Aerodynamic Inverse Design and Shape Optimization via Control Theory

Antony Jameson

1 Thomas V. Jones Professor of Engineering
Department of Aeronautics & Astronautics
Stanford University

SciTech 2015
January 5, 2015
Introduction
- Introduction
- Lighthill’s Method
- Control Theory Approach to Design

Numerical Formulation
- Design using the Euler Equations
- Sobolev Inner Product

Design Process
- Design Process Outline
- Drag Minimization
- Inverse Design
- P51 Racer
- Reduced Sweep: Design for the CRM
- Low Sweep Wing Design
- Natural Laminar Wing
- Numerical Wind Tunnel

Appendix
- Design using the Transonic Potential Flow Equation
INTRODUCTION
Objective of Computational Aerodynamics

1. Capability to predict the flow past an airplane in different flight regimes such as take off, cruise, flutter.
2. Interactive design calculations to allow immediate improvement
3. Automatic design optimization
Early Aerodynamic Design Methods

- 1945 Lighthill (Conformal Mapping, Incompressible Flow)
- 1965 Nieuwland (Hodograph, Power Series)
- 1970 Garabedian - Korn (Hodograph, Complex Characteristics)
- 1974 Boerstoel (Hodograph)
- 1974 Trenen (Potential Flow, Dirichlet Boundary Conditions)
- 1977 Henne (3D Potential Flow, based on FLO22)
- 1985 Volpe-Melnik (2D Potential Flow, Based on FLO36)
- 1976 Sobieczki (Fictitious Gas)
- 1979 Drela-Giles (2D Euler Equations, Streamline Coordinates, Newton Iteration)
LIGHTHILL’S METHOD
Lighthill’s Method

Let the profile P be conformally mapped to an unit circle C

The surface velocity is $q = \frac{1}{h} |\nabla \phi|$ where $\phi$ is the potential in the circle plane, and h is the mapping modulus $h = \left| \frac{dz}{d\sigma} \right| = \frac{ds}{d\theta}$

Choose $q = q_T$

Solve for the mapping modulus $h = \frac{1}{q_T} |\nabla \phi|$
Implementation of Lighthill’s Method

Design Profile C for Specified Surface speed $q_t$. Let a profile C be conformally mapped to a circle by

$$
\log \frac{dz}{d\sigma} = \sum \frac{C_n}{\sigma^n}
$$

$$
\log \frac{ds}{d\theta} + i(\alpha - \theta - \frac{pi}{2}) = \sum (a_n \cos(n\theta) + b_n \sin(n\theta)) + i \sum (b_n \cos(n\theta) - a_n \sin(n\theta))
$$

where

$$
q = \frac{\nabla \Phi}{h}, \ h = \left| \frac{dz}{d\sigma} \right|
$$

and

$$
\Phi = (r + \frac{1}{r})\cos \theta + \frac{\Gamma}{2\pi} \theta \text{ is known}
$$

On C set $q = q_t$

$$
\rightarrow \frac{ds}{d\theta} = \frac{\Phi}{q_t} \rightarrow a_n, b_b
$$
Constraints with Lighthill’s Method

To preserve $q_\infty$

$$c_0 = 0$$

Also, integration around a circuit gives

$$\Delta z = \oint \frac{dz}{d\sigma} = 2\pi i c_1$$

Closure $\rightarrow c_1 = 0$

Thus,

$$\int \log(q_t) d\theta = 0$$

$$\int \log(q_t) \cos(\theta) d\theta = 0$$

$$\int \log(q_t) \sin(\theta) d\theta = 0$$
**CONTROL THEORY APPROACH TO DESIGN**
A wing is a device to control the flow. Apply the theory of control of partial differential equations (J.L.Lions) in conjunction with CFD.

References

- Pironneau (1964) Optimum shape design for subsonic potential flow
- Jameson (1988) Optimum shape design for transonic and supersonic flow modeled by the transonic potential flow equation and the Euler equations
Control Theory Approach to the Design Method

Define a cost function

\[ I = \frac{1}{2} \int_B (p - p_t)^2 d\mathcal{B} \]

or

\[ I = \frac{1}{2} \int_B (q - q_t)^2 d\mathcal{B} \]

The surface shape is now treated as the control, which is to be varied to minimize I, subject to the constraint that the flow equations are satisfied in the domain D.
Choice of Domain

ALTERNATIVES

1. Variable computational domain - Free boundary problem
2. Transformation to a fixed computational domain - Control via the transformation function

EXAMPLES

1. 2D via Conformal mapping with potential flow
2. 2D via Conformal mapping with Euler equations
3. 3D Sheared Parabolic Coordinates with Euler equation
4. ...

