

# Solution of the Inverse Problem of Linear Optimal Control with Positiveness Conditions and Relation to Sensitivity

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## 1 Formulation

Let

$$\dot{x} = Ax + Bu, \quad (1.1)$$

where the dimensions of  $x$  and  $u$  are  $m$  and  $n$ , and let

$$u = Dx, \quad (1.2)$$

be a given control. It is desired to find a performance index

$$J = \int_{t_0}^{t_f} (x^T Qx + u^T Ru) dt + x^T(t_f) Fx(t_f), \quad (1.3)$$

with  $R = R^T > 0$ ,  $Q = Q^T \geq 0$ , which is minimized by  $u$ .

The solution of this problem without positiveness conditions on  $Q$  is given in ([?]). If a performance index (1.3) exists which is minimized by (1.2), then

$$-RD = B^T P, \quad (1.4)$$

where  $P$  is a symmetric matrix satisfying

$$-\dot{P} = A^T P + PA - D^T R D + Q, P(t_f) = F. \quad (1.5)$$

If (1.5) is multiplied on the left by  $x^T$  and on the right by  $x$ , then substituting from (1.1) and (1.4)

$$x^T Qx + u^T Ru = -\frac{d}{dt}(x^T P x) + (u - Dx)^T R(u - Dx)$$

Integrating from  $t_1$  to  $t_2$

$$\int_{t_1}^{t_2} (x^T Qx + u^T Ru) dt + x^T(t_2) P(t_2) x(t_2) = x^T(t_1) P(t_1) x(t_1) + \int_{t_1}^{t_2} ((u - Dx)^T R(u - Dx)) dt. \quad (1.6)$$

Setting  $t_1 = t_0$ ,  $t_2 = t_f$ ,  $P(t_f) = F$ , it is seen that

$$J \geq x^T(t_0)P(t_0)x(t_0) ,$$

since the final term is non-negative. Note also that on setting  $t_2 = t_f$  and  $u = Dx$  it follows that if  $Q \geq C$  and  $F \geq 0$ , then  $P(t_1) \geq 0$  for all  $t_1 < t_f$  because the left side is non-negative. Also multiplying (1.4) on the right by  $B$ , the symmetry of  $P$  is seen to imply the symmetry of  $RDB$ . The condition for the existence of  $R = R^T > 0$  and  $P = P^T > 0$  satisfying (1.4) are given in ([?]). They are:

A1:  $DB$  has  $n$  independent real eigen-vectors.

A2: The eigen-values of  $DB$  are non-positive.

A3:  $r_{DB} = r_D$ , where  $r_D$  denotes the rank of  $D$ , etc.

For  $P > 0$  A3 is replaced by

A3\*:  $r_{DB} = r_D = r_B$ .

If  $R = R^T > 0$  is given, then the conditions for a solution  $P \geq 0$  to (1.4) are

B1:  $RDB$  is symmetric.

B2:  $RDB \leq 0$ .

B3:  $r_{RDB} = r_{RD}$ .

For  $P > 0$  B3 is replaced by

B3\*:  $r_{RDB} = r_{RD} = r_B$

Here B3 and B3\* are simply restatements of A3 and A3\*, but B3 and B3\* would still be needed for a case where  $R$  is only non-negative.

## 2 The Sensitivity Inequality

The property that the feedback control minimizes some performance index (1.3) with  $Q \geq 0$  is of considerable interest because of its connection with the ability of the control to reduce the sensitivity of the system to parameter variations. Let  $\Delta x_c$  be the trajectory deviation resulting from plant variations,  $\Delta A$  and  $\Delta B$ , when the feedback control (1.2) is used, and let  $\Delta x_o$  be the deviation when (1.2) is replaced by an open loop control which would give the same trajectory in the absence of parameter deviations. Also let

$$\Delta A = \epsilon \delta A , \Delta B = \epsilon \delta B ,$$

and define  $\delta x = \lim_{\epsilon \rightarrow 0} \frac{\Delta x}{\epsilon}$ . Using the equivalence of controls when  $\delta A = \delta B = 0$  we have

$$\delta \dot{x}_c = (A + BD)\delta x_c + (\delta A + \delta BD)x , \quad (2.1)$$

