Optimization of Linear Systems of Constrained Configuration

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1 Abstract

For the sake of simplicity it is often desirable to restrict the number of feedbacks in a controller. In this case the optimal feedbacks depend on the disturbance to which the system is subjected. Using a quadratic error integral as a measure of the response, three criteria of optimization are considered:

(1) The response to a given initial disturbance.

(2) The worst response to an initial disturbance of given magniitude.

(3) The worst comparison with the unconstrained optimal system.

It is shown that for each of these criteria the gradient with respect to the feedbacks can be calculated by a uniform method. The solution may then be found either directly or by a descent procedure. The method is illustrated by an example.

2 Introduction

There exists a well-developed theory for the optimal regulation of a linear system (Kalman 1960). The optimal controller incorporates feedbacks from every state variable. It thus generates the control signal from the minimum amount of information necessary to predict the motion of the system. If some of the state variables cannot be measured they may be reconstructed from the available measurements by an observer (Luenberger 1964) or, if the measurements are noisy, by an optimal estimator (Kalman and Bucy 1961). Such systems are complex. The engineer, however, generally wishes to produce the simplest acceptable system. In fact it is often possible to produce an acceptable system, with sufficient stability in all modes, by incorporating just a few feedbacks. Another reason for eliminating feedbacks is that the linear equations may only approximately represent a system which is actually nonlinear. If, for example, the lateral motion of an aircraft is represented by linearized equations, the resulting optimal control includes a feedback from the roll angle. It is evident that such a signal should either be excluded, or else limited, when the roll angle is large. Both because of practical considerations of this kind, and in order to assess the trade-off between performance and complexity, it is thus often desirable to consider the design of a specific optimum controller (Koivuniemi 1967, Rekasius 1967) with a pre-determined feedback configuration. The solution of this problem for several different criteria of optimization is the subject of this paper.

If the performance criterion is a quadratic error integral, the unconstrained problem; reviewed in 4, has a solution which is optimal for all initial conditions of the system. The required feedbacks vary with time unless the system is optimized over an infinite interval. When, however, the configuration of the controller is constrained, a solution which is independent of the initial conditions can no longcr be found, and it is then not fruitful to permit time varying feedbacks. With constant feedbacks the problem is reduced to an ordinary minimum problem in the parameter space of the feedbacks, for which the necessary condition of optimality is that the gradient with respect to each allowed feedback must vanish. It is shown in 5 how to calculate this gradient for a fixed initial condition. If the system is to operate satisfactorily for a range of initial disturbances, it may not be clear which is the most suitable to choose for optimization. This difficulty can be avoided by choosing the worst case as the critical design case. It then becomes necessary to consider min-max criteria. Sections 6 and 7 treat optimization for two such criteria, the worst response to an initial disturbance of given magnitude, and the worst comparison with the unconstrained optimal system. In each case it is shown that the gradient can be represented by a suitable specialization of the formulae derived in 5. The solution for a fixed initial condition and for both the min-max criteria can thus be computed by a uniform method. This is the principal result of the paper. A simple example is solved in 8, to illustrate the influence of the performance criterion, and the application of the results of 5,6,7.

3 Mathematical Formulation

Consider a linear system

$$\dot{x} = Ax + Bu \, , x(0) = x_0 \, , \tag{1}$$

where the n vector x represents the state, and the m vector u is the control. Let the output be the p vector:

$$y = Cx \, , \tag{2}$$

and take as a measure of performance the quadratic integral:

$$J = \int_0^T \left(y^T Q y + u^T R u \right) dt \, , \tag{3}$$

where Q and R are positive definite. Suppose that it is desired to use a feedback control;

$$u = Dx \ , \tag{4}$$

where there may be restrictions on the allowed feedbacks D_{ij}. Then

$$\dot{x} = Fx \ , x(0) = x_0 \ , \tag{5}$$

where

$$F = A + BD \ . \tag{6}$$

Also,

$$J = \int_0^T x^T Sxdt \ , \tag{7}$$

where

$$S = C^T QC + D^T RD \ . \tag{8}$$

If (5) is integrated and substituted in (7), then J can be expressed as a quadratic form in the initial conditions:

$$J = x_0^T P(0) x_0 \ , \tag{9}$$

where it can be verified by differentiating the performance index measured from t to T with respect to t and using (5) that:

$$-\dot{P} = F^T P + PF + S \ , P(T) = 0 \ . \tag{10}$$

If the system is constant and is to be optimized over an infinite interval, then as long as the feedbacks are such that the system is stable, P approaches a constant value which may be determined by setting

$$\dot{P} = 0$$

in (10) and solving the resulting Lyapunov matrix equation.

4 Review of results in the absence of constraints

When a small variation δD is made in the feedback matrix, (9) and (10) yield:

$$\delta J = x_0^T \delta P(0) x_0 \ , \tag{11}$$

where

$$-\delta\dot{P} = F^T \delta P + \delta PF + \delta D^T (B^T P + RD) + (B^T P + RD)^T \delta P \ , \delta P(T) = 0 \ , \tag{12}$$

δP and δJ can then be determined by integrating (12). Note that if:

$$D = -R^{-1} B^T P \ , \tag{13}$$

then according to (12)

$$\delta P = 0$$

This is Kalman's (1960) solution for the case where there is no restriction on the allowed feedbacks. According to (11) it is optimal for all initial conditions. It may be computed by substituting (13) in (10) and integrating backwards the resulting matrix Riccati equation:

$$-\dot{P} = A^T P + PA + C^T QC - P^T BR^{-1} B^T T \ , P(T) = 0 \ . \tag{14}$$

It is easily shown that Kalman's solution is in fact a global minimum. If a variation ΔD, not necessarily small is made in D, then by comparison of (10) and (14):

$$-\Delta\dot{P} = F^T \Delta P + \Delta PF + \Delta D^T R\Delta D \ , \Delta P(T) = 0 \ . \tag{15}$$

Combining (15) and (5)

$$-\frac{d}{dt}\left(x^T Px\right) = x^T \Delta D^T R\Delta Dx \ .$$

Since δP vanishes at the upper boundary, it follows on integrating this equation that:

$$\Delta J = x_0^T \Delta P(0) x_0 = \int_0^T x^T \Delta D^T R\Delta Dxdt \ ,$$

and because R is positive definite:

$$\Delta J \geq 0 \ .$$

In general the optimal feedback gains vary with time. If, however, the system is constant and is optimized over an infinite time interval, the elimination of $\dot{P}$ from (10) leads to its elimination from the variational equations. The optimal feedback gains are then constant, and maybe determined by setting

$$\dot{P} = 0$$

in (14) and finding the positive definite solution of the resulting matrix equation. (Kalman 1960)

5 Optimization with given initial condition

When there is a restriction on the allowable feedbacks Kalman's solution (13)-(14) is no longer available. Then according to (12) δP will not vanish, but it is necessary for optimality that:

$$\delta J = x_0^T \delta P(0) x_0 = 0 \ .$$

The solution will thus depend on the initial condition.

Given a particular initial condition x_0, the optimal control signal $u(t)$ of the open-loop system can be determind. If the number of allowed feedbacks is greater than the dimension of u, and the feedbacks are allowed to vary with time, then this signal can be generated in infinitely many ways by choosing some of the feedbacks arbitrarily and solving for the remainder. We therefore, consider only systems with constant feedbacks. Let G be the gradient matrix with elements:

$$G_{ij} = \frac{\partial J}{\partial D_{ij}} \ .$$

If the constrained configuration is optimal, then it is necessary that G_{ij} vanish for each allowed feedback D_{ij}.