Antony Jameson
Mathematics of Aerodynamic Shape Optimization
Suppose that the surface of the body is expressed by an equation

\[ f(x) = 0 \]

Vary \( f \) to \( f + \delta f \) and find \( \delta I \).

If we can express

\[ \delta I = \int_B g \delta f \, dB = (g, \delta f)_B \]

Then we can recognize \( g \) as the gradient \( \frac{\partial I}{\partial f} \).

Choose a modification

\[ \delta f = -\lambda g \]

Then to first order

\[ \delta I = -\lambda (g, g)_B \leq 0 \]

In the presence of constraints project \( g \) into the admissible trial space.

Accelerate by the conjugate gradient method.
Define the geometry through a set of design parameters, for example, to be the weights $\alpha_i$ applied to a set of shape functions $b_i(x)$ so that the shape is represented as 

$$f(x) = \sum \alpha_i b_i(x).$$

Then a cost function $I$ is selected, for example, to be the drag coefficient or the lift to drag ratio, and $I$ is regarded as a function of the parameters $\alpha_i$. The sensitivities $\frac{\partial I}{\partial \alpha_i}$ may be estimated by making a small variation $\delta \alpha_i$ in each design parameter in turn and recalculating the flow to obtain the change in $I$. Then

$$\frac{\partial I}{\partial \alpha_i} \approx \frac{I(\alpha_i + \delta \alpha_i) - I(\alpha_i)}{\delta \alpha_i}.$$

The gradient vector $G = \frac{\partial I}{\partial \alpha}$ may now be used to determine a direction of improvement. The simplest procedure is to make a step in the negative gradient direction by setting 

$$\alpha^{n+1} = \alpha^n + \delta \alpha,$$

where

$$\delta \alpha = -\lambda G$$

so that to first order

$$I + \delta I = I - G^T \delta \alpha = I - \lambda G^T G < I.$$
Disadvantages

The main disadvantage of this approach is the need for a number of flow calculations proportional to the number of design variables to estimate the gradient. The computational costs can thus become prohibitive as the number of design variables is increased.
For flow about an airfoil or wing, the aerodynamic properties which define the cost function are functions of the flow-field variables \( w \) and the physical location of the boundary, which may be represented by the function \( F \), say. Then

\[
I = I(w, F),
\]

and a change in \( F \) results in a change

\[
\delta I = \left[ \frac{\partial I^T}{\partial w} \right] \delta w + \left[ \frac{\partial I^T}{\partial F} \right] \delta F
\]

in the cost function. Suppose that the governing equation \( R \) which expresses the dependence of \( w \) and \( F \) within the flowfield domain \( D \) can be written as

\[
R(w, F) = 0.
\]

Then \( \delta w \) is determined from the equation

\[
\delta R = \left[ \frac{\partial R}{\partial w} \right] \delta w + \left[ \frac{\partial R}{\partial F} \right] \delta F = 0.
\]

Since the variation \( \delta R \) is zero, it can be multiplied by a Lagrange Multiplier \( \psi \) and subtracted from the variation \( \delta I \) without changing the result.
Formulation of the Adjoint Approach to Optimal Design

\[
\delta I = \frac{\partial I^T}{\partial w} \delta w + \frac{\partial I^T}{\partial F} \delta F - \psi^T \left( \frac{\partial R}{\partial w} \delta w + \frac{\partial R}{\partial F} \delta F \right) \\
= \left\{ \frac{\partial I^T}{\partial w} - \psi^T \left[ \frac{\partial R}{\partial w} \right] \right\} \delta w + \left\{ \frac{\partial I^T}{\partial F} - \psi^T \left[ \frac{\partial R}{\partial F} \right] \right\} \delta F.
\]

(4)

Choosing \( \psi \) to satisfy the adjoint equation

\[
\left[ \frac{\partial R}{\partial w} \right]^T \psi = \frac{\partial I}{\partial w}
\]

(5)

the first term is eliminated, and we find that

\[
\delta I = G^T \delta F,
\]

(6)

where

\[ G = \frac{\partial I^T}{\partial F} - \psi^T \left[ \frac{\partial R}{\partial F} \right]. \]

An improvement can be made with a shape change

\[ \delta F = -\lambda G \]

where \( \lambda \) is positive and small enough that the first variation is an accurate estimate of \( \delta I \).
Advantages

- The advantage is that (6) is independent of $\delta w$, with the result that the gradient of $I$ with respect to an arbitrary number of design variables can be determined without the need for additional flow-field evaluations.