$$\delta \dot{x}_0 = A\delta x_o + (\delta A + \delta BD)x . \quad (2.2)$$

Whence

$$\delta x_c = \delta x_o + \delta \quad (2.3)$$

where

$$\dot{\delta} = A\delta + BD\delta x_c \quad (2.4)$$

Then

$$\int_{t_0}^t \delta x_c^T D^T R D \delta x_c dt \leq \int_{t_0}^t \delta x_0^T D^T R D \delta x_o dt - \int_{t_0}^t \delta^T Y \delta dt, \quad (2.5)$$

for all  $t$  if the following condition is satisfied

C: The sensitivity inequality

$$S_y(t) = \int_{t_0}^t \{(u - Dx)^T R (u - Dx) - u^T R u\} dt - \int_{t_0}^t x^T Y x dt \geq 0 \quad (2.6)$$

holds, where  $x$  is the solution of (1.1) with  $x(t_0) = 0$  under an arbitrary input  $u$ .

This follows on setting  $u = D\delta x_c$  and interpreting  $x$  as  $\delta$ . Now setting  $t_1 = t_0$ ,  $x(t_0) = 0$ , and  $t_2 = t$  in (1.6) it is seen that C holds with  $Y = Q$  when the control (1.2) minimizes the performance index (1.3), provided that  $P(t) \geq 0$ . This is in turn ensured if  $Q \geq 0$  and  $F \geq 0$ . The non-negativeness of  $Q$  and  $F$  thus guarantees a reduction in sensitivity to parameter variations in the sense of 2.5.

### 3 Solution of the Inverse Problem with $Q \geq U$

It was remarked in Section 1 that  $Q \geq 0$  implies  $P \geq 0$ . If  $R$  is not specified, conditions A1, A2, and A3 are necessary for a solution of the inverse problem. Also A1, A2, and A3\* are necessary and sufficient for  $P > 0$ , and it will now be shown that the existence of a solution  $P > 0$  to (1.4) is sufficient for a solution with  $Q \geq 0$ . We thus have

**Theorem 3.1** *Conditions A1, A2 and A3 are necessary for a solution of the inverse problem with  $Q \geq 0$ . Conditions A1, A2 and A3\* are also sufficient.*

**Proof** Necessity has already been established. To prove sufficiency observe that from (1.1)

$$\frac{d}{dt}(xe^{-\alpha t}) = (A - \alpha I)xe^{-\alpha t} + Bue^{-\alpha t}. \quad (3.1)$$

Thus the control (1.2) minimizes the performance index

$$J = \int_{t_0}^{t_f} e^{-2\alpha t} (x^T Q_o x + u^T R_o u) dt + e^{-2\alpha t} x^T(t_f) F_o x(t_f). \quad (3.2)$$

If (1.4) holds together with

$$-\dot{P} = (A - \alpha I)^T P + P(A - \alpha I) - D^T R_o D + Q_o, P(t_f) = F_o. \quad (3.3)$$

Under conditions A1, A2 and A3\* it is possible to construct  $R = R^T > 0$  and  $P = P^T > 0$  satisfying (1.4). Then  $Q_o$  may be constructed as

$$Q_o = Q_1 + 2\alpha P, \quad (3.4)$$

where

$$Q_1 = D^T R_o D - A^T P - P A - \dot{P}. \quad (3.5)$$

Since  $P > 0$  and  $Q_1$  is a fixed function of  $t$ , it is always possible to choose  $\alpha > 0$  sufficiently large that  $Q_o \geq 0$ . Also comparison of (3.3) and (3.4) with (3.3) shows that the control (1.2) minimizes (3.2) and hence (1.3) on setting  $Q = Q_o e^{-2\alpha t}$ ,  $R = R_o e^{-2\alpha t}$ ,  $F = F_o e^{-2\alpha t}$ .

**Corollary 3.2** *If the control (1.2) satisfies conditions A1, A2 and A3\* then it satisfies the criterion (2.5) for sensitivity reduction for some  $Y \geq 0$ .*

**Proof** Observe that the procedure of Theorem (3.1) gives a joint solution for  $Q$  and  $R$ , but cannot be used to construct  $Q \geq 0$  when  $R$  is given. In particular, if the system is constant it leads to a performance with an exponential time weighting factor  $e^{-2\alpha t}$ ,  $\alpha > 0$ , and establishes the sensitivity criterion (2.5) with a similar factor.