To determine G it is convenient to introduce the outer product:

$$X = xx^T \ .$$

In terms of X, the system equations (5) become:

$$\dot{X} = FX + XF^T \ , X(0) = x_0 x_0^T \tag{16}$$

and the expression for the performance index (7) becomes:

$$J = \int_0^T tr(SX)dt \ . \tag{17}$$

Thus

$$\delta J = \int_0^T \left\{ tr(S\delta X) + 2tr(\delta D^T RDX) \right\} dt \ . \tag{18}$$

Also it follows from (16) that:

$$\delta\dot{X} = F\delta X + \delta XF^T + B\delta DX + X\delta D^T B^T \ , \delta X(0) = 0 \ . \tag{19}$$

Equations (10) and (19) are an adjoint pair. Remembering that:

$$trAB = trBA \ ,$$

for any two matrices A and B, they can be combined to give:

$$\frac{d}{dt} tr(P\delta X) = 2tr(\delta D^T B^T PX) - tr(S\delta X) \ .$$

Since $P\delta X$ vanishes at both boundaries, it follows on integrating this equation that:

$$\int_0^T tr(S\delta X)dt = 2\int_0^T tr(\delta D^T B^T PX)dt \ .$$

Substituting this result in (18):

$$\delta J = 2\int_0^T tr\{\delta D^T (B^T P + RD)X\}dt \ ,$$

whence

$$G = 2\int_0^T (B^T P + RD) X dt \ . \tag{20}$$

If the system is constant and is to be optimized over an infinite interval, then P is constant and

$$G = 2(B^T P + RD)W \ , \tag{21}$$

where

$$W = \int_0^\infty X dt \ .$$

Integrating (16), it is found that:

$$FW + WF^T + x_0 x_0^T = 0 \ . \tag{22}$$

For a simple system, the optimal feedbacksmay be found by solving the equations which are obtained when the gradient with respect to each allowed feedback is required to vanish. Generally however this is impractical, and one has to search directly for the minimum in the parameter space of the feedbacks. The gradient may then be used to determine a favorable direction for each step. The most effective procedures of this kind seem to be the conjugate gradient method (Fletcher and Reeves 1964), and the Fletcher-Powell-Davidson method (Fletcher and Powell 1963).

6 Optimization of worst response to an initial disturbance of given magnitude

It has been shown that the optimal feedbacks of a constrained configuration depend on the initial disturbance, and it is not at all certain that a system optimized for one disturbance will be satisfactory for another. For any choice of feedbacks there will be a least favorable initial disturbance of given magnitude which will maximize J. To ensure the zcceptability of the system for all initial disturbances we can use the response in this worst case as a more stringent criterion of optimization. (Kalman and Bertram 1960, Koivuniemi 1967). Instead of J we then minimize:

$$M = \max_{x_0} \frac{J}{x_0^T x_0} = \max_{x_0} \frac{x_0^T P(0) x_0}{x_0^T x_0} \ . \tag{23}$$

It is well known (Bellman 1960) that this ratio is equal to the maximum characteristic value of P. In fact, since P is symmetric, it has real characterictic values λ and its characteristic vectors v may be formed as an orthonormal set. If V is a characteristic matrix with columns v_i, then

$$V^T V = I$$

and

$$V^T P V = \Lambda \ ,$$

where Λ is a diagonal matrix with elements λ_i. Suppose that:

$$x_0 = Vz \ .$$

Then

$$\frac{J}{x_0^T x_0} = \frac{z^T V^T P V z}{z^T V^T V z} = \frac{z^T \Lambda z}{z^T z} = \frac{\sum_i \lambda_i z_i^2}{\sum_i z_i^2}$$

and since

$$\lambda_{min} \sum_i z_i^2 \leq \sum_i \lambda_i z_i^2 \leq \lambda_{max} \sum_i z_i^2 \ ,$$

it can be seen both that:

$$M = \lambda_{max}(P) \tag{24}$$

and that:

$$\min_{x_0} \frac{J}{x_0^T x_0} = \lambda_{min}(P) \ .$$

The gradient of M with respect to the feedbacks can also be easily determined. Consider first the variation of the characteristic values when P is varied (Bellman 1960). If v is a characteristic vector of unit length, then the corresponding characteristic value is:

$$\lambda = v^T P v \ . \tag{25}$$

Let $v + \delta v$ be the new characteristic vector when P is varied. If $v + \delta v$ is also of unit length then:

$$\delta v^T v = 0$$

and it follows from (25) that:

$$\begin{aligned} \delta\lambda &= \delta v^T P v + v^T \delta P v + v^T P \delta v \\ &= \lambda \delta v^T v + v^T \delta P v + \lambda v^T \delta v \\ &= v^T \delta P v \ . \end{aligned}$$

