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TECHNICAL NOTE

On the Non-linear Stability of Flux Reconstruction Schemes

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Abstract The flux reconstruction (FR) approach unifies various high-order schemes, including collocation based nodal discontinuous Galerkin (DG) methods, and all spectral difference methods (at least for a linear flux function), within a single framework. Recently a new range of linearly stable FR schemes have been identified, henceforth referred to as Vincent-Castonguay-Jameson-Huynh (VCJH) schemes. In this short note non-linear stability properties of FR schemes are elucidated via analysis of linearly stable VCJH schemes (so as to focus attention solely on issues of non-linear stability). It is shown that linearly stable VCJH schemes (at least in their standard form) may be unstable if the flux function is non-linear. This instability is due to aliasing errors, which manifest since FR schemes (in their standard form) utilize a collocation projection at the solution points to construct a polynomial approximation of the flux. Strategies for minimizing such aliasing driven instabilities are discussed within the context of the FR approach. In particular, it is shown that the location of the solution points will have a significant effect on non-linear stability. This result is important, since linear analysis of FR schemes implies stability is independent of solution point location. Finally, it is shown that if an exact L2 projection is employed to construct an approximation of the flux (as opposed to a collocation projection), then aliasing errors and hence aliasing driven instabilities will be eliminated. However, performing such a projection exactly, or at least very accurately, would be more costly than performing a collocation projection, and would certainly impact the inherent efficiency and simplicity of the FR approach. It can be noted that in all above regards, non-linear stability properties of FR schemes are similar to those of nodal DG schemes. The findings should motivate further research into the non-linear performance of FR schemes, which have hitherto been developed and analyzed solely in the context of a linear flux function.

Keywords High-order methods · Flux reconstruction · Nodal discontinuous Galerkin method · Spectral difference method · Non-linear stability

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1 Introduction

In recent decades discontinuous Galerkin (DG) methods, and a number of similar variants, have emerged as an attractive alternative to classical finite element and finite volume methods for high-order accurate numerical simulations on unstructured grids. Recently Huynh [1, 2] proposed the flux reconstruction (FR) approach, which encompasses both collocation based nodal DG schemes of the type described by Hesthaven and Warburton [3], and spectral difference (SD) methods (at least for a linear flux function), which were originally proposed by Kopriva and Kolias [4], and later generalized by Liu, Vinokur and Wang [5].

Utilizing the FR approach of Huynh [1, 2], it was proved by Jameson [6] that (for 1D linear advection) a particular SD method is stable for all orders of accuracy in a broken norm of Sobolev type. Recently, this result has been extended by Vincent, Castonguay and Jameson [7], who identified a class of FR schemes which are guaranteed to be linearly stable. These schemes, which are parameterized by a single scalar, will henceforth be referred to as Vincent-Castonguay-Jameson-Huynh (VCJH) schemes. The identification of such schemes offers significant insight into why certain FR schemes are stable, whereas others are not. Also from a practical standpoint the VCJH formulation offers a simple prescription for implementing an infinite range of efficient and linearly stable high-order methods. In this short note non-linear stability properties of FR schemes are elucidated via analysis of linearly stable VCJH schemes (so as to focus attention solely on issues of non-linear stability). To begin, a brief overview of the one-dimensional (1D) FR approach is given, followed by an overview of 1D VCJH schemes. The non-linear stability of 1D VCJH schemes is then analyzed and discussed. Finally conclusions are drawn.

2 Overview of the Flux Reconstruction Approach

Consider solving the following 1D scalar conservation law

$$\frac{\partial u}{\partial t} + \frac{\partial f}{\partial x} = 0 \tag{2.1}$$

within an arbitrary periodic domain $\mathbf{\Omega}$, where x is a spatial coordinate, t is time, $u = u(x,t)$ is a conserved scalar quantity and $f = f(u)$ is the flux of u in the x direction. Further, consider partitioning $\mathbf{\Omega}$ into N distinct elements each denoted $\mathbf{\Omega}_n = \{x | x_n < x < x_{n+1}\}$ such that

$$\mathbf{\Omega} = \bigcup_{n=0}^{N-1} \mathbf{\Omega}_n, \qquad \bigcap_{n=0}^{N-1} \mathbf{\Omega}_n = \emptyset. \tag{2.2}$$

Finally, having partitioned $\mathbf{\Omega}$ into separate elements, consider representing the exact solution u within each $\mathbf{\Omega}_n$ by a polynomial of degree k denoted $u_n^\delta = u_n^\delta(x,t)$ (which is in general piecewise discontinuous between elements), and the exact flux f within each $\mathbf{\Omega}_n$ by a polynomial of degree $k+1$ denoted $f_n^\delta = f_n^\delta(x,t)$ (which is piecewise continuous between elements), such that a total approximate solution $u^\delta = u^\delta(x,t)$ and a total approximate flux $f^\delta = f^\delta(x,t)$ can be defined within $\mathbf{\Omega}$ as

$$u^\delta = \bigoplus_{n=0}^{N-1} u_n^\delta \approx u, \qquad f^\delta = \bigoplus_{n=0}^{N-1} f_n^\delta \approx f. \tag{2.3}$$

