Recent Developments in the Flux Reconstruction Method and Extensions to Large Eddy Simulation

Prof. Antony Jameson

Department of Aeronautics & Astronautics
– Aerospace Computing Laboratory –
Stanford University
jameson@baboon.stanford.edu

SIAM Conference on Computational Science and Engineering
February 25–March 1, 2013
Boston, MA
I. The History of CFD
   - Van Leer’s View
   - Emergence of CFD
   - Multi-Disciplinary Nature of CFD
   - Hierarchy of Governing Equations
   - 50 Years of CFD
   - Advances in Computer Power

II. Complexity of CFD
   - The Cost of the Degrees of Freedom
   - Grid Size for a Transport Aircraft Wing
   - Complexity of CFD in the ‘70s & ‘80s
   - CFD Complexity for Turbulent Flow Simulations

III. Usage of CFD
   - Boeing’s Experience
   - Airbus’ Experience
   - Wing Optimization Using SYN107

IV. Current Status & Future Trends
   - The Current Status of CFD
   - The Future of CFD (?)
   - Large-Eddy Simulation

V. Overview of Numerical Methods
   - Typical Requirements of CFD
   - Classic Numerical Methods
   - A Review of the Literature

VI. The FR Methodology
   - Introduction
   - The Flux Reconstruction Scheme
   - The FR Scheme Graphically Illustrated
   - Energy Stability of the FR Scheme
   - A Family of Energy Stable Schemes
   - Recent Developments and Issues in the FR Approach

VII. Applications
   - Numerical Dissipation
   - High-Order Boundaries
   - Transitional Flow over SD7003 Airfoil
   - Study of Flapping Wing Sections

VIII. Structural LES Modeling
   - Explicit Filtering in the SD Element
   - Discrete Filtering Operators
   - The Restriction-Prolongation Filter
   - Discrete Filters by Gauss Quadrature
   - Discrete Filters for Arbitrary Points

IX. LES Computations
   - Wall-Resolved Turbulent Channel Flow
   - A Wall-Modeling Strategy
   - Wall-Modeled Turbulent Channel Flow
   - Flow past a Square Cylinder

X. Summary and Conclusions
History of CFD in Van Leer’s View

Top level: Jay Boris, Vladimir Kolgan, Bram van Leer, Antony Jameson
Ground level: Richard Courant, Kurt Friedrichs, Hans Lewy, Robert MacCormack, Philip Roe, John von Neumann, Stanley Osher, Amiram Harten, Peter Lax, Sergei Godunov
Emergence of CFD

- In 1960 the underlying principles of fluid dynamics and the formulation of the governing equations (potential flow, Euler, RANS) were well established.
- The new element was the emergence of powerful enough computers to make numerical solution possible – to carry this out required new algorithms.
- The emergence of CFD in the 1965–2005 period depended on a combination of advances in computer power and algorithms.

**Some significant developments in the ‘60s:**
- birth of commercial jet transport – B707 & DC-8
- intense interest in transonic drag rise phenomena
- lack of analytical treatment of transonic aerodynamics
- birth of supercomputers – CDC6600
Multi-Disciplinary Nature of CFD

Mathematics

Fluid Mechanics

Aeronautical Engineering

Numerical Wind Tunnel

Computer Science
The History of CFD

50 Years of CFD

- **1960–1970: Early Developments**
  Riemann-based schemes for gas dynamics (Godunov), 2nd-order dissipative schemes for hyperbolic equations (Lax-Wendroff), efficient explicit methods for Navier-Stokes (MacCormack), panel method (Hess-Smith)

  type-dependent differencing (Murman-Cole), complex characteristics (Garabedian), rotated difference (Jameson), multigrids (Brandt), complete airplane solution (Glowinsky)

- **1980–1990: Euler and Navier-Stokes Equations**
  oscillation control via limiters (Boris-Book), high-order Godunov scheme (van Leer), flux splitting (Steger-Warming), shock capturing via controlled diffusion (Jameson-Schmit-Turkel), approximate Riemann solver (Roe), total variation diminishing (Harten), multigrids (Jameson, Ni), solution of complete airplane (Jameson-Baker-Weatherill)

- **1990–2000: Aerodynamic Shape Optimization**
  adjoint based control theory

- **2000–2010: Discontinuous Finite Element Methods**
  Discontinuous Galerkin, Spectral Difference, Flux Reconstruction, etc.
## The History of CFD

### Advances in Computer Power

<table>
<thead>
<tr>
<th>Year</th>
<th>Machine Type</th>
<th>Peak Performance</th>
<th>Flops</th>
</tr>
</thead>
<tbody>
<tr>
<td>1970</td>
<td>CDC6600</td>
<td>1 Megaflops</td>
<td>$10^6$</td>
</tr>
<tr>
<td>1980</td>
<td>Cray 1</td>
<td>100 Megaflops</td>
<td>$10^8$</td>
</tr>
<tr>
<td>1994</td>
<td>IBM SP2</td>
<td>10 Gigaflops</td>
<td>$10^{10}$</td>
</tr>
<tr>
<td>2007</td>
<td>Linux Clusters</td>
<td>100 Teraflops</td>
<td>$10^{14}$</td>
</tr>
<tr>
<td>2007</td>
<td>(affordable) Box Cluster in my house Four 3 GHz dual core CPUs (24 Gigaflops peak) $10,000</td>
<td>2.5 Gigaflops</td>
<td>$2.5 \times 10^9$</td>
</tr>
<tr>
<td>2009</td>
<td>HP Pavilion Quadcore Notebook $1,099</td>
<td>1 Gigaflops</td>
<td>$10^9$</td>
</tr>
<tr>
<td>2011</td>
<td>MacBook Pro Quadcore Laptop $2,099</td>
<td>2.5 Gigaflops</td>
<td>$2.5 \times 10^9$</td>
</tr>
</tbody>
</table>
Outline

I. The History of CFD
   • Van Leer’s View
   • Emergence of CFD
   • Multi-Disciplinary Nature of CFD
   • Hierarchy of Governing Equations
   • 50 Years of CFD
   • Advances in Computer Power

II. Complexity of CFD
   • The Cost of the Degrees of Freedom
   • Grid Size for a Transport Aircraft Wing
   • Complexity of CFD in the ‘70s & ‘80s
   • CFD Complexity for Turbulent Flow Simulations

III. Usage of CFD
   • Boeing’s Experience
   • Airbus’ Experience
   • Wing Optimization Using SYN107

IV. Current Status & Future Trends
   • The Current Status of CFD
   • The Future of CFD (?)
   • Large-Eddy Simulation

V. Overview of Numerical Methods
   • Typical Requirements of CFD
   • Classic Numerical Methods
   • A Review of the Literature