Thus:

$$\delta M = v^T \delta P(0) v \ ,$$

where v is a characteristic vector of unit length corresponding to the maximum characteristic value of $P(0)$. Comparing this formula with (11), it is apparent that the gradient of M can be determined by (20), where the initial condition of (16) is now taken as:

$$X(0) = vv^T \ .$$

The procedure described in sec. 5 for a solution with fixed initial condition can thus be carried over to the solution for the min-max criterion (23).

7 Optimization of worst comparison with the unconstrained optimal system

The index M is a measure of the response to the worst initial disturbance of given magnitude. The normalization, and consequently M, will depend on the system of units used in formulation the system equations. This difficulty may be avoided by comparing, at the least favorable initial condition, the actual performance index J and the unrestricted optimal index J_{opt} obtained from the Kalman solution, that is, by minimizing (Rekasius 1967):

$$L = \max_{x_0} \frac{J}{J_{opt}} = \max_{x_0} \frac{x_0^T P(0) x_0}{x_0^T P_{opt}(0) x_0} \;, \tag{26}$$

where P_{opt} is determined from the matrix Ricatti equation (14). If the system is controllable and observable P_{opt} is positive definite and can be factored as $K^T K$, where K is non-singular. Setting

$$L = \max \frac{z^T (K^T)^{-1} P K^{-1} z}{z^T z} = \lambda_{max} (K^T)^{-1} P K^{-1} \;.$$

But if

$$(K^T)^{-1} P K^{-1} v = \lambda v \;,$$

then, multiplying by K^{-1}:

$$P_{opt}^{-1} P w = \lambda w \;,$$

where

$$w = K^{-1} v \;.$$

Thus

$$L = \lambda_{max} (P_{opt}^{-1} P) \;.$$

By a similar argument:

$$\min_{x_0} \frac{J}{J_{opt}} = \lambda_{min} (P_{opt}^{-1} P) \;.$$

It is easily verified that these expressions are invariant under a transformation of state variables.

To obtain the gradient of L with respect to the feedbacks we note that in this case if v is a characteristic vector of unit length then:

$$\delta\lambda = v^T (K^T)^{-1} \delta P K^{-1} v = w^T \delta P w \;.$$

Thus (20) can again be used, but now the initial condition of (16) should be:

$$X(0) = w w^T \;,$$

where w is the characteristic vector corresponding to the maximum characteristic value of $P_{opt}^{-1} P$, and the length of w is determined by the requirement that v is of unit length, or

$$v^T v = w^T P_{opt} w = 1 \;.$$

The method of sec. 5 suffices for the criterion of (26) also.

8 Example: modification of a harmonic oscillator

As an example consider the use of feedbacks to modify a harmonic oscillator

$$\ddot{y} + y = 0 \ ,$$

The equations can be formulated as

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \ ,$$

where

$$y = x_2$$

and

$$u = d_1 x_1 + d_2 x_2 = d_1 \dot{y} + d_2 y \ .$$

Let the initial conditions be

$$x_0 = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

and consider the minimization of

$$J = \int_0^\infty (8y^2 + u^2) dt \ .$$

First suppose that there are no constraints on the configuration. Since the optimization interval is infinite the matrix Riccati equation reduces to the matrix quadratic equation

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} + \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 8 \end{bmatrix} - \begin{bmatrix} P_{11}^2 & P_{11}P_{12} \\ P_{21}P_{11} & P_{12}^2 \end{bmatrix} = 0 \ ,$$

whence

$$P_{opt} = \begin{bmatrix} 2 & 2 \\ 2 & 6 \end{bmatrix} \ .$$

The optimal feedbacks are then

$$d_1 = -2 \ , \quad d_2 = -2 \ .$$

The natural frequency of the optimal system is thus raised to $\sqrt{3}$ radians per second and its damping ratio is $1/\sqrt{3}$.