From an implementation perspective, it is advantageous to transform each $\mathbf{\Omega}_n$ to a standard element $\mathbf{\Omega}_S = \{r | -1 \le r \le 1\}$ via the mapping

$$r = \Gamma_n(x) = 2\left(\frac{x - x_n}{x_{n+1} - x_n}\right) - 1, \tag{2.4}$$

which has the inverse

$$x = \Gamma_n^{-1}(r) = \left(\frac{1-r}{2}\right)x_n + \left(\frac{1+r}{2}\right)x_{n+1}. \tag{2.5}$$

Having performed such a transformation, the evolution of u_n^δ within any individual $\mathbf{\Omega}_n$ (and thus the evolution of u^δ within $\mathbf{\Omega}$) can be determined by solving the following transformed equation within the standard element $\mathbf{\Omega}_S$

$$\frac{\partial \hat{u}^\delta}{\partial t} + \frac{\partial \hat{f}^\delta}{\partial r} = 0, \tag{2.6}$$

where

$$\hat{u}^\delta = \hat{u}^\delta(r,t) = u_n^\delta(\Gamma_n^{-1}(r), t) \tag{2.7}$$

is a polynomial of degree k,

$$\hat{f}^\delta = \hat{f}^\delta(r,t) = \frac{f_n^\delta(\Gamma_n^{-1}(r), t)}{h_n}, \tag{2.8}$$

is a polynomial of degree $k+1$, and $h_n = (x_{n+1} - x_n)/2$.

The FR approach to solving (2.6) within the standard element $\mathbf{\Omega}_S$ can be described in five stages. The first stage involves representing $\hat{u}^\delta$ in terms of a nodal basis as follows

$$\hat{u}^\delta = \sum_{i=0}^{k} \hat{u}_i^\delta\, l_i, \tag{2.9}$$

where l_i are Lagrange polynomials defined as

$$l_i = \prod_{j=0, j\neq i}^{k} \left(\frac{r - r_j}{r_i - r_j}\right), \tag{2.10}$$

r_i ($i = 0$ to k) are $k+1$ distinct solution points within $\mathbf{\Omega}_S$, and $\hat{u}_i^\delta = \hat{u}_i^\delta(t)$ ($i = 0$ to k) are values of $\hat{u}^\delta$ at the solution points r_i.

The second stage of the FR approach involves constructing a degree k polynomial $\hat{f}^{\delta D} = \hat{f}^{\delta D}(r,t)$, defined as the approximate transformed discontinuous flux within $\mathbf{\Omega}_S$. Specifically, $\hat{f}^{\delta D}$ is obtained via a collocation projection at the $k+1$ solution points, and can hence be expressed as

$$\hat{f}^{\delta D} = \sum_{i=0}^{k} \hat{f}_i^{\delta D}\, l_i \tag{2.11}$$

where the coefficients $\hat{f}_i^{\delta D} = \hat{f}_i^{\delta D}(t)$ are simply values of the transformed flux at each solution point r_i evaluated directly from the approximate solution. The flux $\hat{f}^{\delta D}$ is termed discontinuous since it is calculated directly from the approximate solution, which is in general piecewise discontinuous between elements.

The third stage of the FR approach involves evaluating the approximate solution at either end of the standard element $\mathbf{\Omega}_S$ (i.e. at $r = \pm 1$). These values, in conjunction with analogous information from adjoining elements, are then used to calculate numerical interface fluxes. In what follows the numerical interface fluxes associated with the left and right hand ends of $\mathbf{\Omega}_S$ (and transformed appropriately for use in $\mathbf{\Omega}_S$) will be denoted $\hat{f}_L^{\delta I}$ and $\hat{f}_R^{\delta I}$ respectively.