VI. The FR Methodology
   • Introduction
   • The Flux Reconstruction Scheme
   • The FR Scheme Graphically Illustrated
   • Energy Stability of the FR Scheme
   • A Family of Energy Stable Schemes
   • Recent Developments and Issues in the FR Approach

VII. Applications
   • Numerical Dissipation
   • High-Order Boundaries
   • Transitional Flow over SD7003 Airfoil
   • Study of Flapping Wing Sections

VIII. Structural LES Modeling
   • Explicit Filtering in the SD Element
   • Discrete Filtering Operators
   • The Restriction-Prolongation Filter
   • Discrete Filters by Gauss Quadrature
   • Discrete Filters for Arbitrary Points

IX. LES Computations
   • Wall-Resolved Turbulent Channel Flow
   • A Wall-Modeling Strategy
   • Wall-Modeled Turbulent Channel Flow
   • Flow past a Square Cylinder

X. Summary and Conclusions
Fluid dynamic problems involve polynomials with large $N$ and fairly large $p$

**Complexity of Fluid Dynamic Simulations - Explicit Schemes**

- With $N \approx n^3$ mesh points in 3D and explicit time stepping, each time step requires $O(n^3)$ operations.
- The time step of a stable scheme is proportional to the mesh interval $h$ divided by the wave speed, and $h \approx 1/n$, giving complexity $Cn^4 \approx N^{4/3}$ with a constant $C$ depending on the algorithm.

**Complexity of Fluid Dynamic Simulations - Implicit Schemes**

- An implicit scheme requires matrix inversion at each time step with complexity $NB^2$ where $B$ is the bandwidth $\approx n^2$, so the cost of a step is $O(n^7)$.
- The time step is not limited by the mesh interval, so the number of time steps is independent of $n$, giving total complexity $\approx n^7$. 
Complexity of CFD

Grid Size for a Transport Aircraft Wing

- 32 cells in the boundary layer
- 512 cells around the wing to limit the mesh aspect ratio (to about 1000)

Surface Mesh
- 256 cells spanwise

Total: 512 x 64 x 256 = 8,388,608 cells
The complexity of a 3D prediction of transonic flow is $O(n^4)$ and reasonable accuracy can be obtained with $n \approx 100$

Calculations could be completed in $O(10^8)$ operations with a CDC 6600 which could achieve $\approx 10^6$ flops

Thus a useful 3D calculation might be possible in $O(10^2)$ seconds

The author recognized this in 1971

Actually FLO22 (Jameson and Caughey), which was the first program which could actually predict transonic flow over a swept wing with engineering accuracy, required about 10,000 seconds for a solution
Complexity of CFD in the ‘80s

- 800,000 mesh cells for a viscous mesh around a wing
- 5,000 flops per solution step using FLO107
- 300 steps for the solution to converge
- \((8 \times 10^5) \times (5 \times 10^3) \times (3 \times 10^2) = 1.2 \times 10^{12}\)

Roughly \(10^{12}\) flops for RANS simulation on 0.8 million mesh cells

With a 1 Gigaflop computer, solution takes about 1,000 seconds...

... About 400 seconds with a 2011 MacBook Pro quadcore at 2.5 Gflops
For a turbulent flow with a Reynolds number $Re$, the length scale of the smallest eddies relative to the integral length scale $\approx Re^{-3/4}$ (Kolmogorov, 1943)

With a comparable time step, the complexity of the simulations $\approx Re^3$

For a jumbo jet such as the Airbus A380, $Re \approx 10^8$

Direct Numerical Simulation (DNS) of the flow over the A380 has a complexity $\approx 10^{24}$ operations

With a Petaflop computer (IBM Roadrunner, 2008), DNS of the A380 has a complexity of about $10^9$ seconds

About 30 Years!
Outline

I. The History of CFD
- Van Leer's View
- Emergence of CFD
- Multi-Disciplinary Nature of CFD
- Hierarchy of Governing Equations
- 50 Years of CFD
- Advances in Computer Power

II. Complexity of CFD
- The Cost of the Degrees of Freedom
- Grid Size for a Transport Aircraft Wing
- Complexity of CFD in the ‘70s & ‘80s
- CFD Complexity for Turbulent Flow Simulations

III. Usage of CFD
- Boeing's Experience
- Airbus’ Experience
- Wing Optimization Using SYN107

IV. Current Status & Future Trends
- The Current Status of CFD
- The Future of CFD (?)
- Large-Eddy Simulation

V. Overview of Numerical Methods
- Typical Requirements of CFD
- Classic Numerical Methods
- A Review of the Literature

VI. The FR Methodology
- Introduction
- The Flux Reconstruction Scheme
- The FR Scheme Graphically Illustrated
- Energy Stability of the FR Scheme
- A Family of Energy Stable Schemes
- Recent Developments and Issues in the FR Approach

VII. Applications
- Numerical Dissipation
- High-Order Boundaries
- Transitional Flow over SD7003 Airfoil
- Study of Flapping Wing Sections

VIII. Structural LES Modeling
- Explicit Filtering in the SD Element
- Discrete Filtering Operators
- The Restriction-Prolongation Filter
- Discrete Filters by Gauss Quadrature
- Discrete Filters for Arbitrary Points

IX. LES Computations
- Wall-Resolved Turbulent Channel Flow
- A Wall-Modeling Strategy
- Wall-Modeled Turbulent Channel Flow
- Flow past a Square Cylinder

X. Summary and Conclusions
Usage of CFD – Boeing's Experience

Impact of CFD on Configuration Lines & Wind Tunnel Testing

NASA Tech
- PANAIR
- FLO22 A411
- Cartesian Grid Tech.
- TLNS3D
- HSR & IWD
- TLNS3D-MB OVERFLOW
- CFL3D OVERFLOW
- Unstructured Adaptive Grid 3-D N-S

Boeing Tools
- A502
- A488
- TRANAIR
- TRANAIR Optimization
- TLNS3D-MB/ZEUS
- CFL3D/ZEUS OVERFLOW
- CFD++

Boeing Products
- 767
- 757
- 737-300
- 777
- 737NG
- 2000
- 2005
- 787

1980 state of the art
- 77

Modern close coupled nacelle installation, 0.02 Mach faster than 737-200
21% thicker faster wing than 757, 767 technology
Highly constrained wing design Faster wing than 737-300
Successful multipoint optimization design Faster and more efficient than previous aircraft
CFD for Loads and Stability and Control

50% Reduction in Wind Tunnel Testing!