## 4 Solution of the Inverse Problem with $Q \geq 0$ for given $R$

We now consider methods of solving for  $Q$  when  $R$  is a given matrix satisfying conditions B1, B2 and B3. To find additional requirements on  $R$  for  $Q \geq 0$  multiply (1.5) on the left by  $B^T$ . Then using (1.4)

$$-B^T \dot{P} = B^T A^T P + R D A - B^T D^T R D + B^T Q.$$

But differentiating (1.4)

$$-B^T \dot{P} - \dot{B}^T P = \frac{d}{dt}(R D).$$

Thus

$$B^T Q = L, \quad (4.1)$$

where

$$L = B^T D^T R D + R D A - (B^T A^T - \dot{B}^T) P + \frac{d}{dt}(R D). \quad (4.2)$$

Also multiplying (4.2) on the right by  $B$  and again using (1.4)

$$B^T Q B = M, \quad (4.3)$$

where

$$M = B^T D^T RDB + RDAB + B^T A^T D^T R + \frac{d}{dt}(RDB) - RD\dot{B} - \dot{B}^T D^T R. \quad (4.4)$$

Since  $M$  depends only on the system matrices and  $R$ , and  $B^T QB \geq 0$  if  $Q \geq 0$ , the necessary condition for  $Q \geq 0$  is:

B4:  $M \geq 0$  where  $M$  is defined in (4.4)

In order to construct  $Q \geq 0$  we shall assume the stronger condition:

B4\*  $M > 0$

As long as (1.4) holds

$$B^T L^T = M. \quad (4.5)$$

Thus if (4.1) is regarded as an equation for  $Q$  it has a symmetric solution

$$Q_o = L^T M^{-1} L.$$

Moreover if  $Q$  is any other solution then

$$B^T(Q - Q_o) = 0.$$

Thus the general symmetric solution for  $Q$  is

$$Q = L^T M^{-1} L + Y, \quad (4.6)$$

where  $Y$  is any symmetric matrix such that

$$B^T Y = 0. \quad (4.7)$$

Condition B4\* ensures that  $Q \geq 0$  if  $Y \geq 0$ . Also let

$$x = (I - BM^{-1}L)z,$$

where  $z$  is an arbitrary vector. Then

$$x^T Q x = z^T Y z.$$

Thus  $Y \geq 0$  is also necessary for  $Q \geq 0$ .

Consider the differential equation that is obtained when (4.6) is substituted for  $Q$  in (1.5):

$$-\dot{P} = A^T P + PA - D^T R D + L^T M^{-1} L + Y. \quad (4.8)$$

Since  $M$  is given and  $L$  is linear in  $P$ , this is a Ricatti equation which can be integrated to determine  $P$  and hence  $L$ . We shall verify that (4.8) has solutions that satisfy (1.4). Define  $K$  by

$$K = B^T P + RD \quad (4.9)$$

so that  $K = 0$  if (1.4) holds. Then when (1.4) is no longer assumed to hold (4.2) and (4.4) yield

$$B^T L^T = M - K(AB - \dot{B}) \quad (4.10)$$

instead of (4.5). Also using (4.8)

$$\begin{aligned} -\dot{K} &= -B^T \dot{P} - \dot{B}^T P - \frac{d}{dt}(RD) \\ &= B^T A^T P + B^T P A - B^T D^T R D + B^T L^T M^{-1} L + B^T Y - \dot{B}^T P - \frac{d}{dt}(RD), \end{aligned}$$

whence in view of (4.2), (4.7) and (4.10)

$$\begin{aligned} -\dot{K} &= -L + K A + \left[ M - K(AB - \dot{B}) \right] M^{-1} L \\ &= K \left[ A - (AB - \dot{B}) M^{-1} L \right]. \end{aligned} \quad (4.11)$$

Under conditions B1, B2 and B3 it is possible to choose  $P(t_f) \geq 0$  satisfying (1.4). Then  $K(t_f) = 0$  and the solution of (4.11) when integrated backwards is  $K = 0$ . The corresponding solution of (4.8) therefore satisfies (1.4).