Suppose now it is desired to optimize the system using rate feedback only, so that

$$u = d_1 \dot{y} = d_1 x_1 \ .$$

The system is stable if

$$d_1 < 0$$

and P can be determined from the Lyapunov matrix equation:

$$\begin{bmatrix} d_1 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} + \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} d_1 & -1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 8 \end{bmatrix} - \begin{bmatrix} d_1^2 & 0 \\ 0 & 0 \end{bmatrix} = 0 .$$

the solution is:

$$P = \begin{bmatrix} -\left(\frac{d_1}{2} + \frac{4}{d_1}\right) & 4 \\ 4 & -\left(\frac{9d_1}{2} + \frac{4}{d_1}\right) \end{bmatrix} .$$

The gradient matrix is

$$\begin{bmatrix} g_1 & g_2 \end{bmatrix} = 2\left\{ \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} + \begin{bmatrix} d_1 & 0 \end{bmatrix} \right\} \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix} ,$$

where

$$\begin{bmatrix} d_1 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix} + \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix} \begin{bmatrix} d_1 & -1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} a_1^2 & a_1 a_2 \\ a_2 a_1 & a_2^2 \end{bmatrix} = 0 .$$

Thus

$$2W = \begin{bmatrix} \frac{a_1^2 + a_2^2}{d_1} & -a_2^2 \\ -a_2^2 & \frac{a_1^2+a_2^2}{d_1} - a_2^2 d_1^2 + 2a_1 a_2 \end{bmatrix}$$

and

$$g_1 = 2(P_{11} + d_1)W_{11} + 2P_{12}W_{21} = \left(\frac{4}{d_1^2} - \frac{1}{2}\right) a_1^2 + \left(\frac{4}{d_1^2} - \frac{9}{2}\right) a_2^2 .$$

If d_1 is optimal then g_1 must vanish, whence

$$d_1 = -2\sqrt{\frac{2a_1^2 + 2a_2^2}{a_1^2 + 9a_2^2}} .$$

If the system has an initial velocity but no initial displacement

$$a_2 = 0 , \quad d_1 = -2\sqrt{2} = -2.828 ,$$

and the damping ratio of the closed loop system is 1.414. If it has an initial displacement but no initial velocity, then

$$a_1 = 0 , \quad d_1 = \frac{-2\sqrt{2}}{3} = -0.943 ,$$

and the damping of the closed loop system is 0.471. The optimal closed loop system thus changes substantially when the initial condition is changed.

To optimize the worst response to an initial disturbance of given magnitude, or the index M is eq. (23), the characteristic values of P may be determined from the characteristic equation:

$$\lambda^2 + \lambda\left(5d_1 + \frac{8}{d_1}\right) + \frac{9d_1^2}{4} + 20 + \frac{16}{d_1^2} = 0 .$$

The roots are:

$$\lambda_{max}(P) = -\left(\frac{5}{2}d_1 + \frac{4}{d_1}\right) + 2\sqrt{(d_1^2+4)} = M \ ,$$

and

$$\lambda_{min}(P) = -\left(\frac{5}{2}d_1 + \frac{4}{d_1}\right) - 2\sqrt{(d_1^2+4)} \ .$$

The characteristic vector of unit length corresponding to λ_{max} is:

$$v = \frac{1}{\sqrt{(c^2+1)}}\begin{bmatrix} c \\ 1 \end{bmatrix} \ ,$$

where

$$c = \frac{d_1}{2} + \sqrt{\left(\frac{d_1^2}{4}+1\right)} \ .$$

The gradient of M with respect to d_1 is found by substituting v for the initial condition. Thus

$$g_1 = \frac{\left(\frac{4}{d_1^2} - \frac{1}{2}\right)c^2 + \left(\frac{4}{d_1^2} - \frac{9}{2}\right)}{c^2+1} = \frac{4}{d_1^2} - \frac{5}{2} - \frac{2}{\sqrt{(4/d_1^2+1)}}$$