Number of Wings Tested
- 77
- 18
- 11
- 11
Computational Methods at Boeing

**TRANAIR:**
- Full Potential with directly coupled Boundary Layer
- Cartesian solution adaptive grid
- Drela lag-dissipation turbulence model
- Multi-point design/optimization

**Navier-Stokes Codes:**
- CFL3D – Structured Multiblock Grid
- TLNS3D – Structured Multiblock Grid, Thin Layer
- OVERFLOW – Overset Grid

**N-S Turbulence Models:**
- S-A Spalart-Allmaras
- Menter’s $k-\omega$ SST
CFD Contributions to B787

- Wind-Tunnel Corrections
  - Vertical Tail and Aft Body Design
  - Control-Surface Failure Analysis
  - APU Inlet and Ducting
  - APU and Propulsion Fire Suppression

- Planform Design
- High-Speed Wing Design
- Wing-Tip Design
- High-Lift Wing Design
- Aeroelastics
- Vortex Generators
- Icing
- Flutter

- Wing Controls
- Reynolds-Number Corrections
- Cab Design

- Air-Data System Location
- ECS Inlet Design
- Engine-Bay Thermal Analysis
- Inlet Design Certification
- Inlet Design

- Engine/ Airframe Integration
- Nacelle Design
- Thrust-Reverser Design
- Community Noise

- Exhaust System Design
- Design For Stability & Control

- Avionics Cooling
- Buffet Boundary

- Design For Stability & Control
- Engine/ Airframe Integration
- Nacelle Design
- Thrust-Reverser Design
- Community Noise

- Air-Data System Location
- ECS Inlet Design
- Engine-Bay Thermal Analysis
- Inlet Design Certification
- Inlet Design
MEGAFLOW / MEGADESIGN
- National CFD Initiative (since 1995)

Development & validation of a national CFD software for complete aircraft applications which

➢ allows computational aerodynamic analysis for 3D complex configurations at cruise, high-lift & off-design conditions
➢ builds the basis for shape optimization and multidisciplinary simulation
➢ establishes numerical flow simulation as a routinely used tool at DLR and in German aircraft industry
➢ serves as a development platform for universities
**Unstructured RANS Capability: TAU**

**Tool for complex configurations**

- hybrid meshes, cell vertex / cell centered
- high-level turbulence & transition models (RSM, DES, linear stability methods)
- state-of-the-art algorithms (JST, multigrid, ...)
- local mesh adaptation
- chimera technique
- fluid / structure coupling
- continuous/discrete adjoint
- extensions to hypersonic flows

**TAU-Code**

- unstructured database
- C-code, Python
- portable code, optimized for cache hardware
- high performance on parallel computer
Numerical Flow Simulation

Relation CFD / wind tunnel

- CFD
- wind tunnel
- improvements algorithms & hardware
- unstructured hybrid grids
- CFD of future
- CFD cost effective alternative

B€
costs

number of simulations > 30,000
CFD Contribution to A380

- Frequent use
- Moderate use
- Growing use

- High Speed Wing Design
- Flutter Prediction
- Sting Corrections
- Flow Control Devices (VG/Strakes)
- Cabin Ventilation
- Performance Prediction
- Cockpit/Avionics Ventilation
- Low Speed Wing Design
- Cabin Noise
- Fuselage Design
- Powerplant Integration
- Nacelle Design
- Inlet Design
- Wing Tip Design
- Spoiler/Control Surfaces
- Tails Design
- Fuel System Design
- APU Inlet/Outlet Design
- External Noise Sources
- Handling Quality Data
- Belly Fairing Design
- Ground Effect
- ECS Inlet/Outlet Design
- Nozzle Design
- Engine Core Compartment
- Static Deformation
- Pack Bay Thermal Analysis
- Aero Loads Data
- Thrust Reverser Design
State of the Art Wing Design Process in 2 Stages, starting from Garabedian-Korn Airfoil and NASA Common Research Model.
Outline

I. The History of CFD
   ▸ Van Leer’s View
   ▸ Emergence of CFD
   ▸ Multi-Disciplinary Nature of CFD
   ▸ Hierarchy of Governing Equations
   ▸ 50 Years of CFD
   ▸ Advances in Computer Power

II. Complexity of CFD
   ▸ The Cost of the Degrees of Freedom
   ▸ Grid Size for a Transport Aircraft Wing
   ▸ Complexity of CFD in the ‘70s & ‘80s
   ▸ CFD Complexity for Turbulent Flow Simulations

III. Usage of CFD
   ▸ Boeing’s Experience
   ▸ Airbus’ Experience
   ▸ Wing Optimization Using SYN107

IV. Current Status & Future Trends
   ▸ The Current Status of CFD
   ▸ The Future of CFD (??)
   ▸ Large-Eddy Simulation

V. Overview of Numerical Methods
   ▸ Typical Requirements of CFD
   ▸ Classic Numerical Methods
   ▸ A Review of the Literature

VI. The FR Methodology
   ▸ Introduction
   ▸ The Flux Reconstruction Scheme
   ▸ The FR Scheme Graphically Illustrated
   ▸ Energy Stability of the FR Scheme
   ▸ A Family of Energy Stable Schemes
   ▸ Recent Developments and Issues in the FR Approach

VII. Applications
   ▸ Numerical Dissipation
   ▸ High-Order Boundaries
   ▸ Transitional Flow over SD7003 Airfoil
   ▸ Study of Flapping Wing Sections

VIII. Structural LES Modeling
   ▸ Explicit Filtering in the SD Element
   ▸ Discrete Filtering Operators
   ▸ The Restriction-Prolongation Filter
   ▸ Discrete Filters by Gauss Quadrature
   ▸ Discrete Filters for Arbitrary Points

IX. LES Computations
   ▸ Wall-Resolved Turbulent Channel Flow
   ▸ A Wall-Modeling Strategy
   ▸ Wall-Modeled Turbulent Channel Flow
   ▸ Flow past a Square Cylinder

X. Summary and Conclusions
The Current Status of CFD

- Worldwide commercial and government codes are based on algorithms developed in the ‘80s and ‘90s.
- These codes can handle complex geometry but are generally limited to 2nd order accuracy.
- They cannot handle turbulence without modeling.
- Unsteady simulations are very expensive, and questions over accuracy remain.
CFD has been on a plateau for the past 15 years

- Representations of current state of the art:
  - Formula 1 cars
  - Complete aircrafts

- The majority of current CFD methods are not adequate for vortex dominated and transitional flows:
  - Rotorcraft
  - High-lift systems
  - Formation flying
The Future of CFD
The Future of CFD
The Future of CFD
The number of DoF for an LES of turbulent flow over an airfoil scales as $Re_c^{1.8}$ (resp. $Re_c^{0.4}$) if the inner layer is resolved (resp. modeled).