- The cost of solving the adjoint equation is comparable to that of solving the flow equations. Thus the gradient can be determined with roughly the computational costs of two flow solutions, independently of the number of design variables, which may be infinite if the boundary is regarded as a free surface.

- When the number of design variables becomes large, the computational efficiency of the control theory approach over traditional approach, which requires direct evaluation of the gradients by individually varying each design variable and recomputing the flow fields, becomes compelling.
DESIGN USING THE EULER EQUATIONS
Design using the Euler Equations

In a fixed computational domain with coordinates, $\xi$, the Euler equations are

$$ J \frac{\partial w}{\partial t} + R(w) = 0 \quad (7) $$

where $J$ is the Jacobian (cell volume),

$$ R(w) = \frac{\partial}{\partial \xi_i} (S_{ij} f_j) = \frac{\partial F_i}{\partial \xi_i} \quad (8) $$

and $S_{ij}$ are the metric coefficients (face normals in a finite volume scheme). We can write the fluxes in terms of the scaled contravariant velocity components

$$ U_i = S_{ij} u_j $$

as

$$ F_i = S_{ij} f_j = \begin{bmatrix} \rho U_i \\ \rho U_i u_1 + S_{i1} p \\ \rho U_i u_2 + S_{i2} p \\ \rho U_i u_3 + S_{i3} p \\ \rho U_i H \end{bmatrix}. $$

where $p = (\gamma - 1) \rho (E - \frac{1}{2} u_i^2)$ and $\rho H = \rho E + p$. 
A variation in the geometry now appears as a variation $\delta S_{ij}$ in the metric coefficients. The variation in the residual is

$$\delta R = \frac{\partial}{\partial \xi_i} (\delta S_{ij} f_j) + \frac{\partial}{\partial \xi_i} \left(S_{ij} \frac{\partial f_j}{\partial w} \delta w\right)$$

(9)

and the variation in the cost $\delta I$ is augmented as

$$\delta I - \int_D \psi^T \delta R \, d\xi$$

(10)

which is integrated by parts to yield

$$\delta I - \int_B \psi^T n_i \delta F_i d\xi_B + \int_D \frac{\partial \psi^T}{\partial \xi} (\delta S_{ij} f_j) \, d\xi + \int_D \frac{\partial \psi^T}{\partial \xi_i} S_{ij} \frac{\partial f_j}{\partial w} \delta w \, d\xi$$
Design using the Euler Equations

For simplicity, it will be assumed that the portion of the boundary that undergoes shape modifications is restricted to the coordinate surface $\xi_2 = 0$. Then equations for the variation of the cost function and the adjoint boundary conditions may be simplified by incorporating the conditions

$$n_1 = n_3 = 0, \quad n_2 = 1, \quad B_\xi = d\xi_1 d\xi_3,$$

so that only the variation $\delta F_2$ needs to be considered at the wall boundary. The condition that there is no flow through the wall boundary at $\xi_2 = 0$ is equivalent to

$$U_2 = 0, \quad \text{so that} \quad \delta U_2 = 0$$

when the boundary shape is modified. Consequently the variation of the inviscid flux at the boundary reduces to

$$\delta F_2 = \delta p \begin{bmatrix} 0 \\ S_{21} \\ S_{22} \\ S_{23} \\ 0 \end{bmatrix} + p \begin{bmatrix} 0 \\ \delta S_{21} \\ \delta S_{22} \\ \delta S_{23} \\ 0 \end{bmatrix}.$$

\[(11)\]
In order to design a shape which will lead to a desired pressure distribution, a natural choice is to set

$$I = \frac{1}{2} \int_{\mathcal{B}} (p - p_d)^2 dS$$

where $p_d$ is the desired surface pressure, and the integral is evaluated over the actual surface area. In the computational domain this is transformed to

$$I = \frac{1}{2} \iint_{\mathcal{B}_w} (p - p_d)^2 |S_2| d\xi_1 d\xi_3,$$

where the quantity

$$|S_2| = \sqrt{S_{2j}S_{2j}}$$

denotes the face area corresponding to a unit element of face area in the computational domain.
In the computational domain the adjoint equation assumes the form

\[ C_i^T \frac{\partial \psi}{\partial \xi_i} = 0 \]  \hspace{1cm} (12)

where

\[ C_i = S_{ij} \frac{\partial f_j}{\partial w}. \]