as may be verified in this case by direct differentiation. Substituting

$$d_1 = \frac{-2}{\sqrt{\left(\frac{4}{3}+\frac{7}{3}\cos\theta\right)}}$$

It is found that the gradient vanishes when

$$\cos 3\theta = \frac{89}{343}$$

yielding for the optimal rate feedback:

$$d_1 = -1.077 \ .$$

The closed loop system then has a damping ratio of 0.539. Also

$$\lambda_{max}(P) = 10.950$$

and

$$\lambda_{min}(P) = 1.863 \ .$$

thus over the range of all initial vectors of unit length:

$$1.863 \le J \le 10.950 \ .$$

Note that when both feedbacks are allowed:

$$\lambda_{max}(P_{opt}) = 4 + 2\sqrt{2} = 6.828$$

and

$$\lambda_{min}(P_{opt}) = 4 - 2\sqrt{2} = 1.172$$

so that over the same range:

$$1.172 \le J_{opt} \le 6.828 \ .$$

The worst comparison with the unconstrained optimal system is represented by the index L (26), or the maximum characteristic value of:

$$\begin{aligned} P_{opt}^{-1}P &= \begin{bmatrix} \frac{3}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} -\left(\frac{d_1}{2} + \frac{4}{d_1}\right) & 4 \\ 4 & -\left(\frac{9d_1}{2} + \frac{4}{d_1}\right) \end{bmatrix} \\ &= \begin{bmatrix} -\left(\frac{3d_1}{8} + 1 + \frac{3}{d_1}\right) & \left(\frac{9d_1}{8} + 3 + \frac{1}{d_1}\right) \\ \left(\frac{d_1}{8} + 1 + \frac{1}{d_1}\right) & -\left(\frac{9d_1}{8} + 1 + \frac{1}{d_1}\right) \end{bmatrix} \end{aligned}$$

The characteristic equation is then:

$$\lambda^2 + \lambda\left(\frac{3d_1}{2} + 2 + \frac{4}{d_1}\right) + \frac{9d_1^2}{32} + \frac{1}{2} + \frac{2}{d_1^2} = 0 \ ,$$

with roots:

$$\lambda_{max}\left(P_{opt}^{-1}P\right) = -\left(2 + \sqrt{2}\right)\left(\frac{3d_1}{8} + \frac{1}{\sqrt{2}} + \frac{1}{d_1}\right) = L$$

and

$$\lambda_{min}\left(P_{opt}^{-1}P\right) = -\left(2 - \sqrt{2}\right)\left(\frac{3d_1}{8} - \frac{1}{\sqrt{2}} + \frac{1}{d_1}\right)$$

the characteristic vector w corresponding to λ_{max} with length such that:

$$w^T P_{opt} w = 1 \ ,$$

is

$$w = \frac{1}{\sqrt{(2c^2 + 4c + 6)}} \begin{bmatrix} c \\ 1 \end{bmatrix} \ ,$$

where

$$c = -(\sqrt{2} - 1)\frac{3d_1 + 4 + 2\sqrt{2}}{d_1 + 4 - 2\sqrt{2}} \ .$$

Substituting w for the initial condition, the gradient with respect to d_1 is found to be:

$$g_1 = \frac{\left(\frac{4}{d_1^2} - \frac{1}{2}\right)c^2 + \left(\frac{4}{d_1^2} - \frac{9}{2}\right)}{2c^2 + 4c + 6} = (2 + \sqrt{2})\left(\frac{1}{d_1^2} - \frac{3}{8}\right) \ .$$

The optimal rate feedback according to this criterion is:

$$d_1 = -2\sqrt{\frac{2}{3}} = -1.633$$

and the corresponding damping ratio is 0.816. Also then:

$$\lambda_{max}\left(P_{opt}^{-1}P\right) = (\sqrt{2}+1)(\sqrt{3}-1) = 1.767$$

and

$$\lambda_{min}\left(P_{opt}^{-1}P\right) = (\sqrt{2}-1)(\sqrt{3}-1) = 1.132$$

so that as the initial condition is varied

$$1.132 \leq \frac{J}{J_{opt}} \leq 1.767\ .$$

The figure shows the variation of J over the range of initial vectors of unit length when the system is optimized for different criteria. The unconstrained optimal solution sets the lower limit of J throughout the range. It can be seen that if the system is allowed to use only a rate feedback, and it is optimized for a unit initial velocity, then J becomes quite large for other initial disturbances.