Rapid advances in computer hardware should make LES feasible within the foreseeable future for industrial problems at high Reynolds numbers. To realize this goal requires:

- high-order algorithms for unstructured meshes (complex geometries)
- Sub-Grid Scale models applicable to wall bounded flows
- massively parallel implementation

Chapman (1979), AIAA J. 17(12)
Typical Requirements of CFD

- **Accuracy**: solution must be right
- **Small numerical dissipation**: unsteady flow features
- **Unstructured grids**: complex geometries
- **Numerical flux**: wave propagation problems
- **High resolution capabilities**: transitional and turbulent flows
- **Efficiency**: code parallelism
- ...
Past Research on DG Schemes:

Recent Research:
Attempts to reduce complexity and avoid quadrature:
• Spectral Difference (SD) scheme by Kopriva & Kolias [1996], Liu, Vinokur & Wang [2006]
• Nodal Discontinuous Galerkin (NDG) scheme by Atkins & Shu [1998], Hesthaven & Warburton [2007]
• Flux Reconstruction (FR) scheme by Huynh [2007,2009]
Outline

I. The History of CFD
   - Van Leer's View
   - Emergence of CFD
   - Multi-Disciplinary Nature of CFD
   - Hierarchy of Governing Equations
   - 50 Years of CFD
   - Advances in Computer Power

II. Complexity of CFD
   - The Cost of the Degrees of Freedom
   - Grid Size for a Transport Aircraft Wing
   - Complexity of CFD in the ‘70s & ‘80s
   - CFD Complexity for Turbulent Flow Simulations

III. Usage of CFD
   - Boeing’s Experience
   - Airbus’ Experience
   - Wing Optimization Using SYN107

IV. Current Status & Future Trends
   - The Current Status of CFD
   - The Future of CFD (?)
   - Large-Eddy Simulation

V. Overview of Numerical Methods
   - Typical Requirements of CFD
   - Classic Numerical Methods
   - A Review of the Literature

VI. The FR Methodology
   - Introduction
   - The Flux Reconstruction Scheme
   - The FR Scheme Graphically Illustrated
   - Energy Stability of the FR Scheme
   - A Family of Energy Stable Schemes
   - Recent Developments and Issues in the FR Approach

VII. Applications
   - Numerical Dissipation
   - High-Order Boundaries
   - Transitional Flow over SD7003 Airfoil
   - Study of Flapping Wing Sections

VIII. Structural LES Modeling
   - Explicit Filtering in the SD Element
   - Discrete Filtering Operators
   - The Restriction-Prolongation Filter
   - Discrete Filters by Gauss Quadrature
   - Discrete Filters for Arbitrary Points

IX. LES Computations
   - Wall-Resolved Turbulent Channel Flow
   - A Wall-Modeling Strategy
   - Wall-Modeled Turbulent Channel Flow
   - Flow past a Square Cylinder

X. Summary and Conclusions
The FR Methodology

Introduction

- The following presentation emphasizes development of Huynh's FR approach, and energy stability
- Energy stability analysis versus Fourier stability analysis
  - Energy method is more general and rigorous
  - Energy method enables stability proofs for all orders of accuracy
  - Energy method applies to non-uniform meshes
  - Fourier analysis provides more detailed information about the distribution of dispersive and diffusive errors
  - Fourier analysis identifies super accuracy for linear problems

The Energy Stable FR scheme (ESFR):
- Until recently, stable FR schemes identified on an ad hoc basis
- We have identified a range of correction functions that guarantee linear stability for all orders of accuracy
- Achieved by extending Jameson’s proof of stability of an SD scheme for the linear advection equation for all orders of accuracy
The Flux Reconstruction Scheme

The solution is locally represented by Lagrange polynomial of degree $n - 1$ on the solution points:

$$u_h = \sum_{j=1}^{n} u_j l_j(x) \quad f_h^D = \sum_{j=1}^{n} f_j^D l_j(x)$$

The flux is discontinuous and needs to be corrected in a suitable way

$$\Delta_L = \tilde{f}_L - f_h^D (-1) \quad \Delta_R = \tilde{f}_R - f_h^D (1)$$

$$h_L(-1) = 1, \quad h_L(1) = 0 \quad h_R(1) = 1, \quad h_R(-1) = 0$$

The continuous flux is obtained from the discontinuous counterpart by adding the correction functions of degree $n$ weighted by the flux corrections

$$f_h^C = \sum_{j=1}^{n} f_j^D l_j(x) + h_L(x) \Delta_L + h_R(x) \Delta_R$$

The continuous flux is finally differentiated at the solution points and the solution is advanced in time

$$\frac{\partial u_i}{\partial t} + \left[ \sum_{j=1}^{n} f_j^D \frac{dl_j}{dx} (x_i) + \Delta_L \frac{dh_L}{dx} (x_i) + \Delta_R \frac{dh_R}{dx} (x_i) \right] = 0$$

The solution is locally represented by Lagrange polynomial of degree $n-1$ on the $n$ solution points:

$$u_h = \sum_{j=1}^{n} u_j l_j(x)$$

The discontinuous flux is constructed

\[ f_h^D = \sum_{j=1}^{n} f_j^D l_j(x) \]
The FR Methodology

The FR Scheme Graphically Illustrated

Solution is evaluated at element boundaries

\[ u_L = \sum_{j=1}^{n} u_j l_j (-1) \quad u_R = \sum_{j=1}^{n} u_j l_j (+1) \]

The common interface flux is computed from multiply defined values at each interface (FV-type numerical flux such as approximate Riemann flux)
The common interface flux is computed from multiply defined values at each interface (FV-type numerical flux such as approximate Riemann flux)

The FR Methodology

The FR Scheme Graphically Illustrated

Correction functions of degree $n$ are introduced

\[ h_L(-1) = 1, \quad h_L(1) = 0 \]

The FR Methodology

The FR Scheme Graphically Illustrated

The correction functions are scaled

\[ \Delta_L = \tilde{f}_L - h_D(-1) \]

The correction is added to the discontinuous flux

\[ f_h^* = \sum_{j=1}^{n} f^D_j l_j(x) + h_L(x) \Delta L \]
The right boundary is corrected the same way

\[ h_R(1) = 1, \quad h_R(-1) = 0 \]
The correction is scaled...

\[ \Delta R = \tilde{f}_R - f^D_h (+1) \]
And added to the discontinuous flux

\[
f_h^C = \sum_{j=1}^{n} f_j^D l_j(x) + h_L(x)\Delta_L + h_R(x)\Delta_R
\]
Total approximate continuous flux

\[ f_h^C = \sum_{j=1}^{n} f_j^D l_j(x) + h_L(x) \Delta_L + h_R(x) \Delta_R \]
The divergence of the flux is evaluated at the solution points

\[ \frac{\partial f^C}{\partial x}(x_i) = \sum_{j=1}^{n} f_j^D \frac{dl_j}{dx}(x_i) + \Delta_L \frac{dh_L}{dx}(x_i) + \Delta_R \frac{dh_R}{dx}(x_i) \]
The solution is advanced in time

\[
\frac{\partial u_i}{\partial t} + \left[ \sum_{j=1}^{n} f_j^D \frac{dl_j}{dx}(x_i) + \Delta_L \frac{dh_L}{dx}(x_i) + \Delta_R \frac{dh_R}{dx}(x_i) \right] = 0
\]
The FR method defines a family of energy stable schemes in the norm

\[ \| U_{\delta D} \|_{p,2} = \left[ \sum_{n=1}^{N} \int_{x_n}^{x_{n+1}} (U_{n}^{\delta D})^2 + \frac{c}{2} (J_n)^{2p} \left( \frac{\partial p U_{n}^{\delta D}}{\partial x_p} \right)^2 \, dx \right]^{1/2} \]

