Large Eddy Simulation with Unstructured High-Order Numerical Methods for Real-World Turbulent Flows

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NASA AMS Seminar
November 7th 2013
The Aerospace Computing Lab

Figure 2: Fourth-order accurate solution of the viscous flow over a sphere at \( Re = 118 \) and \( Ma = 0.2 \).

Figure 3: Propagation of an Euler vortex through a dynamically deforming mesh: the starting vortex on undeformed mesh at \( t = 0 \) s (a), and the Euler vortex under deforming mesh at \( t = 25 \) s (b).

The method that updates the mesh at every time step. This method has the property of preserving grid orthogonality near the surface under substantial deformation, which is very desirable for high Reynolds number viscous flow simulation. The inviscid 2D compressible Euler vortex propagation problem has been used to demonstrate that the high-order numerical solver can maintain its design spatial order under dynamic deforming meshes.

Under dynamic mesh deformation, conservation of geometric property becomes another important consideration. In our investigation, we have derived the Geometric Conservation Law, studied its effect, and developed algorithms that enforce geometric conservation. This study is essential for ensuring the conservative property of the high-order scheme for the Navier-Stokes equations under dynamic mesh.

Study of flapping wing sections. Simulations of flow over a plunging NACA0012 airfoil have been undertaken using our in-house two-dimensional (2D) SD Navier Stokes flow solver. The plunging airfoil simulations were based on water tunnel experiments performed by Jones et al. at a Reynolds number of 1850. Although both the airfoil and the plunging motion are symmetric, the flow exhibits the remarkable property that the vortices are shed asymmetrically, either upward or downward, depending on the direction of the first stroke. Figure 4 shows that the pattern of vortical structures obtained via our numerical simulations compares well with experimental results.

Figure 4: Vorticity over a plunging NACA0012 airfoil at \( Re = 1850 \) calculated using a fourth-order SD scheme (a) is compared with an analogous experimental result from Jones et al. (b). The numerical results compare well with the experimental results. In particular, the simulations were able to reproduce the fine structures occurring in the wake of the plunging airfoil.

Three-dimensional simulations of flow over a plunging SD7003 wing section have also been performed using the 3D SD flow solver. Fourth-order accurate simulations were undertaken on meshes with \( 1.7 \times 10^6 \) DoF at \( Re = 4 \times 10^4 \). The time history of the resulting aerodynamic forces is in excellent agreement with the results obtained by Visbal. A key feature is the dynamic transition of the leading edge laminar bubble into turbulent structures, as illustrated in Figure 5.

Study of computational flapping wing aerodynamics. Using the 3D SD flow solver with moving deforming meshes, simulations have been performed first for a full size flapping wing, and subsequently for a geometrically more complex wing-body configuration. Simulation of flow over a flapping NACA0012 wing has been performed with the fifth-order SD method on a mesh with \( 4.7 \times 10^6 \) DoF at \( Re = 2000 \). Figure 6(a) shows the instantaneous entropy iso-surface, colored by the magnitude of Mach number, over the flapping wing. The Strouhal number of the flapping wing is 0.3. At this Strouhal number, the symmetric flapping motion generates net thrust and lift. Simulation of flow over a flapping wing-body configuration has been performed with the fourth-order SD method with \( 2.1 \times 10^7 \) DoF at \( Re = 5000 \). The flapping motion is symmetrical and the Strouhal number of the flapping wing is also equal to 0.3. Figure 6(b) shows the corresponding...
Speaker Profile

- May 2013: Joined Prof. Antony Jameson’s Aerospace Computing Laboratory (ACL)
  - Combining large-eddy simulation (LES) with high-order discontinuous numerical methods
- March 2013: Completed PhD at Imperial College London, UK
  - Joint project with Applied Modelling and Computation Group (AMCG) and Rolls-Royce Submarines
  - Developed RANS & LES turbulence modelling in finite element code ‘Fluidity’ featuring unstructured adaptive meshing
- 2007: Masters in Mechanical Engineering from University of Manchester, UK
Motivation