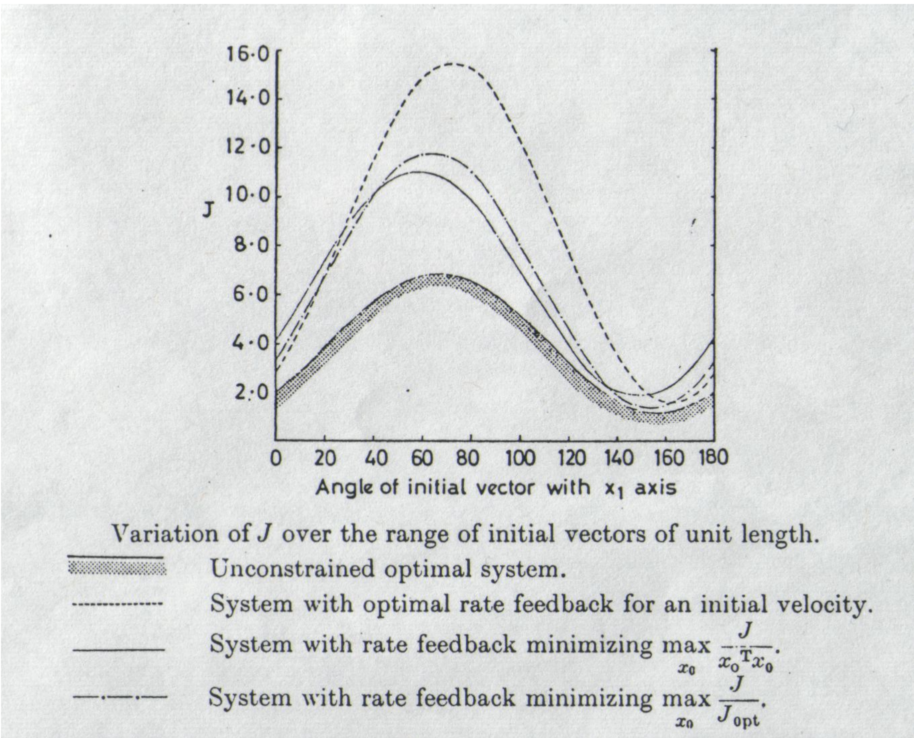

Figure 1: Variation of J over the range of intial vectors of unit length

The high level of damping which limits the effect of an initial velocity in fact causes a slow return to equilibrium after an initial displacement. The optimal systems for the min-max criteria have a better balanced response over the range of initial disturbances. Comparing the two, it can be seen that the slightly higher level of damping of the system which optimizes the worst comparison with the unconstrained optimal system results in a better response to an initial velocity, but a worse response to an initial displacement.

9 Conclusion

If there are restrictions on the feedback configuration the feedbacks which minimize a quadratic error integral depend on the initial state of the system. It is then better to optimize the worst response to an initial disturbance of given magnitude, or else the worst comparison with the unconstrained optimal system. If the error integral is represented as a quadratic form in the initial conditions with matrix P, then these measures can be computed as the maximum characteristic values of P and $P_{opt}^{-1}P$. The gradient of any of these measures with respect to the feedbacks can be represented in terms of P and the outer product xx^T. The solution for a simple system may then be found by equating to zero the gradient with respect to each allowed feedback. For more complex systems it is necessary to resort to a descent method using the gradient. It remains an open question whether any of these measures may possess local minima as well as a global minimum in the feedback space.

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