The schemes have the form

\[ \frac{\partial u_i}{\partial t} + \left[ \sum_{j=1}^{n} f_j^D \frac{dl_j}{dx}(x_i) + \Delta_L \frac{dh_L}{dx}(x_i) + \Delta_R \frac{dh_R}{dx}(x_i) \right] = 0 \]

where the correction functions in terms of Legendre polynomials are

\[ h_L = \frac{(-1)^p}{2} \left[ L_p - \left( \frac{\eta_p(c)L_{p-1} + L_{p+1}}{1 + \eta_p(c)} \right) \right] \]
\[ h_R = \frac{(+1)^p}{2} \left[ L_p + \left( \frac{\eta_p(c)L_{p-1} + L_{p+1}}{1 + \eta_p(c)} \right) \right] \]

with a single parameter \( c \)

\[ \eta_p(c) = \frac{c(2p + 1)(a_{p+1})^2}{2} \]

The FR Methodology

A Family of Energy Stable Schemes

Nodal DG:

\[ c = 0 \Rightarrow \eta_p = 0 \]
\[ g_L = \frac{(-1)^p}{2} [L_p - L_{p+1}], \quad g_R = \frac{(+1)^p}{2} [L_p + L_{p+1}] \]

Spectral Difference:

\[ c = \frac{2p}{(2p + 1)(p + 1)(a_p p!)^2} \Rightarrow \eta_p = \frac{p}{p + 1} \]
\[ g_L = \frac{(-1)^p}{2} (1 - x)L_p, \quad g_R = \frac{(+1)^p}{2} (1 + x)L_p \]

G2 Scheme by Huynh [2007]:

\[ c = \frac{2(p + 1)}{(2p + 1)p(a_p p!)^2} \Rightarrow \eta_p = \frac{p + 1}{p} \]
\[ g_L = \frac{(-1)^p}{2} \left[ L_p - \frac{(p + 1)L_{p-1} + pL_{p+1}}{2p + 1} \right], \quad g_R = \frac{(+1)^p}{2} \left[ L_p + \frac{(p + 1)L_{p-1} + pL_{p+1}}{2p + 1} \right] \]
ESFR schemes have been extended to triangles (Castonguay et al. 2011) and tetrahedra (Williams et al. 2013) with proofs for advection/diffusion.

In practice they have proved successful for quadrilateral and hexahedral elements with a tensor product formulation, but so far there is a stability proof only for the nodal DG case (Jameson 2011).

A general hybrid grid would require pyramid elements for which further research is needed (Jameson 2011).

Nonlinear stability is improved by the best choice of quadrature points, but additional filtering may be needed.

Recent Developments and Issues in the FR Approach

- The stability proofs have been developed for semi-discrete schemes, and carry over to fully discrete schemes only in the case of an implicit time-stepping scheme of Crank-Nicolson type (Jameson, 2011)
- An FR formulation for space-time elements remains to be developed
- Further research on shock capturing methods is needed, particularly for moving shocks
- In the case of steady-state problems convergence is excessively slow. Further work on multi h-p schemes is needed, and also on iterative methods and preconditioners for implicit time-stepping schemes

Jameson, (2011) AIAA P. 2011-3226
## Outline

### I. The History of CFD
- Van Leer’s View
- Emergence of CFD
- Multi-Disciplinary Nature of CFD
- Hierarchy of Governing Equations
- 50 Years of CFD
- Advances in Computer Power

### II. Complexity of CFD
- The Cost of the Degrees of Freedom
- Grid Size for a Transport Aircraft Wing
- Complexity of CFD in the ‘70s & ‘80s
- CFD Complexity for Turbulent Flow Simulations

### III. Usage of CFD
- Boeing’s Experience
- Airbus’ Experience
- Wing Optimization Using SYN107

### IV. Current Status & Future Trends
- The Current Status of CFD
- The Future of CFD (?)
- Large-Eddy Simulation

### V. Overview of Numerical Methods
- Typical Requirements of CFD
- Classic Numerical Methods
- A Review of the Literature

### VI. The FR Methodology
- Introduction
- The Flux Reconstruction Scheme
- The FR Scheme Graphically Illustrated
- Energy Stability of the FR Scheme
- A Family of Energy Stable Schemes
- Recent Developments and Issues in the FR Approach

### VII. Applications
- Numerical Dissipation
- High-Order Boundaries
- Transitional Flow over SD7003 Airfoil
- Study of Flapping Wing Sections

### VIII. Structural LES Modeling
- Explicit Filtering in the SD Element
- Discrete Filtering Operators
- The Restriction-Prolongation Filter
- Discrete Filters by Gauss Quadrature
- Discrete Filters for Arbitrary Points

### IX. LES Computations
- Wall-Resolved Turbulent Channel Flow
- A Wall-Modeling Strategy
- Wall-Modeled Turbulent Channel Flow
- Flow past a Square Cylinder

### X. Summary and Conclusions
Numerical Dissipation

Temporal Mixing-Layer

N=6

N=2

60×60 DoF

N=5

100×200×10 DoF
Numerical Dissipation

Iso-$Q$

Vorticity magnitude

$N=6, 60 \times 60 \times 12 \text{ DoF}$
High-Order Boundaries

Transitional Flow over SD7003 Airfoil

<table>
<thead>
<tr>
<th></th>
<th>Freestream Turbulence</th>
<th>Separation $x_{sep}/c$</th>
<th>Transition $x_{tr}/c$</th>
<th>Reattach. $x_r/c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Radespiel et al.</td>
<td>0.08%</td>
<td>0.30</td>
<td>0.53</td>
<td>0.64</td>
</tr>
<tr>
<td>Ol et al.</td>
<td>0.10%</td>
<td>0.18</td>
<td>0.47</td>
<td>0.58</td>
</tr>
<tr>
<td>Galbraith Visbal</td>
<td>0%</td>
<td>0.23</td>
<td>0.55</td>
<td>0.65</td>
</tr>
<tr>
<td>Uranga et al.</td>
<td>0%</td>
<td>0.23</td>
<td>0.51</td>
<td>0.60</td>
</tr>
<tr>
<td>Present ILES*</td>
<td>0%</td>
<td>0.23</td>
<td>0.53</td>
<td>0.64</td>
</tr>
</tbody>
</table>

Experiments in green

16 hours on 16 C2070s!