• Features of real-world turbulent flows:
  • Complex geometry
  • Very high Reynolds numbers ($Re > 10^7$)
  • Separation, reattachment, vortex interactions
  • Importance of small scales e.g. aeroacoustics

• Need unstructured meshes to fit geometry
• Need LES to make accurate predictions of complex phenomena at high $Re$
• Need high-order methods to reduce numerical error at small scales

• L: rotor-vortex interaction, M: high.lift wing, R: jet stalling
Outline

Introduction

High-Order Discontinuous Numerical Schemes

High-Order LES
  Advanced SGS Models
  Discrete Filters
  Current Work

Summary and Future Directions
Introduction

High-Order Discontinuous Numerical Schemes

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Summary and Future Directions
LES On Unstructured Meshes

- Easier to fit to complex geometry
- Easier to vary resolution
- Fewer degrees of freedom
LES On Unstructured Meshes

- Easier to fit to complex geometry
- Easier to vary resolution
- Fewer degrees of freedom

- $Re = 5100$
- 2nd order finite element code Fluidity
- Tetrahedral mesh, 250k elements
- Adaptive meshing
- Dynamic LES
- Reattachment length within 4% of DNS
- Greater accuracy with 60% fewer elements vs. structured mesh

High-Order Discontinuous Numerical Methods

- Many applications where resolution of small scales is important
- 2nd-order schemes: highly dissipative, errors in small scales
- High order (> 2): lower dissipation, improved small-scale resolution
- Discontinuous high-order schemes have local stencil: efficient parallelisation
- Greater accuracy per degree of freedom (DoF)

Blue dash-dot: exact solution; purple: discrete representation by second-order finite volume (L) and high-order discontinuous Galerkin (R) schemes. Dots are solution points (DoF).
Introduction

*High-Order Discontinuous Numerical Schemes*

High-Order LES
- Advanced SGS Models
- Discrete Filters
- Current Work

Summary and Future Directions
**High-Order Flux Reconstruction (FR) Schemes**

- Discretise $u, f$ using degree $N - 1$ Lagrange polynomials at $N$ solution points:
  \[
  u_h^D = \sum_{j=1}^{N} u_j l_j(\xi), \quad f_h^D = \sum_{j=1}^{N} f_j^D l_j(\xi), \quad \xi \in [-1 : 1]
  \]

- $u_h^D, f_h^D$ discontinuous across element interfaces

---

1. **Represent $u, f$ by Lagrange polynomial of degree $N - 1$ in element**

High-Order Flux Reconstruction (FR) Schemes

- Compute common interface fluxes $\tilde{f}_L, \tilde{f}_R$, e.g. Roe flux method

• Compute common interface fluxes $\tilde{f}_L, \tilde{f}_R$, e.g. Roe flux method

2. Get common interface fluxes $\tilde{f}_L, \tilde{f}_R$

**High-Order Flux Reconstruction (FR) Schemes**

- **correction functions** $g_L, g_R$ satisfying:

$$
g_L(-1) = g_R(1) = 1, \quad g_L(1) = g_R(-1) = 0, \quad g_L(\xi) = g_R(-\xi),$$

---

**3. Correction function $g_L$**

High-Order Flux Reconstruction (FR) Schemes

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- **Weight by flux corrections** $\Delta_L, \Delta_R$:

  $$\Delta_L = \tilde{f}_L - f_D^h (-1), \quad \Delta_R = \tilde{f}_R - f_D^h (1)$$

### 3. Correction function $g_L$

5. Add scaled correction to $f_D$
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6. Do steps 3-5 on right interface

High-Order Flux Reconstruction (FR) Schemes

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6. Do steps 3-5 on right interface