**Theorem 4.1** *Given  $R = R^T > 0$ , conditions B1-B4 are necessary for a solution of the inverse problem with  $Q \geq 0$ . Conditions B1-B3 are sufficient for a solution over some finite time interval. Every solution with  $Q \geq 0$  is then given by the solution of (4.2) and (4.8) for some  $Y \geq 0$  satisfying (4.7).*

If equation (4.11) is unstable when integrated backwards then the integration of (4.8) would tend to drift away from satisfying (1.4). To overcome this difficulty we may introduce instead of  $L$  and  $M$  the matrices

$$L_1 = B^T D^T R D + R D A_1 + B^T P(A_1 - A) - (B^T A^T - \dot{B}^T) P + \frac{d}{dt}(RD), \quad (4.12)$$

and

$$M_1 = B^T L_1 \quad (4.13)$$

where  $A_1$  is a matrix to be selected. Then

$$L_1 = L + K(A_1 - A),$$

and from (4.10)

$$\begin{aligned} M_1 &= B^T L^T + B^T (A_1^T - A^T) K^T \\ &= M - K(AB - \dot{B}) - (B^T A^T - B^T A_1^T) K^T. \end{aligned}$$

Thus  $L_1 = L$  and  $M_1 = M$  when  $K = 0$ . We now integrate

$$-\dot{P} = A^T P + P A - D^T R D + L_1 M_1^{-1} L_1 + Y \quad (4.14)$$

Then from (4.7) and (4.13)

$$-\dot{K} = B^T A^T P + B^T P A - B^T D^T R D + L_1 - \dot{B}^T P - \frac{d}{dt}(R D)$$

whence (4.12) yields

$$-\dot{K} = K A_1$$

Thus if  $A_1$  is chosen as any stable matrix, backward integration of (4.14) will preserve  $K = 0$  without danger of drift. But then  $L_1 = L$  and  $M_1 = M$ , so that along the integration path  $M_1 = M_1^T$  and under condition B4\*,  $M_1 > 0$ .

While conditions B1-B3 and B4\* establish the existence of  $Q \geq 0$  over some finite time interval, and every  $Q$  can then be constructed from equation (4.6) or (4.14), these conditions do not establish the existence of  $Q \geq 0$  over an arbitrarily large time interval, because equation (4.8) or (4.14) which follows the same integration path, may have a finite escape time.

In particular, conditions B1-B3 and B4\* are not sufficient for the existence of a solution with constant  $Q \geq 0$  and  $R > 0$ , when  $A$ ,  $B$  and  $D$  are constant. This is easily seen from an example. Consider the system

$$\dot{x} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u ,$$

with

$$u = \begin{bmatrix} -2 & \frac{1}{2} \end{bmatrix} x ,$$

where is is desired to find a performance index

$$J = \int_0^{\infty} (x^T Q x + u^2) dt ,$$

which is minimized by  $u$ .  $RDB$  and  $M$  are scalars,

$$RDB = -2 ,$$

and

$$M = (B^T D^T)^2 + 2DAB = 5 .$$

So conditions B1 - B3 and B4\* all hold. On the other hand (1.4) may be solved for  $P$  to give

$$P = \begin{bmatrix} 2 & -\frac{1}{2} \\ -\frac{1}{2} & P_{22} \end{bmatrix} ,$$

where  $P_{22}$  is the only undetermined element in  $P$ . Then substituting for  $P$  in (1.5) with  $\dot{P} = 0$  gives

$$\begin{aligned} Q &= D^D - A^T P - P A \\ &= \begin{bmatrix} 5 & 1 - P_{22} \\ 1 - P_{22} & -\frac{3}{4} \end{bmatrix} , \end{aligned}$$

so that it is not possible to obtain  $Q \geq 0$  by choice of  $P_{22}$  when  $P$  is constant.