The forth stage of the FR approach involves adding a correction flux $\hat{f}^{\delta C} = \hat{f}^{\delta C}(r, t)$ of degree $k + 1$ to $\hat{f}^{\delta D}$, such that their sum equals the transformed numerical interface flux at $r = \pm 1$, yet remains close to $\hat{f}^{\delta D}$ within the interior of $\mathbf{\Omega}_S$. To construct $\hat{f}^{\delta C}$ such that the above requirements are satisfied, consider first defining $g_L = g_L(r)$ and $g_R = g_R(r)$ as degree $k + 1$ correction functions that have oscillations of small amplitude within $\mathbf{\Omega}_S$ (and hence approximate zero in some sense), as well as satisfying

$$g_L(-1) = 1, \qquad g_L(1) = 0, \tag{2.12}$$

$$g_R(-1) = 0, \qquad g_R(1) = 1, \tag{2.13}$$

and

$$g_L(r) = g_R(-r). \tag{2.14}$$

A suitable expression for $\hat{f}^{\delta C}$ can now be written in terms of g_L and g_R as

$$\hat{f}^{\delta C} = (\hat{f}_L^{\delta I} - \hat{f}_L^{\delta D})g_L + (\hat{f}_R^{\delta I} - \hat{f}_R^{\delta D})g_R, \tag{2.15}$$

where $\hat{f}_L^{\delta D} = \hat{f}^{\delta D}(-1, t)$ and $\hat{f}_R^{\delta D} = \hat{f}^{\delta D}(1, t)$. Using this expression, the degree $k + 1$ approximate transformed total flux $\hat{f}^{\delta}$ within $\mathbf{\Omega}_S$ can be constructed from the discontinuous and correction fluxes as follows

$$\hat{f}^{\delta} = \hat{f}^{\delta D} + \hat{f}^{\delta C} = \hat{f}^{\delta D} + (\hat{f}_L^{\delta I} - \hat{f}_L^{\delta D})g_L + (\hat{f}_R^{\delta I} - \hat{f}_R^{\delta D})g_R. \tag{2.16}$$

The final stage of the FR approach involves evaluating the divergence of $\hat{f}^{\delta}$ at each solution point r_i using the expression

$$\frac{\partial \hat{f}^{\delta}}{\partial r}(r_i) = \sum_{j=0}^{k} \hat{f}_j^{\delta D} \frac{\mathrm{d}l_j}{\mathrm{d}r}(r_i) + (\hat{f}_L^{\delta I} - \hat{f}_L^{\delta D})\frac{\mathrm{d}g_L}{\mathrm{d}r}(r_i) + (\hat{f}_R^{\delta I} - \hat{f}_R^{\delta D})\frac{\mathrm{d}g_R}{\mathrm{d}r}(r_i). \tag{2.17}$$

These values can then be used to advance $\hat{u}^{\delta}$ in time via a suitable temporal discretization of the following semi-discrete expression

$$\frac{\mathrm{d}\hat{u}_i^{\delta}}{\mathrm{d}t} = -\frac{\partial \hat{f}^{\delta}}{\partial r}(r_i). \tag{2.18}$$

The nature of a specific FR scheme depends solely on three factors, namely the location of the solution points r_i, the methodology for calculating the transformed numerical interface fluxes $\hat{f}_L^{\delta I}$ and $\hat{f}_R^{\delta I}$, and the form of the flux correction functions g_L (and thus g_R). It was shown by Huynh [1] that a collocation based (under integrated) nodal DG scheme is recovered in 1D if the corrections functions g_L and g_R are the right and left Radau polynomials respectively. Also, it has been shown that SD type methods can be recovered (at least for a linear flux function) if the corrections g_L and g_R are set to zero at a set of k points within $\mathbf{\Omega}_S$ (located symmetrically about the origin) [1]. Several additional forms of g_L (and thus g_R) have also been suggested, leading to the development of new schemes, with various stability and accuracy properties. For further details of these new schemes see the articles by Huynh [1, 2].

3 Vincent-Castonguay-Jameson-Huynh Schemes

VCJH schemes [7] can be recovered if the left and right corrections functions g_L and g_R respectively are defined as

$$g_L = \frac{(-1)^k}{2}\left[L_k - \left(\frac{\eta_k L_{k-1} + L_{k+1}}{1+\eta_k}\right)\right], \tag{3.1}$$

and

$$g_R = \frac{1}{2}\left[L_k + \left(\frac{\eta_k L_{k-1} + L_{k+1}}{1+\eta_k}\right)\right], \tag{3.2}$$

where

$$\eta_k = \frac{c(2k+1)(a_k k!)^2}{2}, \quad a_k = \frac{(2k)!}{2^k (k!)^2}, \tag{3.3}$$

L_k is a Legendre polynomial of degree k, and c is a free scalar parameter that must lie within the range

$$\frac{-2}{(2k+1)(a_k k!)^2} < c < \infty. \tag{3.4}$$

Such correction functions satisfy

$$\int_{-1}^{1} g_L \frac{\partial \hat{u}^\delta}{\partial r}\mathrm{dr} - c\left(\frac{\partial^k \hat{u}^\delta}{\partial r^k}\right)\left(\frac{\mathrm{d}^{k+1} g_L}{\mathrm{d}r^{k+1}}\right) = 0, \tag{3.5}$$