7. Globally $C_0$-continuous flux $f_C$

8. Differentiate continuous flux at solution points and advance solution in time:

\[
\frac{\partial f^C}{\partial x}(x_i) = \sum_{j=1}^{n} f_j^D \frac{dl_j}{dx}(x_i) + \Delta_L \frac{dg_L}{dx}(x_i) + \Delta_R \frac{dg_R}{dx}(x_i)
\]

Energy-Stable FR Schemes

- Scheme is energy-stable (for linear $f$) if energy norm does not increase in time:

$$\frac{d}{dt} \| u_h \|^2_{p,2} = \frac{d}{dt} \left[ \sum_{n=0}^{N-1} \int_{x_n}^{x_{n+1}} u_{h,n}^2 + \frac{c}{2} (J_n)^{2p} \left( \frac{\partial u_h}{\partial x} \right)^2 \right]_{x_n}^{x_{n+1}} \leq 0, \quad (1)$$

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$$

- Choose correction functions $g_L, g_R$ to satisfy (1):

$$
g_L = \frac{(-1)^p}{2} \left[ L_p - \left( \frac{\eta_p(c) L_{p-1} + L_{p+1}}{1 + \eta_p(c)} \right) \right],
$$

$$
g_R = \frac{(-1)^p}{2} \left[ L_p - \left( \frac{\eta_p(c) L_{p-1} + L_{p+1}}{1 + \eta_p(c)} \right) \right],
$$

- $L_p = \text{Legendre polynomial}, \ \eta_p(c) = \frac{c(2p+1) (a_p p!)^2}{2}$

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g_L = \frac{(-1)^p}{2} \left[ L_p - \left( \frac{\eta_p(c) L_{p-1} + L_{p+1}}{1 + \eta_p(c)} \right) \right], \quad g_R = \frac{(-1)^p}{2} \left[ L_p - \left( \frac{\eta_p(c) L_{p-1} + L_{p+1}}{1 + \eta_p(c)} \right) \right],
\]

- \( L_p = \) Legendre polynomial, \( \eta_p(c) = \frac{(c(2p + 1)(a_p p!))^2}{2} \)

- Family of energy-stable schemes known as VCJH Schemes

A Family of Energy-Stable Schemes

- Nonlinear flux: **aliasing instability**
- Set \( c > 0 \) to damp instability (similar to filtered Discontinuous Galerkin (DG))
- Choose \( c \) to compromise between (1) **accuracy**, (2) **damping**, (3) **CFL limit**
- Choose Gauss-Legendre points as solution points to minimise aliasing error

<table>
<thead>
<tr>
<th>( c )</th>
<th>Scheme</th>
<th>Damping</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 0 )</td>
<td>nodal DG</td>
<td>no damping</td>
</tr>
<tr>
<td>( \frac{2p}{(2p+1)(p+1)(p!a_p)^2} )</td>
<td>Spectral Difference</td>
<td>damped</td>
</tr>
<tr>
<td>( \frac{2(p+1)}{(2p+1)p(p!a_p)^2} )</td>
<td>( g_2 ) (Huynh 2007)</td>
<td>damped</td>
</tr>
<tr>
<td>tabulated(^1)</td>
<td>( c^+ )</td>
<td>damped</td>
</tr>
<tr>
<td>( \rightarrow \infty )</td>
<td>(-)</td>
<td>totally damped</td>
</tr>
</tbody>
</table>

\(^1\) Determined experimentally for each order \( p \) and RK timestepping scheme

**Taylor-Green Vortex at \( Re = 1600 \)**

- Analytical solution develops into turbulent spectrum in prescribed manner
- Allows quantification of numerical dissipation