This example indicates the need to examine the conditions under which (4.8) or (4.14) can be integrated over an arbitrary interval. Since (4.14) follows the same path as (4.8) when integrated from a final value of  $P$  which satisfies (1.4), it suffices to consider (4.8). Substituting from (4.2), it may be written as

$$-\dot{P}_1 = A^T P_1 + P_1 A + D^T R D - Y - D_1^T M D_1 , \quad (4.15)$$

where

$$P_1 = -P , \quad (4.16)$$

and

$$D_1 = -M^{-1}(B^T A^T P_1 + \dot{B}^T D^T R D + R D A + \frac{d}{dt}(R D) - \dot{B}^T P_1) \quad (4.17)$$

Let

$$\dot{x}_1 = A x_1 + A B u_1 - \dot{B} u_1 . \quad (4.18)$$

Then equations (4.15) and (4.17) are equations for determining the control

$$u_1 = D_1 x_1 , \quad (4.19)$$

which minimizes

$$\begin{aligned} J_1 &= \int_{t_1}^{t_f} \{ x_1^T (D^T R D - Y) x_1 \\ &+ 2u_1^T (B^T D^T R D + R D A + \frac{d}{dt}(R D)) x_1 + u_1^T M u_1 \} dt \\ &+ x_1^T(t_f) P_1(t_f) x_1(t_f) . \end{aligned} \quad (4.20)$$

Substituting for  $M$  from (4.4) and using (4.7), (4.20) becomes

$$\begin{aligned} J_1 &= \int_{t_1}^{t_f} (x_1 + B u_1)^T (D^T R D - Y) (x_1 + B u_1) dt \\ &+ 2 \int_{t_1}^{t_f} u_1^T (R D A + \frac{d}{dt}(R D)) (x_1 + B u_1) dt \\ &- \int_{t_1}^{t_f} u_1^T \frac{d}{dt}(R D B) u_1 dt + x_1^T(t_f) P_1(t_f) x_1(t_f) . \end{aligned} \quad (4.21)$$

Set

$$x = x_1 + B u_1 . \quad (4.22)$$

Then  $x$  satisfies (1.1) where  $\dot{u}_1 = u$ , or

$$u_1 = \int_{t_1}^{t_f} u dt , \quad (4.23)$$

there being no constant if (4.18) and (1.1) are both to be in equilibrium with zero control.



The second and third terms in (4.21) become

$$\begin{aligned}
2 \int_{t_1}^{t_f} u_1^T (RDA + \frac{d}{dt}(RD))(x) dt &- \int_{t_1}^{t_f} u_1^T \frac{d}{dt}(RDB)u_1 dt \\
&= \int_{t_1}^{t_f} \left\{ 2u_1^T RD\dot{x} + 2u_1^T \frac{d}{dt}(RD)x - 2u_1^T RDB\dot{u}_1 - u_1^T \frac{d}{dt}(RDB)u_1 \right\} dt \\
&= [2u_1^T RDx - u_1^T RDBu_1]_{t_1}^{t_f} - 2 \int_{t_1}^{t_f} u RDx dt .
\end{aligned}$$

Also using (1.4) and (4.16)

$$x_1^T P_1 x_1 + 2u_1^T RDx_1 + u_1^T RDBu_1 = x^T P_1 x = -x^T P x .$$

Thus (4.21) becomes

$$J_1 = S_y - x^T(t_f)P(t_f)x(t_f) + I(t_1) , \quad (4.24)$$

where  $S_y$  is defined by (2.6) and

$$I = -2u_1 RDx_1 - u_1 RDBu_1 . \quad (4.25)$$

Now (4.23) shows that a non-zero value of  $u_1$  at  $t = t_1$  corresponds to an impulse at  $t = t_1$  in  $u$

$$u = u_1(t_1)\delta(t - t_1) .$$

But under such an impulse  $x$  is shifted from  $x_0^-$  to  $x_0^+ = x_0^- + Bu_1(t_1)$  with a contribution to  $S_y$  exactly equal to  $I(t_1)$ , since  $x_1$  is continuous so that  $x_1(t_1) = x_0^-$ . Thus the first and third terms in  $J_1$  equal  $S_y$  evaluated from  $t_1^-$  in case of an initial impulse in  $u$ . It follows that  $-x^T(t_1)P(t_1)x(t_1)$  is the minimum value of  $S_y$  when  $x(t_1) = x_0^-$ . Suppose that this quantity is not bounded. Then since the system is controllable, it can be brought from  $x(t_0) = 0$  to  $x(t_1) = x_0^-$  with a finite contribution to  $S_y$ , and hence over the interval  $(t_0, t_f)$  condition C would be violated. We deduce that condition C is sufficient for the existence of a solution to (4.8).