To cancel the dependence of the boundary integral on \( \delta p \), the adjoint boundary condition reduces to

\[ \psi_j n_j = p - p_d \]  \hspace{1cm} (13)

where \( n_j \) are the components of the surface normal

\[ n_j = \frac{S_{2j}}{|S_2|}. \]
This amounts to a transpiration boundary condition on the co-state variables corresponding to the momentum components. Note that it imposes no restriction on the tangential component of $\psi$ at the boundary. We find finally that

$$\delta I = - \int_D \frac{\partial \psi^T}{\partial \xi_i} \delta S_{ij} f_j dD$$

$$- \iint_{BW} \left( \delta S_{21} \psi_2 + \delta S_{22} \psi_3 + \delta S_{23} \psi_4 \right) p d\xi_1 d\xi_3. \quad (14)$$

Here the expression for the cost variation depends on the mesh variations throughout the domain which appear in the field integral. However, the true gradient for a shape variation should not depend on the way in which the mesh is deformed, but only on the true flow solution. In the next section we show how the field integral can be eliminated to produce a reduced gradient formula which depends only on the boundary movement.
Consider the case of a mesh variation with a fixed boundary. Then \( \delta I = 0 \) but there is a variation in the transformed flux,

\[
\delta F_i = C_i \delta w + \delta S_{ij} f_j.
\]

Here the true solution is unchanged. Thus, the variation \( \delta w \) is due to the mesh movement \( \delta x \) at each mesh point. Therefore

\[
\delta w = \nabla w \cdot \delta x = \frac{\partial w}{\partial x_j} \delta x_j \quad (= \delta w^*)
\]

and since \( \frac{\partial}{\partial \xi_i} \delta F_i = 0 \), it follows that

\[
\frac{\partial}{\partial \xi_i} (\delta S_{ij} f_j) = - \frac{\partial}{\partial \xi_i} (C_i \delta w^*). \tag{15}
\]

It has been verified by Jameson and Kim\(^\star\) that this relation holds in the general case with boundary movement.

The Reduced Gradient Formulation

Now

\[ \int_{D} \phi^T \delta R \, dD = \int_{D} \phi^T \frac{\partial}{\partial \xi_i} C_i \left( \delta w - \delta w^* \right) \, dD \]
\[ = \int_{B} \phi^T C_i \left( \delta w - \delta w^* \right) \, dB \]
\[ - \int_{D} \frac{\partial \phi^T}{\partial \xi_i} C_i \left( \delta w - \delta w^* \right) \, dD. \]  

(16)

Here on the wall boundary

\[ C_2 \delta w = \delta F_2 - \delta S_{2j} f_j. \]  

(17)

Thus, by choosing \( \phi \) to satisfy the adjoint equation and the adjoint boundary condition, we reduce the cost variation to a boundary integral which depends only on the surface displacement:

\[ \delta I = \int_{B_W} \psi^T \left( \delta S_{2j} f_j + C_2 \delta w^* \right) \, d\xi_1 d\xi_3 \]
\[ - \int_{B_W} \left( \delta S_{21} \psi_2 + \delta S_{22} \psi_3 + \delta S_{23} \psi_4 \right) \, p \, d\xi_1 d\xi_3. \]  

(18)
**Sobolev Inner Product**
Another key issue for successful implementation of the continuous adjoint method is the choice of an appropriate inner product for the definition of the gradient. It turns out that there is an enormous benefit from the use of a modified Sobolev gradient, which enables the generation of a sequence of smooth shapes. This can be illustrated by considering the simplest case of a problem in the calculus of variations.