**Isosurfaces of z-component of vorticity at \( t = 0 \) (L) and \( t = 20 \) (R)**

ONERA benchmark solution (2012)
Taylor-Green Vortex at $Re = 1600$

- 4th order accurate SD on hex meshes: $16^3$, $32^3$, $64^3$ elements
- Comparing to 4th order filtered DG (Beck and Gassner (2013)), DNS (Debonis (2012))

\[ \text{Average kinetic energy } \langle k \rangle = \frac{1}{V} \int k dV \]

\[ \text{Dissipation rate } -\frac{d}{dt}(\langle k \rangle) \]
Taylor-Green Vortex - Tetrahedral Meshes

- Tetrahedral meshes formed by splitting hex elements
- Identical results to hex meshes

**Average kinetic energy**
\[
\langle k \rangle = \frac{1}{V} \int k dV
\]

**Dissipation rate**
\[
-\frac{d}{dt}(\langle k \rangle)
\]
Taylor-Green Vortex - Effect of Scheme Order

- 3rd - 7th/6th order accurate SD on 16³ hex/tet meshes
- Same results with $g_2$ scheme (Huynh 2007)
- Aliasing instability appears for > 5 order on hexes
- Odd/even disparity, predicted by Jameson and Lodato (2013)
Spectral Difference Applied to Separated Turbulent Flows

- $L \to R$: SD7003 airfoil (Q-criterion isosurfaces, $Re = 6 \times 10^4$); sphere (vorticity isosurfaces, $Re = 1 \times 10^4$); flapping wing-body (entropy isosurfaces, $Re = 5000$)

Spectral Difference Applied to Separated Turbulent Flows

- **L to R:** SD7003 airfoil (Q-criterion isosurfaces, $Re = 6 \times 10^4$); sphere (vorticity isosurfaces, $Re = 1 \times 10^4$); flapping wing-body (entropy isosurfaces, $Re = 5000$)

- Numerical dissipation sufficient to stabilise separated flows at **moderate** $Re$
- Wider range of scales at real-world conditions ($Re > 10^7$)
- Under-resolution is inevitable - numerical dissipation insufficient
- Need **SGS model** to increase accuracy and stability

Introduction

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Summary and Future Directions
Advanced Subgrid-Scale (SGS) Models

(a) Functional SGS models: pure dissipation, e.g. eddy-viscosity
(b) Structural SGS models: recreation of SGS stress tensor, e.g. Similarity
• Mixed models combine advantages of functional and structural models
• Dynamic models scale functional model to suit local flow conditions

1 Lodato et al. (2009), Phys. Fl, 21(3). 2 Moin et al. (1991), Phys, Fl-A, (3)
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- WALE-Similarity Mixed Model (WSM):\(^1\)
  \[ \tau_{ij} = 2\bar{\rho}[\nu_t \hat{S}_{ij} - (\hat{u}_i \hat{u}_j - \hat{u}_i \hat{u}_j)] \]
  (a) WALE Eddy Viscosity
  (b) Similarity

- Compressible dynamic eddy-viscosity model: \(^2\)
  \[ \nu_t = C^2_S(x, t) \Delta^2 |\hat{S}|, \quad C_S(x, t) = f (\hat{u}_i \hat{u}_j, \hat{u}_i \hat{u}_j, |\hat{S}| \hat{S}_{ij}) \]

- \( \tilde{g} = \bar{\rho}g/\bar{\rho} = \) Favre-filtered quantity, \( \tilde{g} = F(\tilde{g}) = \) explicitly filtered quantity
- Explicit filter \( F \) filters out resolved scales \( L \leq \hat{\Delta} \)
- Nature of \( F \) is critical to model behaviour and accuracy

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\(^1\) Lodato et al. (2009), Phys. Fl, 21(3).
\(^2\) Moin et al. (1991), Phys, Fl. A, (3)
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Summary and Future Directions
Types of Filter

2D Filter kernels
- Gaussian
- Box
- Spectral

Box-filtered DNS
- Unfiltered
- Filtered (narrow)
- Filtered (wide)

Gaussian-blurred image
- Original
- Three pixel radius
- Ten pixel radius
Explicit Filter Operators For High-Order FR Schemes