128 hours (> 5 days)
on 16 Xeon x5670 CPUs

Re=6×10^4, AoA=4°, 2.2×10^7 DoF

*1.7×10^6 DoF

Study of Flapping Wing Sections

SD, 2D, N=5 on deforming grid

Experiment (Jones, et al.)

NACA0012, Re=1850, Ma=0.2, St=1.5, ω=2.46, h=0.12c

Outline

I. The History of CFD
   - Van Leer's View
   - Emergence of CFD
   - Multi-Disciplinary Nature of CFD
   - Hierarchy of Governing Equations
   - 50 Years of CFD
   - Advances in Computer Power

II. Complexity of CFD
   - The Cost of the Degrees of Freedom
   - Grid Size for a Transport Aircraft Wing
   - Complexity of CFD in the ‘70s & ‘80s
   - CFD Complexity for Turbulent Flow Simulations

III. Usage of CFD
   - Boeing’s Experience
   - Airbus’ Experience
   - Wing Optimization Using SYN107

IV. Current Status & Future Trends
   - The Current Status of CFD
   - The Future of CFD (?)
   - Large-Eddy Simulation

V. Overview of Numerical Methods
   - Typical Requirements of CFD
   - Classic Numerical Methods
   - A Review of the Literature

VI. The FR Methodology
   - Introduction
   - The Flux Reconstruction Scheme
   - The FR Scheme Graphically Illustrated
   - Energy Stability of the FR Scheme
   - A Family of Energy Stable Schemes
   - Recent Developments and Issues in the FR Approach

VII. Applications
   - Numerical Dissipation
   - High-Order Boundaries
   - Transitional Flow over SD7003 Airfoil
   - Study of Flapping Wing Sections

VIII. Structural LES Modeling
   - Explicit Filtering in the SD Element
   - Discrete Filtering Operators
   - The Restriction-Prolongation Filter
   - Discrete Filters by Gauss Quadrature
   - Discrete Filters for Arbitrary Points

IX. LES Computations
   - Wall-Resolved Turbulent Channel Flow
   - A Wall-Modeling Strategy
   - Wall-Modeled Turbulent Channel Flow
   - Flow past a Square Cylinder

X. Summary and Conclusions
Explicit Filtering in the SD Element

Key issues:
- non-uniform and staggered distribution of points
- the filter stencil shall not lie across elements
- filter width shall be prescribed and constant

Filtering Strategy:
1. The filtered solution is computed at solution points
2. The SGS model term is evaluated at solution points
3. The SGS model term is extrapolated at flux points via Lagrange basis
The filtering operator for the 1D standard element is defined as

$$\bar{\phi}_s = \sum_{i=1}^{N} w_i^s \phi_i, \quad (s = 1, \ldots, N)$$

The kernel of the above discrete filter can be written as

$$\hat{G}_s(k) = \sum_{i=1}^{N} w_i^s \exp(-j\beta_i^s k \Delta), \quad \text{with} \quad \beta_i^s = \frac{\xi_i - \xi_s}{\Delta}$$

$\Delta = 1/N$ is assumed to be the actual resolution within the SD element.
**The Restriction-Prolongation Filter**

**Sharp cutoff in modal space:**

The solution is first projected on a lower order polynomial (restriction step) and then extrapolated back to the original solution points (prolongation step).

---

Gauss-Legendre quadrature points:

- The discrete filter is obtained by analytical integration of a selected filter kernel
- Cutoff is enforced iteratively by checking the filter’s 2nd moment in physical space

Lodato, Castonguay, Jameson (in preparation)
Structural LES Modeling

Discrete Filters for Arbitrary Points

**Generalized method of Vasilyev et al. (1998):**

- Value and slope at cutoff are enforced using a selected filter kernel (2)
- Higher moments are set to zero (N-3) + preservation of constant variable (1)

Outline

I. The History of CFD
   - Van Leer’s View
   - Emergence of CFD
   - Multi-Disciplinary Nature of CFD
   - Hierarchy of Governing Equations
   - 50 Years of CFD
   - Advances in Computer Power

II. Complexity of CFD
   - The Cost of the Degrees of Freedom
   - Grid Size for a Transport Aircraft Wing
   - Complexity of CFD in the ‘70s & ‘80s
   - CFD Complexity for Turbulent Flow Simulations

III. Usage of CFD
   - Boeing’s Experience
   - Airbus’ Experience
   - Wing Optimization Using SYN107

IV. Current Status & Future Trends
   - The Current Status of CFD
   - The Future of CFD (?)
   - Large-Eddy Simulation

V. Overview of Numerical Methods
   - Typical Requirements of CFD
   - Classic Numerical Methods
   - A Review of the Literature

VI. The FR Methodology
   - Introduction
   - The Flux Reconstruction Scheme
   - The FR Scheme Graphically Illustrated
   - Energy Stability of the FR Scheme
   - A Family of Energy Stable Schemes
   - Recent Developments and Issues in the FR Approach

VII. Applications
   - Numerical Dissipation
   - High-Order Boundaries
   - Transitional Flow over SD7003 Airfoil
   - Study of Flapping Wing Sections

VIII. Structural LES Modeling
   - Explicit Filtering in the SD Element
   - Discrete Filtering Operators
   - The Restriction-Prolongation Filter
   - Discrete Filters by Gauss Quadrature
   - Discrete Filters for Arbitrary Points

IX. LES Computations
   - Wall-Resolved Turbulent Channel Flow
   - A Wall-Modeling Strategy
   - Wall-Modeled Turbulent Channel Flow
   - Flow past a Square Cylinder

X. Summary and Conclusions
**LES Computations**

**Wall-Resolved Turbulent Channel Flow**

\[ Re_\tau = 180 \quad (\Delta^+ : 38, 2–10, 19) \]

\[ Re_\tau = 395 \quad (\Delta^+ : 39, 1–40, 26) \]

\[ Re_\tau = 590 \quad (\Delta^+ : 60, 3–33, 26) \]

Wall-Resolved Turbulent Channel Flow

Breuer and Rodi (1996)
Wall-Modeled Turbulent Channel Flow

**LES Computations**

**Reτ = 590** $(\Delta^+ : 58, 24–47, 58)$

**Reτ = 2000** $(\Delta^+ : 98, 22–102, 98)$

Iso-Q colored by velocity magnitude

- LES with SD+WSM+LW
- DNS Moser, et al., 1999 (Reτ 590)
- DNS Hoyas, Jiménez, 2006 (Reτ 2000)
- Modeled region

Wall-Modeled Turbulent Channel Flow

$Re_\tau = 590$

- **Wall-Resolved**
  - $u$ at $y^+ = 100$
- **Wall-Modeled**

$Re_\tau = 2000$

- **Wall-Resolved**
  - $u$ at $y^+ = 100$
- **Wall-Modeled**

Where $Re_\tau$ is the Reynolds number based on the friction velocity ($\tau$) and the channel half-width ($\Delta$), and $y^+$ is the wall-normal distance normalized by the friction velocity.
Flow Past Square Cylinder: $Re = 21400$

- Time integration: RK3
- Nº of elements: 35760 ($2.3 \times 10^6$ DoF)
- Grid dimensions: $21D \times 12D \times 3.2D$
- Reynolds: 21400
- Mach: 0.3
- Statistics: 16 $T_0$
Flow past a Square Cylinder: \( \text{Re}_D = 21400 \)

Preliminary results (work in progress)

 iso-\( Q \) colored by velocity
The early development of CFD in the Aerospace Industry was primarily driven by the need to calculate steady transonic flows: this problem is quite well solved.