Suppose that we wish to find the path \( y(x) \) which minimizes

\[
I = \int_a^b F(y, y') \, dx
\]

with fixed end points \( y(a) \) and \( y(b) \). Under a variation \( \delta y(x) \),

\[
\delta I = \int_1^b \left( \frac{\partial F}{\partial y} \delta y + \frac{\partial F}{\partial y'} \delta y' \right) \, dx
\]

\[
= \int_1^b \left( \frac{\partial F}{\partial y} - \frac{d}{dx} \frac{\partial F}{\partial y'} \right) \delta y \, dx
\]
The Need for a Sobolev Inner Product in the Definition of the Gradient

Thus defining the gradient as

$$ g = \frac{\partial F}{\partial y} - \frac{d}{dx} \frac{\partial F}{\partial y'} $$

and the inner product as

$$ (u, v) = \int_a^b uv \, dx $$

we find that

$$ \delta I = (g, \delta y). $$

If we now set

$$ \delta y = -\lambda g, \quad \lambda > 0, $$

we obtain an improvement

$$ \delta I = -\lambda (g, g) \leq 0 $$

unless $g = 0$, the necessary condition for a minimum.
The Need for a Sobolev Inner Product in the Definition of the Gradient

Note that $g$ is a function of $y, y', y''$,

$$g = g(y, y', y'')$$

In the well known case of the Brachistrone problem, for example, which calls for the determination of the path of quickest descent between two laterally separated points when a particle falls under gravity,

$$F(y, y') = \sqrt{\frac{1 + y'^2}{y}}$$

and

$$g = -\frac{1 + y'^2 + 2yy''}{2 \left[ y(1 + y'^2) \right]^{3/2}}$$

It can be seen that each step

$$y^{n+1} = y^n - \lambda^n g^n$$

reduces the smoothness of $y$ by two classes. Thus the computed trajectory becomes less and less smooth, leading to instability.
The Need for a Sobolev Inner Product in the Definition of the Gradient

In order to prevent this we can introduce a weighted Sobolev inner product

\[ \langle u, v \rangle = \int (uv + \epsilon u'v') \, dx \]

where \( \epsilon \) is a parameter that controls the weight of the derivatives. We now define a gradient \( \bar{g} \) such that \( \delta I = \langle \bar{g}, \delta y \rangle \). Then we have

\[
\delta I = \int (\bar{g} \delta y + \epsilon \bar{g}' \delta y') \, dx = \int \left( \bar{g} - \frac{\partial}{\partial x} \epsilon \frac{\partial \bar{g}}{\partial x} \right) \delta y \, dx = (g, \delta y)
\]

where

\[
\bar{g} - \frac{\partial}{\partial x} \epsilon \frac{\partial \bar{g}}{\partial x} = g
\]

and \( \bar{g} = 0 \) at the end points.
Therefore $\bar{g}$ can be obtained from $g$ by a smoothing equation. Now the step

$$y^{n+1} = y^n - \lambda^n g^n$$

gives an improvement

$$\delta I = -\lambda^n \langle g^n, \bar{g}^n \rangle$$

but $y^{n+1}$ has the same smoothness as $y^n$, resulting in a stable process.
OUTLINE OF THE DESIGN PROCESS
Outline of the Design Process

The design procedure can finally be summarized as follows:

1. Solve the flow equations for $\rho, u_1, u_2, u_3$ and $p$.
2. Solve the adjoint equations for $\psi$ subject to appropriate boundary conditions.
3. Evaluate $\mathcal{G}$ and calculate the corresponding Sobolev gradient $\overline{\mathcal{G}}$.
4. Project $\overline{\mathcal{G}}$ into an allowable subspace that satisfies any geometric constraints.
5. Update the shape based on the direction of steepest descent.
6. Return to 1 until convergence is reached.
Design Cycle

- Flow Solution
- Adjoint Solution
- Gradient Calculation
- Sobolev Gradient
- Shape & Grid Modification
  
  Repeat the Design Cycle until Convergence
Constraints

- Fixed $C_L$.
- Fixed span load distribution to present too large $C_L$ on the outboard wing which can lower the buffet margin.
- Fixed wing thickness to prevent an increase in structure weight.
  - Design changes can be limited to a specific spanwise range of the wing.
  - Section changes can be limited to a specific chordwise range.

