{2}\right)(G-1)$$

$$= -4\,(M^2-1)\,e^{-i\xi}\sin^2\frac{\xi}{2} - 4\left(\frac{\Delta x}{\Delta y}\right)^2 \sin^2\frac{}{2}$$

whence

$$G = \frac{\{(M_\infty^2-1)\,e^{i\xi}+1\}\sin^2\frac{\xi}{2}}{\sin^2\frac{\xi}{2} + \left(\frac{\Delta x}{\Delta y}\right)^2 \sin^2\frac{\eta}{2}}$$

For a harmonic with a low frequency in y and a moderate frequency in x, $|G|$ exceeds unity, indicating divergence. This conclusion is confirmed by an analysis which includes the proper boundary conditions (see Appendix 1).

It thus appears that the simple Poisson iteration is unpromising for transonic flow calculations. Some kind of stabilization scheme is required if it is to be used as the basis of a fast iterative scheme. One possibility is to apply the Poisson iteration only in the subsonic zone, and to use a marching method in the supersonic zone. This would require the solution of Poisson's equation on an irregular region with a boundary which is altered after each iteration. The capacitance matrix method [20] might be used, but since the capicatance matrix would have to be recalculated at each iteration, the operation count would be very large. It is more attractive, therefore, to retain the rectangular region and try to stabilize the method either by

(1) using some operator other than the Laplacian which can still be solved by a fast method or

(2) introducting a separate stabilization scheme to be applied after each Poisson iteration.

If we use the first approach we may be guided by the fact that the operator L is not symmetric when there is a supersonic zone. It is natural therefore to consider a desymmetrized operator N. We can still use fast direct methods such as odd even reduction or reduction by fast Fourier transform if N has the same block structure as Q (equation (2.6)), with a matrix T repeated in every diagonal block, but with T having a general tridiagonal form. This condition is satisfied by a discrete approximation to the operator

$$\alpha_1 \frac{\partial^2}{\partial x^2} + \frac{\alpha_2}{\Delta x}\frac{\partial}{\partial x} + \frac{\partial^2}{\partial y^2}$$

where $\alpha_1$ and $\alpha_2$ are independent of y. Then equation (2.8) for the correction is replaced by

$$\begin{aligned} &\alpha_1(C_{i+1,j} - 2C_{ij} + C_{i-1,j}) + \alpha_2(C_{ij} - C_{i-1,j}) \\ (2.10) \qquad &+ \left(\frac{\Delta x}{\Delta y}\right)^2 (C_{i,j+1} - 2C_{ij} + C_{i,j-1}) = - R_{ij} \end{aligned}$$

Considering the case of periodic boundary conditions, and making the substitution (2.9), we now find that

$$(-\alpha_1\sigma\bar{\sigma} - \alpha_2\sigma - r)(G-1) = (M_\infty^2-1)\sigma^2 + 4\left(\frac{\Delta x}{\Delta y}\right)^2 \sin^2\frac{\eta}{2}$$

where

$$\sigma = 1-e^{-i\xi}, \quad \mathrm{Re}(\sigma) \geq 0 .$$

Thus

$$G = \frac{\alpha_1\sigma\bar{\sigma} + \alpha_2\sigma - (M_\infty^2-1)\sigma^2}{\alpha_1\sigma\bar{\sigma} + \alpha_2\sigma + 4\left(\frac{\Delta x}{\Delta y}\right)^2 \sin^2\frac{\eta}{2}}$$

Since the real parts of all terms of the denominator are nonnegative, the worst case is when $\eta \to 0$. Then G approaches

$$\frac{\alpha_2+\alpha_1\bar{\sigma} - (M_\infty^2-1)\sigma}{\alpha_2 + \alpha_1\bar{\sigma}}$$

Also

$$|\alpha_2+\alpha_1\bar{\sigma}|^2 - |\alpha_2 + \alpha_1\bar{\sigma} - (M_\infty^2-1)\sigma|^2 = 2(1-\cos\xi)(\alpha_2-\beta-2\alpha_1\cos\xi).$$

It follows that $|G| < 1$ if

$$\alpha_2 > 2\alpha_1 + M_\infty^2 - 1$$

Thus we can expect the iteration to converge for supersonic flow if we add a sufficiently large non-symmetric term. A similar analysis indicates that the desymmetrized iteration would also converge for supersonic flow, but at a lower rate than the symmetric Poisson iteration. Unfortunately it is not possible to vary the coefficients $\alpha_1$ and $\alpha_2$ to suit the local flow conditions without destroying the regular structure required for a fast direct method of solution. This leads to difficulties in treating the full transonic potential equation, for which the direction of desymmetrization should be the local upwind direction, which is no longer fixed.

The second approach of using a separate stabilization scheme is more flexible. For this purpose we can use one or more steps of an alternative iteration scheme after each Poisson iteration. The alternative scheme should be designed to give fast convergence in the supersonic zone to overbalance the divergence of the Poisson iteration. The usual relaxation method described in the introduction is just such a scheme, since a marching procedure is used in the supersonic zone. Thus an exact solution could be obtained in one step if correct values of the nonlinear coefficients and correct data at the sonic line could be inserted.

The following scheme is therefore proposed: use a two stage iteration in which the first stage is a Poisson step with the discrete Laplacian on the left-hand side, and the second stage consists of a fixed number $p$ of relaxation steps to sweep out the errors from the supersonic zone. Considering the linearized equation at any stage, we should expect the scheme to converge if

$$\| M_1 \, M_2^p \| < 1$$

where $M_1$ and $M_2$ are the iteration operators of the two schemes, defined as in equation (2.4). The complexity of the equations is such that it is easier to resort to numerical experiments to test the convergence. These confirm that for the small disturbance equation a fast rate of convergence is obtained by using one relaxation step after each Poisson step.