and

$$\int_{-1}^{1} g_R \frac{\partial \hat{u}^\delta}{\partial r}\mathrm{dr} - c\left(\frac{\partial^k \hat{u}^\delta}{\partial r^k}\right)\left(\frac{\mathrm{d}^{k+1} g_R}{\mathrm{d}r^{k+1}}\right) = 0, \tag{3.6}$$

within the standard element $\mathbf{\Omega}_S$ for any transformed solution $\hat{u}^\delta$, and ensure that the resulting VCJH scheme will be linearly stable in the broken Sobolev type norm $\|u^\delta\|_{k,2}$, defined as

$$\|u^\delta\|_{k,2} = \left[\sum_{n=1}^{N}\int_{x_n}^{x_{n+1}} (u_n^\delta)^2 + \frac{c}{2}(J_n)^{2k}\left(\frac{\partial^k u_n^\delta}{\partial x^k}\right)^2 \mathrm{d}x\right]^{1/2}. \tag{3.7}$$

It can be noted that several existing methods are encompassed by the new class of VCJH schemes. In particular if $c = 0$ then a collocation based nodal DG scheme is recovered [7]. Alternatively, if

$$c = \frac{2k}{(2k+1)(k+1)(a_k k!)^2}, \tag{3.8}$$

an SD method is recovered (at least for a linear flux function) [7]. It is in fact the only SD type scheme that can be recovered from the range of VCJH schemes. Further, it is identical to the SD scheme that Jameson [6] proved to be linearly stable, which is the same as the only SD scheme that Huynh found to be devoid of weak instabilities [1]. Finally, if

$$c = \frac{2(k+1)}{(2k+1)k(a_k k!)^2}, \tag{3.9}$$

then a so called g_2 FR method is recovered [7], which was originally identified by Huynh [1] to be particularly stable.

4 Non-linear Stability of Vincent-Castonguay-Jameson-Huynh Schemes

To gain insight into the non-linear stability of VCJH schemes consider substituting (2.16) into (2.6), to obtain

$$\frac{\partial \hat{u}^\delta}{\partial t} = -\frac{\partial \hat{f}^{\delta D}}{\partial r} - (\hat{f}_L^{\delta I} - \hat{f}_L^{\delta D})\frac{\mathrm{d}g_L}{\mathrm{d}r} - (\hat{f}_R^{\delta I} - \hat{f}_R^{\delta D})\frac{\mathrm{d}g_R}{\mathrm{d}r}. \tag{4.1}$$

On multiplying (4.1) by the approximate transformed solution $\hat{u}^\delta$ and integrating over $\mathbf{\Omega}_S$ one obtains

$$\begin{aligned}\int_{-1}^{1} \hat{u}^\delta \frac{\partial \hat{u}^\delta}{\partial t}\mathrm{d}r = &-\int_{-1}^{1} \hat{u}^\delta \frac{\partial \hat{f}^{\delta D}}{\partial r}\mathrm{d}r - (\hat{f}_L^{\delta I} - \hat{f}_L^{\delta D})\int_{-1}^{1} \hat{u}^\delta \frac{\mathrm{d}g_L}{\mathrm{d}r}\mathrm{d}r \\ &- (\hat{f}_R^{\delta I} - \hat{f}_R^{\delta D})\int_{-1}^{1} \hat{u}^\delta \frac{\mathrm{d}g_R}{\mathrm{d}r}\mathrm{d}r,\end{aligned} \tag{4.2}$$

and thus

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{-1}^{1} (\hat{u}^\delta)^2\mathrm{d}r = &\,2\int_{-1}^{1} \hat{f}^{\delta D}\frac{\partial \hat{u}^\delta}{\partial r}\mathrm{d}r + 2(\hat{f}_L^{\delta I} - \hat{f}_L^{\delta D})\int_{-1}^{1} g_L \frac{\partial \hat{u}^\delta}{\partial r}\mathrm{d}r \\ &+ 2(\hat{f}_R^{\delta I} - \hat{f}_R^{\delta D})\int_{-1}^{1} g_R \frac{\partial \hat{u}^\delta}{\partial r}\mathrm{d}r + 2(\hat{f}_L^{\delta I}\hat{u}_L^\delta - \hat{f}_R^{\delta I}\hat{u}_R^\delta),\end{aligned} \tag{4.3}$$

where $\hat{u}_L^\delta = \hat{u}^\delta(-1,t)$ and $\hat{u}_R^\delta = \hat{u}^\delta(1,t)$. On differentiating (4.1) k times (in space) one obtains

$$\frac{\partial}{\partial t}\left(\frac{\partial^k \hat{u}^\delta}{\partial r^k}\right) = -\frac{\partial^{k+1}\hat{f}^{\delta D}}{\partial r^{k+1}} - (\hat{f}_L^{\delta I} - \hat{f}_L^{\delta D})\frac{\mathrm{d}^{k+1}g_L}{\mathrm{d}r^{k+1}} - (\hat{f}_R^{\delta I} - \hat{f}_R^{\delta D})\frac{\mathrm{d}^{k+1}g_R}{\mathrm{d}r^{k+1}}, \tag{4.4}$$