- Smooth curvature variations via the use of Sobolev gradient.
Application of Thickness Constraints

- Prevent shape change penetrating a specified skeleton (colored in red).
- Separate thickness and camber allow free camber variations.
- Minimal user input needed.
### Computational Cost

**Cost of Search Algorithm**

<table>
<thead>
<tr>
<th>Method</th>
<th>Cost</th>
<th>Steps</th>
</tr>
</thead>
<tbody>
<tr>
<td>Steepest Descent</td>
<td>$O(N^2)$</td>
<td></td>
</tr>
<tr>
<td>Quasi-Newton</td>
<td>$O(N)$</td>
<td></td>
</tr>
<tr>
<td>Smoothed Gradient</td>
<td>$O(K)$</td>
<td></td>
</tr>
</tbody>
</table>

**Note:** $K$ is independent of $N$.

---

### Total Computational Cost of Design

<table>
<thead>
<tr>
<th>Method</th>
<th>Computational Cost</th>
</tr>
</thead>
<tbody>
<tr>
<td>Finite difference gradients</td>
<td>$O(N^3)$</td>
</tr>
<tr>
<td>Steepest descent</td>
<td></td>
</tr>
<tr>
<td>Finite difference gradients</td>
<td>$O(N^2)$</td>
</tr>
<tr>
<td>Quasi-Newton</td>
<td></td>
</tr>
<tr>
<td>Adjoint gradients</td>
<td>$O(N)$</td>
</tr>
<tr>
<td>Quasi-Newton</td>
<td></td>
</tr>
<tr>
<td>Adjoint gradients</td>
<td>$O(K)$</td>
</tr>
<tr>
<td>Smoothed gradient</td>
<td></td>
</tr>
</tbody>
</table>

Note: $K$ is independent of $N$.

Drag Minimization
Viscous Optimization of Korn Airfoil

Mesh size=512 x 64, Mach number=0.75, $C_{L_{target}}=0.63$, Reynolds number=20 Million
The Effect of Applying Smoothed Gradient

Unsmoothed

Smoothed

KORN AIRFOIL
MACH 0.750  ALPHA 1.853  RE 0.305E+08
CL 0.0382  CD 0.0118  CM 0.1257  CLV 0.0000  CDV 0.0040
GRID 512x64  NDEE 1  RESOL.2843600  GMAX 0.535E-01

KORN AIRFOIL
MACH 0.750  ALPHA 0.853  RE 0.305E+08
CL 0.6382  CD 0.0118  CM 0.1257  CLV 0.0000  CDV 0.0040
GRID 512x64  NDEE 1  RESOL.2843600  GMAX 0.306E-02
**Inverse Design**

Recovering of ONERA M6 Wing from its pressure distribution

A. Jameson 2003–2004
This is a difficult problem because of the presence of the shock wave in the target pressure and because the profile to be recovered is symmetric while the target pressure is not.
The pressure distribution of the final design match the specified target, even inside the shock.
Aircraft competing in the Reno Air Races reach speeds above 500 MPH, encountering compressibility drag due to the appearance of shock waves.

Objective is to delay drag rise without altering the wing structure. Hence try adding a bump on the wing surface.
Partial Redesign

- Allow only outward movement.
- Limited changes to front part of the chordwise range.
DO WE NEED SWEPT WINGS ON COMMERCIAL JETS?
Background for Studies of Reduced Sweep

- Current Transonic Transports
  - Cruise Mach: $0.76 \leq M \leq 0.86$
  - C/4 Sweep: $25^\circ \leq \Lambda \leq 35^\circ$
  - Wing Planform Layout Knowledge Base
    - Heavily Influenced By *Design Charts*
    - Data Developed From *Cut-n-Try* Designs
    - Data Aumented With Parametric Variations
    - Data Collected Over The Years
    - Includes Shifts Due To Technologies
e.g., Supercritical Airfoils, Composites, etc.
Evolution of Pressures for $\Lambda = 10^\circ$ Wing during Optimization
### Pure Aerodynamic Optimizations

<table>
<thead>
<tr>
<th>Mach</th>
<th>Sweep</th>
<th>$C_L$</th>
<th>$C_D$</th>
<th>$C_{D,\text{tot}}$</th>
<th>$ML/D$</th>
<th>$\sqrt{ML/D}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.85</td>
<td>35°</td>
<td>0.500</td>
<td>153.7</td>
<td>293.7</td>
<td>14.47</td>
<td>15.70</td>
</tr>
<tr>
<td>0.84</td>
<td>30°</td>
<td>0.510</td>
<td>151.2</td>
<td>291.2</td>
<td>14.71</td>
<td>16.05</td>
</tr>
<tr>
<td>0.83</td>
<td>25°</td>
<td>0.515</td>
<td>151.2</td>
<td>291.2</td>
<td>14.68</td>
<td>16.11</td>
</tr>
<tr>
<td>0.82</td>
<td>20°</td>
<td>0.520</td>
<td>151.7</td>
<td>291.7</td>
<td>14.62</td>
<td>16.14</td>
</tr>
<tr>
<td>0.81</td>
<td>15°</td>
<td>0.525</td>
<td>152.4</td>
<td>292.4</td>
<td>14.54</td>
<td>16.16</td>
</tr>
<tr>
<td>0.80</td>
<td>10°</td>
<td>0.530</td>
<td>152.2</td>
<td>292.2</td>
<td>14.51</td>
<td>16.22</td>
</tr>
<tr>
<td>0.79</td>
<td>5°</td>
<td>0.535</td>
<td>152.5</td>
<td>292.5</td>
<td>14.45</td>
<td>16.26</td>
</tr>
</tbody>
</table>