1D High-Order Element, $\xi = [-1 : 1]$

- Limit kernel to element interior for compactness/parallelisability
- Asymmetric filter kernel
- Well-defined physical/spectral cut-off
- Consistent behaviour for different $N$

Generic Discrete Nodal Filter

- Weighted sum over $N$ nodes:
  $$u_i(\xi) = \sum_{j=1}^{N} \frac{1}{w_{ij}} u_j(\xi), \quad (i = 1, \ldots, N)$$

- Filter kernel $F$ has a characteristic filter width
- Transfer function measures effect on spectrum
Explicit Filter Operators For High-Order FR Schemes

1D High-Order Element, \( \xi = [-1 : 1] \)

![Solution Points and Flux Points](image)

**Specification**

- Limit kernel to element interior for compactness/parallelisability
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Explicit Filter Operators For High-Order FR Schemes

1D High-Order Element, $\xi = [-1 : 1]$

Solution Points
Flux Points

Generic Discrete Nodal Filter

- Weighted sum over $N$ nodes:
  \[
  \overline{u}_i(\xi) = \int_{-\infty}^{\infty} F(\xi_i - \xi) u(\xi) d\xi
  \approx \sum_{j=1}^{N} w_{ij} u_j, \quad (i = 1, \ldots, N)
  \]

- Filter kernel $F$ has a characteristic filter width $\Delta$
- Transfer function measures effect on spectrum

Specification

- Limit kernel to element interior for compactness/parallelisability
- Asymmetric filter kernel
- Well-defined physical/spectral cutoff
- Consistent behaviour for different $N$
Constrained Discrete Filter 1 (CD1)

- Integrate Gaussian filter kernel over Gauss-Legendre quadrature points:

$$\bar{u}_i(x) \approx K \sum_{j=1}^{N} w_i^G u_j \exp \left[ -6(\beta_{ij})^2 \right], \quad \beta_{ij} = (\xi_i - \xi_j) / \Delta_i, \quad \Delta_i = \alpha_i \Delta$$

- $w_i^G$ = Gauss quadrature weights, $\alpha_s = 1.5$ = filter width ratio, $\Delta = 2/N$

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Transfer function of CD1 for \( N = 3, 4, 5 \)

Constrained Discrete Filter 2 (CD2)

- Can be applied to arbitrarily spaced points
- Based on high-order-commuting filters of Vasilyev et al (1998)
- Determine $N$ weights $w_i^s$ for $s^{th}$ node from $N$ constraints:
  1. Weights sum to 1;
  2. Cutoff lengthscale $\Delta = \alpha \Delta$;
  3, $\ldots$, $N$. Enforce $(N - 1)^{th}$-order filtering error
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Transfer function of CD2 for $N = 3, 4, 5$

Pseudo-Spectral Methods and Modal Filtering

- VCJH schemes use Legendre polynomials $\phi(\xi)$ to represent a pseudo-spectrum
- Define modal solution as:

$$u_j(\xi) = \sum_{k=1}^{N} \hat{u}_{j,k} \phi_k(\xi)$$

- $\hat{u}_{j,k} =$ modal solution coefficients
- Can use a modal filter to damp high-order modes (short wavelengths)\(^1\)

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\(^1\) Metais & Lesieur (1992), J. Fl Mech. 239. \(^2\) Blackburn & Schmidt (2003), J. Comp. Phys. 186. \(^3\) Lodato et al. (2012), Int. J. Num. Meth. Fl. \(^4\) Presamuthan et al. (2009), AIAA P.
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Restriction-Prolongation (RP) or sharp modal cutoff filter\(^2,3,4\)

1. ‘Restrict’ (aggregate) solution to lower-order basis
2. ‘Prolongate’ (interpolate) solution back to original basis.