where it can be noted that since $\hat{f}^{\delta D}$ is a polynomial of degree k

$$\frac{\partial^{k+1}\hat{f}^{\delta D}}{\partial r^{k+1}} = 0, \tag{4.5}$$

and thus

$$\frac{\partial}{\partial t}\left(\frac{\partial^k \hat{u}^\delta}{\partial r^k}\right) = -(\hat{f}_L^{\delta I} - \hat{f}_L^{\delta D})\frac{\mathrm{d}^{k+1}g_L}{\mathrm{d}r^{k+1}} - (\hat{f}_R^{\delta I} - \hat{f}_R^{\delta D})\frac{\mathrm{d}^{k+1}g_R}{\mathrm{d}r^{k+1}}. \tag{4.6}$$

On multiplying (4.6) by the kth derivative of the approximate transformed solution $\hat{u}^\delta$ and integrating over $\mathbf{\Omega}_S$ one obtains

$$\int_{-1}^{1}\left(\frac{\partial^k \hat{u}^\delta}{\partial r^k}\right)\frac{\partial}{\partial t}\left(\frac{\partial^k \hat{u}^\delta}{\partial r^k}\right)\mathrm{d}r = -(\hat{f}_L^{\delta I} - \hat{f}_L^{\delta D})\int_{-1}^{1}\left(\frac{\partial^k \hat{u}^\delta}{\partial r^k}\right)\left(\frac{\mathrm{d}l^{k+1} g_L}{\mathrm{d}r^{k+1}}\right)\mathrm{d}r$$
$$- (\hat{f}_R^{\delta I} - \hat{f}_R^{\delta D})\int_{-1}^{1}\left(\frac{\partial^k \hat{u}^\delta}{\partial r^k}\right)\left(\frac{\mathrm{d}^{k+1} g_R}{\mathrm{d}r^{k+1}}\right)\mathrm{d}r, \tag{4.7}$$

and thus since $\hat{u}^\delta$ is a polynomial of degree k, and g_L and g_R are polynomials of degree $k+1$, one obtains

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{-1}^{1}\left(\frac{\partial^k \hat{u}^\delta}{\partial r^k}\right)^2 \mathrm{d}r = -2(\hat{f}_L^{\delta I} - \hat{f}_L^{\delta D})\left(\frac{\partial^k \hat{u}^\delta}{\partial r^k}\right)\left(\frac{\mathrm{d}^{k+1} g_L}{\mathrm{d}r^{k+1}}\right)$$
$$- 2(\hat{f}_R^{\delta I} - \hat{f}_R^{\delta D})\left(\frac{\partial^k \hat{u}^\delta}{\partial r^k}\right)\left(\frac{\mathrm{d}^{k+1} g_R}{\mathrm{d}r^{k+1}}\right). \tag{4.8}$$

On multiplying (4.8) by the scalar quantity c (which lies in the range defined by (3.4)) and summing with (4.3), one obtains

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{-1}^{1}(\hat{u}^\delta)^2 + \frac{c}{2}\left(\frac{\partial^k \hat{u}^\delta}{\partial r^k}\right)^2 \mathrm{d}r \\
&\quad = 2\int_{-1}^{1}\hat{f}^{\delta D}\frac{\partial \hat{u}^\delta}{\partial r}\mathrm{d}r + 2(\hat{f}_L^{\delta I}\hat{u}_L^\delta - \hat{f}_R^{\delta I}\hat{u}_R^\delta) \\
&\qquad + 2(\hat{f}_L^{\delta I} - \hat{f}_L^{\delta D})\left[\int_{-1}^{1} g_L \frac{\partial \hat{u}^\delta}{\partial r}\mathrm{d}r - c\left(\frac{\partial^k \hat{u}^\delta}{\partial r^k}\right)\left(\frac{\mathrm{d}^{k+1} g_L}{\mathrm{d}r^{k+1}}\right)\right] \\
&\qquad + 2(\hat{f}_R^{\delta I} - \hat{f}_R^{\delta D})\left[\int_{-1}^{1} g_R \frac{\partial \hat{u}^\delta}{\partial r}\mathrm{d}r - c\left(\frac{\partial^k \hat{u}^\delta}{\partial r^k}\right)\left(\frac{\mathrm{d}^{k+1} g_R}{\mathrm{d}r^{k+1}}\right)\right],
\end{aligned} \tag{4.9}$$

which for VCJH type schemes, due to (3.5) and (3.6), can be written as

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{-1}^{1}(\hat{u}^\delta)^2 + \frac{c}{2}\left(\frac{\partial^k \hat{u}^\delta}{\partial r^k}\right)^2 \mathrm{d}r = \int_{-1}^{1}\hat{f}^{\delta D}\frac{\partial \hat{u}^\delta}{\partial r}\mathrm{d}r + \hat{f}_L^{\delta I}\hat{u}_L^\delta - \hat{f}_R^{\delta I}\hat{u}_R^\delta. \tag{4.10}$$