## 3. Application to the Transonic Potential Flow Equation

The small disturbance approximation is not valid near the stagnation point on a blunt leading edge. To treat blunt nosed profiles with reasonable accuracy it is necessary to bunch the mesh points near the body by using stretched coordinates $\xi = \xi(x)$ and $\eta = \eta(y)$. The linear part of the equation is then no longer so well approximated by the Laplacian in the new coordinates $\xi$ and $\eta$. There is also a difficulty in treating exterior flows, for which the domain is infinite. If we truncate the domain to a finite rectangle to allow the use of a fast Poisson solver we must provide appropriate data at the boundary by some other means. For example, we can use an analytic solution of the Prandtl Glauert equation as an approximation to the proper boundary values [18].

In the analysis of the flow over an airfoil these difficulties can be circumvented by treating the full transonic potential flow equation, and mapping the exterior of the profile onto the interior of a circle by a conformal transformation. Since the Laplacian is invariant under a conformal transformation, we can now use a fast solver for Poisson's equation in polar coordinates $r$ and $\theta$ for the first stage of the iteration. For this purpose a scheme using the Buneman algorithm in the $\theta$ direction has been programmed.

If $2\pi E$ is the circulation it is convenient to use a reduced potential G defined by

$$\phi = G + \frac{\cos\theta}{r} - E\,\theta \tag{3.1}$$

Then G is finite and single valued. The quasilinear form (1.1) of the potential equation becomes

$$\begin{aligned}(a^2-u^2)G_{\theta\theta} - 2uvrG_{\theta r} + (a^2-v^2)r\frac{\partial}{\partial r}(rG_r)\\ - 2uv(G_\theta - E) + (u^2-v^2)rG_r + (u^2+v^2)\left(\frac{u}{r}H_\theta + vH_r\right) = 0\ ,\end{aligned} \tag{3.2}$$

where H is the modulus of the transformation onto the exterior of the circle, and u and v are the velocity components in the $\theta$ and r directions

$$u = \frac{r(G_\theta - E) - \sin\theta}{H}\ , \qquad v = \frac{r^2G_r - \cos\theta}{H} \tag{3.3}$$

The boundary condition at the profile becomes

$$G_r = \cos\theta \quad \text{at} \quad r = 1 \tag{3.4}$$

and in the far field it reduces to

$$G = E\{\theta - \tan^{-1}(\sqrt{1-M_\infty^2}\tan\theta)\} \tag{3.5}$$

The circulation is determined by the Kutta condition, which requires that u is finite at the trailing edge, and hence that

$$E = G_\theta - \sin\theta \quad \text{at} \quad r = 1\ ,\ \theta = 0\ . \tag{3.6}$$

In conservation form the potential equation becomes

$$\frac{\partial}{\partial\theta}(\rho U) + r\frac{\partial}{\partial r}(\rho V) = 0 \tag{3.7}$$

where

$$(3.8) \quad U = \frac{Hu}{r} = G_\theta - E - \frac{\sin\theta}{r}, \qquad V = \frac{Hv}{r} = rG_r - \frac{\cos\theta}{r}$$

and $\rho$ is evaluated by equation (1.6).

Within either form of the equation the Poisson step consists of calculating a correction $C_{ij}$ at each point by solving the discrete approximation to

$$C_{\theta\theta} + r \frac{\partial}{\partial r}(r\, C_r) = -Q$$

where $Q$ is a suitably scaled residual. With the quasilinear form we set

$$Q = \frac{R}{a^2}$$

where $R$ is the residual of equation (3.2), and with the conservation form we set

$$Q = \frac{R}{\rho}$$

where $R$ is the residual of equation (3.7). In evaluating the residual an appropriate upwind difference scheme must be used in the supersonic zone. At each point $G_{ij}$ is then replaced by $G_{ij} + \omega C_{ij}$, where $\omega$ is a relaxation parameter. After the Poisson step we use one or more relaxation steps with the same difference scheme to correct the supersonic zone.

When the supersonic flow is confined to a pocket over the forward part of the profile, the upwind direction is almost coincident with the $\theta$ coordinate direction in the supersonic zone. It is then possible to use a simple difference scheme with retarded difference formulas restricted to the $\theta$ direction. This has the advantage that the relaxation method becomes a marching scheme in the supersonic zone. As the free stream Mach number is increased towards unity the supersonic zone extends rearward until finally there is a shock wave at the trailing edge. Then the upwind direction can no longer be assumed to be the $\theta$ coordinate direction, and it becomes necessary to use a rotated difference scheme [9]. In this case the relaxation method ceases to be a marching scheme in the supersonic zone, and we can expect to need several relaxation steps after each Poisson step to assure convergence.

While the conservation form gives a better approximation to shock waves, it has the disadvantage of exhibiting larger discretization errors in smooth compressive regions of supersonic flow. Such regions appear over shock-free supercritical airfoils. Also, when the effect of the boundary layer is represented by a simple correction for the displacement thickness, better agreement with the experimental data is often obtained with the quasilinear form. Since no single scheme gives the best results in all cases, the composite iterative method has been tested in four variants:

(1) Simple quasilinear scheme

(2) Rotated quasilinear scheme

(3) Simple conservative scheme

(4) Rotated conservative scheme

In all four schemes the circulation constant E is adjusted to satisfy equation (3.6) after each Poisson step and after each relaxation step. Thus the correction due to the change in circulation is immediately spread throughout the flow.

The details of the four schemes are given in Appendix 2. A listing of the computer program used for the numerical experiments is given in Appendix 3. A description of the program's input and output is also included in Appendix 4.

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