---

RP Filter Behaviour

- Gaussian curves in black for comparison
- Varying $\Delta$ across element
- Transfer function $> 1$: amplification at points near element interfaces

Comparison of Filters in LES of Turbulent Channel Flow

- Filtering extended to 2D and 3D (quadrilateral/hexahedral elements) by tensor products

Comparison of Filters in LES of Turbulent Channel Flow

- Filtering extended to 2D and 3D (quadrilateral/hexahedral elements) by tensor products

- Turbulent channel flow, $Re_\tau = 395$
- Compared to DNS of Moser et al.
- 4th-order-accurate SD scheme ($N=4$)
- WSM with (○) CD1, (×) CD2, (●) RP filters
- Element interfaces shown by vertical dotted lines

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LES of Flow Past a Square Cylinder at $Re = 21400$

- SGS model: WSM model, CD2 filter
- Numerical scheme: Spectral Difference, 4rd order accurate
- Time integration: 3rd order Runge Kutta
- No. of elements: 35760 ($2.3 \times 10^6$ DoF)
- Grid dimensions: $21D \times 12D \times 3.2D$
- Mach: 0.3

Lodato and Jameson (2012), 7th Int. Conf. CFD, Hawaii
LES of Flow Past a Square Cylinder at $Re = 21400$
LES of Flow Past a Square Cylinder at $Re = 21400$

Mean velocity and second moments in cylinder wake

- ○ experimental data, — SD with WSM model, ... SD without WSM model

Effect of using LES Model

- SD alone was unsatisfactory in these under-resolved computations
- SD + SGS significantly improved accuracy of mean velocity
- Less improvement in accuracy of second moments
- SD + SGS on finer mesh (not shown) – much better second moment accuracy
- At higher $Re$, hard to avoid under-resolution – SGS model is essential

Lodato and Jameson (2012), 7th Int. Conf. CFD, Hawaii
Introduction

*High-Order Discontinuous Numerical Schemes*

*High-Order LES*
- Advanced SGS Models
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- Current Work

*Summary and Future Directions*
Nodal Filters on Triangles

**Generic triangular kernel**

- At each solution point, filter is a weighted sum over all points

\[
 r_{sj} = \left( \frac{1}{4} \right) q_{\left( \cdot \right) j - s}^2 + \left( \cdot \right) j - s^2
\]
Nodal Filters on Triangles

**Generic triangular kernel**

- At each solution point, filter is a weighted sum over all points

- Apply CD1/CD2 nodal filters to $N$ triangle quadrature points
- Define filter kernel in **radial** coordinates:
  \[
  r_j^s = \frac{1}{\Delta} \sqrt{(\xi_j - \xi_s)^2 + (\eta_j - \eta_s)^2}
  \]

- What is the appropriate definition of $\Delta$?
- How will their transfer functions behave?
- Currently investigating properties
Extension of 1D Modal Filters to Triangles

- Tensor product not applicable to simplices (triangles, tetrahedra)
- Dubiner basis: warped product of Jacobi polynomials
- Apply modal filter to Dubiner modes
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**Dubiner transformation** \([-1, 1]^2 \rightarrow T^2\)

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**Dubiner transformation** $[-1, 1]^2 \to T^2$

**Dubiner transformation** $[-1, 1]^3 \to T^3$


Li and Wang (2010)
Other Modal Filters

Illustration of generic modal filter

Other Modal Filters

Illustration of generic modal filter

Examples

<table>
<thead>
<tr>
<th>Name</th>
<th>Filter $\hat{F}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Box</td>
<td>$0.5\pi k/k_c \sin(0.5\pi k/k_c)$</td>
</tr>
<tr>
<td>Gaussian</td>
<td>$\exp \left( \frac{-(\pi k/k_c)^2}{24} \right)$</td>
</tr>
<tr>
<td>Bouffanais$^2$</td>
<td>$\frac{1}{1+(k/k_c)^2}$</td>
</tr>
<tr>
<td>Boyd-Vandeven$^3$</td>
<td>...</td>
</tr>
</tbody>
</table>