To proceed, consider writing (4.10) as

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{-1}^{1}(\hat{u}^\delta)^2 + \frac{c}{2}\left(\frac{\partial^k \hat{u}^\delta}{\partial r^k}\right)^2 \mathrm{d}r = \int_{-1}^{1}\hat{f}\frac{\partial \hat{u}^\delta}{\partial r}\mathrm{d}r + \hat{f}_L^{\delta I}\hat{u}_L^\delta - \hat{f}_R^{\delta I}\hat{u}_R^\delta + \hat{\epsilon}, \tag{4.11}$$

where

$$\hat{f} = \hat{f}(r,t) = \frac{f(u_n^\delta(\Gamma_n^{-1}(r),t))}{J_n} \tag{4.12}$$

is the transformed (true) flux function, and

$$\hat{\epsilon} = \int_{-1}^{1}(\hat{f}^{\delta D} - \hat{f})\frac{\partial \hat{u}^\delta}{\partial r}\mathrm{d}r \tag{4.13}$$

is a transformed error term (to be discussed in more detail shortly). On transforming (4.10) back to the physical space element $\mathbf{\Omega}_n$, and summing over all elements within the periodic domain $\mathbf{\Omega}$, one obtains

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u^\delta\|_{k,2}^2 \\ = \sum_{n=0}^{N-1}\left[\int_{x_n}^{x_{n+1}} f(u_n^\delta)\frac{\partial u_n^\delta}{\partial x}\mathrm{d}x + f_n^{\delta I}u_n^\delta(x_n) - f_{n+1}^{\delta I}u_n^\delta(x_{n+1}) + \epsilon_n\right],\end{aligned} \tag{4.14}$$

where

$$f_n^{\delta I} = J_n \hat{f}_L^{\delta I}, \qquad f_{n+1}^{\delta I} = J_n \hat{f}_R^{\delta I}, \tag{4.15}$$

are numerical interface fluxes in physical space evaluated at x_n and x_{n+1} respectively, and

$$\epsilon_n = J_n \hat{\epsilon} \tag{4.16}$$

are error terms in physical space within each $\mathbf{\Omega}_n$.

If one now defines $G = G(u)$ such that

$$\frac{\partial G}{\partial u} = f, \tag{4.17}$$

then (4.14) can be written as

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u^\delta\|_{k,2}^2 \\ = \sum_{n=0}^{N-1}\left[\int_{x_n}^{x_{n+1}} \frac{\partial G}{\partial u}(u_n^\delta)\frac{\partial u_n^\delta}{\partial x}\mathrm{d}x + f_n^{\delta I}u_n^\delta(x_n) - f_{n+1}^{\delta I}u_n^\delta(x_{n+1}) + \epsilon_n\right],\end{aligned} \tag{4.18}$$

and thus

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u^\delta\|_{k,2}^2 \\ = \sum_{n=0}^{N-1}\left[G(u_n^\delta(x_{n+1})) - G(u_n^\delta(x_n)) + f_n^{\delta I}u_n^\delta(x_n) - f_{n+1}^{\delta I}u_n^\delta(x_{n+1}) + \epsilon_n\right],\end{aligned} \tag{4.19}$$

which can be cast (partially) in terms of a summation over interfaces within the periodic domain $\mathbf{\Omega}$ as

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u^\delta\|_{k,2}^2 \\ = \sum_{n=0}^{N-1}\left[f_n^{\delta I}(u_+^\delta(x_n) - u_-^\delta(x_n)) - G(u_+^\delta(x_n)) + G(u_-^\delta(x_n))\right] + \sum_{n=0}^{N-1}\epsilon_n,\end{aligned} \tag{4.20}$$

where $u_+^\delta(x_n) = u_n^\delta(x_n)$ and (to account for the periodicity of the domain)

$$u_-^\delta(x_n) = \begin{cases} u_{N-1}^\delta(x_N), & n = 0, \\ u_{n-1}^\delta(x_n), & n \neq 0. \end{cases} \tag{4.21}$$