CFD has been on a plateau for the last 15 years with 2nd-order accurate FV methods for the RANS equations almost universally used in both commercial and government codes which can treat complex configurations.

These methods cannot reliably predict complex separated, unsteady and vortex dominated flows.

Ongoing advances in both numerical algorithms and computer hardware and software should enable an advance to LES for industrial applications within the foreseeable future.

Research should focus on high-order methods with minimal numerical dissipation for unstructured meshes to enable the treatment of complex configurations.

Eventually DNS may become feasible for high Reynolds number flows hopefully with a smaller power requirement than a wind tunnel.
The current research is a combined effort by

- **Postdocs**: Charlie Liang, Peter Vincent, Guido Lodato
- **Ph.D. students**: Sachin Premasuthan, Kui Ou, Patrice Castonguay, David Williams, Yves Allenau, Lala Li, Manuel López, and Andy Chan

It is made possible by the support of

- the **Airforce Office of Scientific Research** under grant FA9550-10-1-0418 by Dr. Fariba Fahroo
- the **National Science Foundation** under grants 0708071 and 0915006 monitored by Dr. Leland Jameson


5. Lodato, G., P. Castonguay, and A. Jameson, Structural LES modeling with high-order spectral difference schemes. In *Annual Research Briefs* (Center for Turbulence Research, Stanford University, 2011)


ESFR Schemes for Quadrilateral Elements
ESFR Schemes for Quads

In the case of quadrilateral elements a natural approach is to use a tensor product to represent the discrete solution

\[ u_h = \sum_{j=1}^{p+1} \sum_{n=1}^{p+1} u_{jn} l_j(x) n_n(y) \]

where \( l_j(x) \) and \( n_n(y) \) are Lagrange basis polynomials of degree \( p \). In the case \( p = 2 \) this can be expanded as

\[ u_h = a_1 + a_2 x + a_3 y + a_4 x^2 + a_5 x y + a_6 y^2 + a_7 x^2 y + a_8 x y^2 + a_9 x^2 y^2 \]

Now

\[ \nabla u_h = \begin{bmatrix} a_2 + 2 a_4 x + a_5 y + 2 a_7 x y + a_8 y^2 + 2 a_9 x y^2 \\ a_3 + a_5 x + 2 a_6 y + a_7 x^2 + 2 a_8 x y + 2 a x^2 y \end{bmatrix} \]

Now we assume that the divergence of the correction function can be represented by a polynomial with the same form as \( u_h \):

\[ \nabla \cdot g = b_1 + b_2 x + b_3 y + b_4 x^2 + b_5 x y + b_6 y^2 + b_7 x^2 y + b_8 x y^2 + b_9 x^2 y^2 \]
ESFR Schemes for Quads

Now, differentiating twice in both \( x \) and \( y \), we find that

\[
\frac{\partial}{\partial t} u_{hxxxy} + \frac{\partial^2}{\partial x^2} \frac{\partial^2}{\partial y^2} \nabla \cdot \mathbf{g} = 0
\]

(4)

Now choose \( \mathbf{g} \) such that

\[
\int_D \mathbf{g} \cdot \nabla u_h dA = c u_{hxxxy} \frac{\partial^2}{\partial x^2} \frac{\partial^2}{\partial y^2} \nabla \cdot \mathbf{g} = \int \frac{c}{4} u_{hxxxy} \frac{\partial^2}{\partial x^2} \frac{\partial^2}{\partial y^2} \nabla \cdot \mathbf{g} dA
\]

since both terms are constants. Thus using equation (4)

\[
\int \mathbf{g} \cdot \nabla u_h dA = -\frac{c}{4} \int u_{hxxxy} \frac{\partial}{\partial t} u_{hxxxy} dA = -\frac{c}{4} \frac{d}{dt} \int u_{hxxxy}^2 dA
\]

and substituting for \( \int \mathbf{g} \cdot \nabla u_h dA \) in equation (2) to get the following. Now on summing over the elements, a proper choice of the interface fluxes yields energy stability as before.

\[
\frac{d}{dt} \int_D \left( \frac{u_h^2}{2} + \frac{c}{4} u_{hxxxy}^2 \right) dA + \int_B \mathbf{n} \cdot \mathbf{a} \frac{u_h^2}{2} dS + \int_B \mathbf{n} \cdot \mathbf{g} u_h dS = 0
\]
ESFR Schemes for Quads

Hence

\[ u_{hxxyy} = 4a_9 \]

\[ \frac{\partial^2}{\partial x^2} \frac{\partial^2}{\partial y^2} \nabla \cdot g = 4b_9 \]

and

\[ \int_D g \cdot \nabla u_h dA = a_2 \int g_x dA + a_3 \int g_y dA \]

\[ + 2a_9 \int (g_x xy^2 + g_y x^2 y) dA \]

Hence to satisfy (4) all lower moments of \( g \) must vanish while

\[ \int_D (g_x xy^2 + g_y x^2 y) dA = 8cb_9 \]
ESFR Schemes for Quads

Now we can integrate by parts to obtain conditions directly for the moments of $\nabla \cdot \mathbf{g}$

\[
\int_D \nabla \cdot \mathbf{g} dV = \int g \cdot \mathbf{n} dS + 0
\]
\[
\int_D \nabla \cdot \mathbf{g} x dV = \int g \cdot \mathbf{n} x dS + 0
\]
\[
\int_D \nabla \cdot \mathbf{g} y dV = \int g \cdot \mathbf{n} y dS + 0
\]
\[
\int_D \nabla \cdot \mathbf{g} x^2 y^2 dV = \int g \cdot \mathbf{n} x^2 y^2 dS + 16cb_9
\]