Modal Filter Behaviours

Transfer function of Gaussian, Box and Boyd-Vandeven Modal Filters for $N = 3$

- Gaussian and Boyd-Vandeven filters show well-behaved transfer functions
Hyperviscosity and Modal Filtering

Spectral Vanishing Viscosity (SVV)\(^1\)

\[ \frac{\partial u_h}{\partial t} + \frac{\partial}{\partial x} (f(u_h)) = \epsilon_N (-1)^{p+1} \frac{\partial^{2p}}{\partial x^{2p}} u_h, \quad \epsilon_N = C_p / N^{2p-1}, \quad (2) \]

3 Meister et al. (2009), Math. Schriften Kassel.
**Hyperviscosity and Modal Filtering**

### Spectral Vanishing Viscosity (SVV)

- In 1D, add ‘hyperviscosity’ with strength $\epsilon_N$ to damp aliasing instability:

$$
\frac{\partial u_h}{\partial t} + \frac{\partial}{\partial x} (f(u_h)) = \epsilon_N (-1)^{p+1} \frac{\partial^{2p}}{\partial x^{2p}} u_h, \quad \epsilon_N = C_p / N^{2p-1}, \quad (2)
$$

### SVV as a Modal Filter

- (2) is equivalent to applying an exponential modal filter to solution:

$$
\tilde{u} = \sum_{|k| < N} \sigma(k/N) \hat{u}_k e^{ikx}, \quad \sigma(\eta) = \exp(-C_p N \Delta t \eta^{2p}) \quad (3)
$$

- $\sigma$ is exponential filter of order $2p$ and strength $\alpha = C_p N \Delta t$
- Choose $C_p$ so that $\alpha$ is constant for different $N$ ($\Delta t$ depends on $N$)
- Similar expression can be derived for Dubiner basis

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Connections Between Filtering, SGS Modeling and Stabilisation

- Stability of VCJH schemes due to implicitly containing an SVV-type term
- SVV term is purely dissipative - cannot model complex SGS physics
- Still need an SGS model e.g. WSM
- Investigating use of SVV-type modal filter as explicit filter in SGS model

Stabilisation

High-order Filter ↔ SGS model
Connections Between Filtering, SGS Modeling and Stabilisation

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Stabilisation

\[ \begin{array}{c}
\text{High-order Filter} \\
\leftrightarrow \\
\text{SGS model}
\end{array} \]

- Some combination of filtering, stabilisation and SGS modeling techniques will result in a stable and accurate method for real-world high-Re turbulent flows!
Introduction

High-Order Discontinuous Numerical Schemes

High-Order LES
  Advanced SGS Models
  Discrete Filters
  Current Work

Summary and Future Directions
Summary

- Need unstructured meshes, high-order numerics and LES modeling to simulate complex problems at realistic Reynolds numbers
- Developing explicit modal and nodal filters on triangles and tetrahedra
- Investigating SGS models appropriate for high-order schemes
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ACL Activities

- Developing an open-source high-order LES code
- Extending family of energy-stable Flux Reconstruction (VCJH) schemes
- Hybrid RANS/LES modeling
- Implicit timestepping
- Polynomial adaptivity (increase/decrease polynomial order)
- Polynomial multigrid convergence acceleration
- Moving meshes
Acknowledgements

The presented research is a combined effort by...

- Principal Investigator: Professor Antony Jameson
- Postdocs: Jonathan Bull, Guido Lodato, Peter Vincent
- PhD students: David Williams, Manuel R. Lopez Morales, Andy Chan, Patrice Castonguay, Yves Alleneau, Kui Ou, Sachin Presamuthan

It was made possible by the generous support of...

- The Airforce Office of Scientific Research under grant FA9550-10-1-0418 by Dr. Fariba Fahroo
- The National Science Foundation under grants 0708071 and 0915006 monitored by Dr. Leland Jameson
Thank you for listening