Finally, using the mean value theorem, (4.20) can be written as

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u^{\delta}\|_{k,2}^{2} \\
&\quad = \sum_{n=0}^{N-1}\left[f_n^{\delta I}(u_+^{\delta}(x_n)-u_-^{\delta}(x_n)) - \frac{\partial G}{\partial u}(\eta_n^{\delta})(u_+^{\delta}(x_n)-u_-^{\delta}(x_n))\right] + \sum_{n=0}^{N-1}\epsilon_n, \quad (4.22)
\end{aligned}$$

for some η_n^{δ} between $u_-^{\delta}(x_n)$ and $u_+^{\delta}(x_n)$, thus

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u^{\delta}\|_{k,2}^{2} = \sum_{n=0}^{N-1}(f_n^{\delta I} - f(\eta_n^{\delta}))(u_+^{\delta}(x_n)-u_-^{\delta}(x_n)) + \sum_{n=0}^{N-1}\epsilon_n. \quad (4.23)$$

If each interface flux is now considered to be an E-flux [8], then all interface contributions will be negative (following the definition of an E-flux), and hence (4.23) can be written as

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u^{\delta}\|_{k,2}^{2} = \Theta + \sum_{n=0}^{N-1}\epsilon_n, \quad (4.24)$$

where $\Theta \le 0$. For energy stability in the norm $\|u^{\delta}\|_{k,2}$, it is therefore required that the sum of ϵ_n is less than or equal to zero.

5 The Error Terms ϵ_n

The nature of the error terms ϵ_n (which clearly determine whether the scheme is stable) can be understood by analyzing the transformed error $\hat{\epsilon}$ within Ω_S. Since $\hat{u}^{\delta}$ is a polynomial of degree k, it has a spatial derivative of degree $k-1$, which can be expanded as

$$\frac{\partial \hat{u}^{\delta}}{\partial r} = \sum_{i=0}^{k-1}\left[\frac{(2i+1)}{2}\int_{-1}^{1}\frac{\partial \hat{u}^{\delta}}{\partial r}L_i\,\mathrm{d}r\right]L_i, \quad (5.1)$$

where L_i are Legendre polynomials of degree i. On substituting (5.1) into (4.13) one obtains

$$\hat{\epsilon} = \sum_{i=0}^{k-1}\int_{-1}^{1}(\hat{f}^{\delta D} - \hat{f})\left[\frac{(2i+1)}{2}\int_{-1}^{1}\frac{\partial \hat{u}^{\delta}}{\partial r}L_i\,\mathrm{d}r\right]L_i\,\mathrm{d}r, \quad (5.2)$$

and hence

$$\hat{\epsilon} = \sum_{i=0}^{k-1}\hat{\epsilon}_i\left[\frac{(2i+1)}{2}\int_{-1}^{1}\frac{\partial \hat{u}^{\delta}}{\partial r}L_i\,\mathrm{d}r\right], \quad (5.3)$$

where

$$\hat{\epsilon}_i = \int_{-1}^{1}\hat{f}^{\delta D}L_i\,\mathrm{d}r - \int_{-1}^{1}\hat{f}L_i\,\mathrm{d}r. \quad (5.4)$$

Neither the sign nor magnitude of the integral term in (5.3) can be guaranteed (since it depends on the transformed approximate solution $\hat{u}^{\delta}$). Therefore, in order to in general minimize $\hat{\epsilon}$ and thus ϵ_n, one should ensure the magnitude of all $\hat{\epsilon}_i$ are as small as possible.

If the flux function is linear then $\hat{f}$ will be a polynomial of degree k. Hence it will be represented exactly by $\hat{f}^{\delta D}$ (formed by a collocation projection at the $k+1$ solution points). It is therefore clear that $\hat{\epsilon}$, and hence ϵ_n, are guaranteed to be zero. Hence by (4.24) stability is guaranteed as expected [7]. However, if the flux function is non-linear, then the collocation projection employed to construct $\hat{f}^{\delta D}$ will introduce aliasing errors; that is to say the modal energies of $\hat{f}^{\delta D}$ (given by the first term on the right hand side of (5.4)) will be different to the corresponding modal energies in $\hat{f}$ (given by the second term on the right hand side of (5.4)). Such a phenomenon occurs because the collocation projection will in general under-sample $\hat{f}$. Consequently high-frequency (under-resolved) modes of $\hat{f}$ will contribute (erroneously) to the energies of lower-frequency resolved modes (for further details see, for example, the article of Kirby and Sherwin [9], or the textbooks of Karniadakis and Sherwin [10], and Hesthaven and Warburton [3]). As a result of these aliasing errors $\hat{\epsilon}_i$ will in general be non-zero, and thus in general the sign and magnitude of $\hat{\epsilon}$ (and hence ϵ_n) cannot be guaranteed. Therefore by (4.24) stability of VCJH schemes can no longer be guaranteed if the flux function is non-linear. Such an instability is often referred to as an aliasing driven instability.