These moment conditions are sufficient to define $\nabla \cdot \mathbf{g}$ uniquely. The nodal DG scheme is recovered by setting $c = 0$. In this case the equations for the coefficients of $\nabla \cdot \mathbf{g}$ can be solved with $\nabla \cdot \mathbf{g}$ expressed in the form of a tensor product. However, if $c \neq 0$ the solution for $\nabla \cdot \mathbf{g}$ can no longer be expressed as a tensor product.
ESFR Schemes for 3D Pyramids
ESFR Schemes for Pyramids

For pyramids, using polynomials of degree 2, we can represent $u_h$ and $\nabla \cdot g$ as

$$u_h = a_1 + a_2 x + a_3 y + a_4 x^2 + a_5 xy + a_6 y^2 + a_7 x^2 y + a_8 xy^2 + a_9 x^2 y^2 + a_{10} z + a_{11} xz + a_{12} yz + a_{13} z^2 + a_{14} xyz$$

and

$$\nabla \cdot g = b_1 + b_2 x + b_3 y + b_4 x^2 + b_5 xy + b_6 y^2 + b_7 x^2 y + b_8 xy^2 + b_9 x^2 y^2 + b_{10} z + b_{11} xz + b_{12} yz + b_{13} z^2 + b_{14} xyz$$

The energy stability for 3D pyramids requires all moments of $g$ to vanish except for

$$\int_D (g_x y^2 + g_y x^2 y) dV = 8 c_1 b_9$$

$$\int_D (g_z z) dV = 2 c_2 b_{13}$$

$$\int_D (g_x yz + g_y xz + g_z xy) dV = c_3 b_{14}$$
ESFR Schemes for Pyramids

Then

\[
\frac{\partial}{\partial t} \mathbf{u}_{hxx} + \frac{\partial^2}{\partial x^2} \frac{\partial^2}{\partial y^2} \nabla \cdot \mathbf{g} = 0
\]

\[
\frac{\partial}{\partial t} \mathbf{u}_{hzz} + \frac{\partial^2}{\partial z^2} \nabla \cdot \mathbf{g} = 0
\]

\[
\frac{\partial}{\partial t} \mathbf{u}_{hxyz} + \frac{\partial}{\partial x} \frac{\partial}{\partial y} \frac{\partial}{\partial z} \nabla \cdot \mathbf{g} = 0
\]

The requirement to ensure energy stability becomes

\[
\int_D \mathbf{g} \cdot \nabla \mathbf{u}_h dA = \frac{1}{2} \frac{d}{dt} \int_D \left( c_1 \mathbf{u}_{hxx}^2 + c_2 \mathbf{u}_{hzz}^2 + c_3 \mathbf{u}_{hxyz}^2 \right) dV
\]
ESFR Schemes for Pyramids

After integration by parts $\nabla \cdot g$ can be determined from 14 moments

$$\int_D \nabla \cdot g \, dV = \int g \cdot n \, dS + 0$$

$$\int_D \nabla \cdot gx \, dV = \int g \cdot nx \, dS + 0$$

$$\int_D \nabla \cdot gy \, dV = \int g \cdot ny \, dS + 0$$

$$\int_D \nabla \cdot gz \, dV = \int g \cdot nz \, dS + 0$$

$$\int_D \nabla \cdot gx^2y^2 \, dV = \int g \cdot nx^2y^2 \, dS + 16c_1b_9$$

$$\int_D \nabla \cdot gz^2 \, dV = \int g \cdot nz^2 \, dS + 4c_2b_{13}$$

$$\int_D \nabla \cdot gxyz \, dV = \int g \cdot nxyz \, dS + c_3b_{14}$$
Energy Stability Analysis for High-order Schemes
Review of Time Integration Methods
Applications

Energy Stable Implicit Crank-Nicolson Type Scheme
Energy Stable Implicit Crank-Nicolson

In order to preserve true energy stability with the time integration scheme one ought to use an implicit scheme of Crank-Nicolson type. Using a superscript \( n \) to denote the time level, let

\[
\bar{u}_h = \frac{1}{2}(u_h^{n+1} + u_h^n)
\]

Suppose now that all terms in the space discretization are evaluated using \( \bar{u}_h \). Then the scheme is

\[
u_h^{n+1} = u_h^n - \Delta t(a \frac{\partial \bar{u}_h}{\partial x} + f_{cl} g'_l + f_{cr} g'_r)
\]

where \( f_{cl} \) and \( f_{cr} \) are calculated from \( \bar{u}_h \) in the adjacent cells. Moreover, differentiating \( p \) times

\[
u_h^{(p)n+1} - u_h^{(p)n} = -\Delta t(f_{cl} g_l^{p+1} + f_{cr} g_r^{p+1})
\]
Energy Stable Implicit Crank-Nicolson

Now, multiplying by $\bar{u}_h$ and integrating over the element

$$\frac{1}{2} \int_{-1}^{1} u_h^{n+12} dx - \frac{1}{2} \int_{-1}^{1} u_h^{n2} dx = -\Delta t \left[ a \int_{-1}^{1} \bar{u}_h \frac{\partial \bar{u}_h}{\partial x} dx + f_{cl} \int_{-1}^{1} \bar{u}_h g'_l dx + f_{cr} \int_{-1}^{1} \bar{u}_h g'_r dx \right]$$

$$= -\Delta t \left[ \frac{a}{2} (\bar{u}_h(1)^2 - \bar{u}_h(-1)^2) + f_{cr} \bar{u}_h(1) - f_{cl} \bar{u}_h(-1) \right]$$

$$+ \Delta t \left[ f_{cr} \int_{-1}^{1} g_r \frac{\partial \bar{u}_h}{\partial x} dx + f_{cl} \int_{-1}^{1} g_l \frac{\partial \bar{u}_h}{\partial x} dx \right]$$

Choosing $g_r$ and $g_l$ as in section II such that

$$\int g_r \frac{\partial \bar{u}_h}{\partial x} = c g_r^{(p+1)} \bar{u}_h^{(p)}, \quad \int g_l \frac{\partial \bar{u}_h}{\partial x} = c g_l^{(p+1)} \bar{u}_h^{(p)}$$

The last term is equivalent to

$$-\frac{c}{2} \left( u_h^{(p)^{n+1}} - u_h^{(p)^{n}} \right) \left( u_h^{(p)^{n+1}} + u_h^{(p)^{n}} \right) = -\frac{c}{2} \left( u_h^{(p)^{n+12}} - u_h^{(p)^{n2}} \right)$$
Energy Stable Implicit Crank-Nicolson

Accordingly on summing over the elements with a proper choice of the interface flux, and transforming the reference element to the physical elements, the broken norm

\[
\sum \int_{x_n}^{x_{n+1}} \left( \frac{u_h^2}{2} + \frac{c}{4} h_n^{2p} u_h^{(p)^2} \right)^2 dx
\]

is strictly non-increasing.