There are various important points that should be noted about the aliasing driven instabilities that manifest when the flux function is non-linear:

- The instabilities are of the same form as those which afflict collocation based nodal DG schemes if the solution is under-resolved.
- If the solution (and hence $\hat{f}$) is well resolved, then aliasing errors, and hence aliasing driven instabilities, are effectively eliminated.
- The location of the solution points (at which the collocation projection is performed) will have a significant impact on aliasing errors, and hence on aliasing driven instabilities. A sensible choice is to locate solution points at abscissa of the Gauss-Legendre quadrature rule. To understand why, consider expanding (5.4) as

$$\hat{\epsilon}_i = \sum_{j=0}^{k} \hat{f}_j^{\delta D} \int_{-1}^{1} l_j L_i \mathrm{d}r - \int_{-1}^{1} \hat{f} L_i \mathrm{d}r. \tag{5.5}$$

Since l_j is of order k and L_i is at most of order $k-1$, (5.5) can be written exactly as

$$\hat{\epsilon}_i = \sum_{j=0}^{k} \hat{f}_j^{\delta D} \sum_{m=0}^{k} l_j(\zeta_m) L_i(\zeta_m)\omega_m - \int_{-1}^{1} \hat{f} L_i \mathrm{d}r \tag{5.6}$$

where ζ_m and ω_m are the abscissa and weights respectively of the Gauss-Legendre quadrature rule. If it is now assumed that the solution points are located at the abscissa ζ_m, then

$$\hat{\epsilon}_i = \sum_{j=0}^{k} \hat{f}(\zeta_j) \sum_{m=0}^{k} \delta_{jm} L_i(\zeta_m)\omega_m - \int_{-1}^{1} \hat{f} L_i \mathrm{d}r \tag{5.7}$$

and hence

$$\hat{\epsilon}_i = \sum_{j=0}^{k} \hat{f}(\zeta_j) L_i(\zeta_j)\omega_j - \int_{-1}^{1} \hat{f} L_i \mathrm{d}r. \tag{5.8}$$

The summation in (5.8) can be recognized as the Gauss-Legendre approximation of the integral term in (5.8). Such an approximation is of optimal accuracy (given a sampling of the integrand at $k+1$ points). Specifically, the approximation is exact for integrands up to order $2k+1$. The use of Gauss-Legendre abscissa as solution points will therefore in general minimize the coefficients $\hat{\epsilon}_i$, and thus minimize any aliasing errors. It can be noted that a similar argument follows for the Gauss-Lobatto-Legendre abscissa. However, for such abscissa the approximation is only exact for integrands up to order $2k-1$. Hence in general aliasing errors will be larger than if Gauss-Legendre abscissa were employed. The fact that non-linear stability depends on solution point location is significant, since until now (based on linear analysis) the stability of FR schemes was considered to be independent of solution point location.

- In addition to minimizing aliasing errors, and hence aliasing driven instabilities, the solution points should also define a well conditioned basis set with which to represent the solution. In 1D (and hence via tensor product extensions in quadrilaterals and hexahedra) Gauss-Legendre and Gauss-Lobatto-Legendre abscissa are suitable from this perspective (in fact Gauss-Lobatto-Legendre abscissa can be viewed as optimal [3]). However, when selecting solution points in triangles, there is a conflict between the requirements of reduced aliasing and good conditioning.

Finally, it can be noted that if the transformed discontinuous flux is obtained via an exact L2 projection (as opposed to a collocation projection), such that $\hat{f}^{\delta D} - \hat{f}$ is orthogonal to all polynomials of degree k, then according to (4.13) there will be no aliasing errors, since the spatial derivative of the approximate solution $\hat{u}^{\delta}$ is of degree $k-1$. Consequently, the resulting VCJH schemes will be non-linearly stable. However, it should be noted that performing such an L2 projection exactly (or at least very accurately) would be more costly than performing a collocation projection, and would certainly impact the inherent efficiency and simplicity of the FR approach.

6 Conclusions

It has been shown that VCJH schemes (at least in their standard form) may be unstable if the flux function is non-linear. Such instability is due to aliasing errors, which manifest since FR schemes (in their standard form) utilize a collocation projection at the solution points to construct a polynomial approximation of the flux. It has also been shown that the location of the solution points (at which the collocation projection is performed) will have a significant effect on non-linear stability. This result is important, since linear analysis of FR schemes implies that stability is independent of solution point location. Finally, it has been shown that if an exact L2 projection is employed to construct an approximation of the flux, then aliasing errors will be eliminated, and non-linear stability will be recovered. However, performing such a projection exactly (or at least very accurately) would be more costly than performing a collocation projection, and would certainly impact the inherent efficiency and simplicity of the FR approach. It can be noted that in all above regards, non-linear stability properties of FR schemes are similar to those of nodal DG schemes. The findings should motivate further research into the non-linear performance of FR schemes, which have hitherto been developed and analyzed solely in the context of a linear flux.

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