- $C_D$ in counts
- $C_{D,\text{tot}} = C_D + 140$ counts
- Lowest Sweep Favors $\sqrt{ML/D} \simeq 4.0\%$
Conclusion of Swept Wing Study

- An unswept wing at Mach 0.80 offers slightly better range efficiency than a swept wing at Mach 0.85.
- It would also improve TO, climb, descent and landing.
- Perhaps B737/A320 replacements should have unswept wings.
DEMO OF LOW SWEEP DESIGN
Application II: Low Sweep Wing Redesign using RK-SGS Scheme

Mesh size=256x64x48, Design Steps=15, Design variables=127x33=4191 surface mesh points

Antony Jameson  Mathematics of Aerodynamic Shape Optimization
WING DESIGN FOR NATURAL LAMINAR FLOW
Airplane Geometry

(a) Wing-body Geometry

(b) Wing-body Surface Mesh

Mesh size=256x64x48
Airplane Mesh

Mesh size=256x64x48
Wing Planform and Sectional Profiles

(a) Cross Sectional Profiles

(b) Wing Planform Shape
Design Points: \( M = 0.40, \ CL = 0.4, 0.6, 0.9, 1.0 \)

At \( M = 0.40 \), the lower limit of the operating range is set by the appearance of suction peak. This suction peak appears at the lower surface when \( CL < 0.40 \), and at the upper surface when \( CL > 0.90 \).
Design Points: $M=0.50$, CL=0.4, 0.6, 0.9, 1.0

At $M = .50$, the lower limit of the operating range is set by the appearance of suction peak. This suction peak appears at the lower surface when $CL < .40$, and at the upper surface when $CL > 1.0$. 

Antony Jameson
Mathematics of Aerodynamic Shape Optimization
Design Points: $M=0.60$, $CL=0.4, 0.6, 0.9, 1.0$

At $M = 0.60$, the lower limit of the operating range is set by the appearance of suction peak at the lower surface, and the formation of shock at the upper surface. The suction peak appears at the lower surface when $CL < 0.40$, and the shock starts to form at the upper surface when $CL > 0.90$. 
Design Points: \( M = 0.65, \ CL = 0.4, 0.5 \)

At \( M = .65 \), the lower limit of the operating range is set by the appearance of suction peak at the lower surface, and the formation of shock at the upper surface. The operating range has become too narrow to be useful. In particular, the suction peak appears at the lower surface when \( CL \leq 40 \), and the shock of moderate strength has already started to form at the upper surface when \( CL = .50 \).
The figure shows the operating boundaries within which favorable pressure distribution can be maintained.
Numerical Wind Tunnel
Concept of Numerical Wind Tunnel

*MDO: Multi-Disciplinary Optimization
**Advanced Numerical Wind Tunnel**

**Numerically Intensive**
- Initial Design
- Requirements
- Master Definition
- Central Database
- High-Level Redesign
- Monitor Results

**Human Intensive**
- Automatic Mesh Generation
- Flow Solution
- Aeroelastic Solution
- Loads
- Geometry Modification
- Optimization
- Quantitative Assessment

*Antony Jameson*  
*Mathematics of Aerodynamic Shape Optimization*
Traditional Engineering Offices
Grumman Aerodynamics Section in 1968
APPENDIX
DESIGN USING THE TRANSONIC POTENTIAL FLOW EQUATION
Airfoil Design For Potential Flow using Conformal Mapping

Consider the case of two-dimensional compressible inviscid flow. In the absence of shock waves, an initially irrotational flow will remain irrotational, and we can assume that the velocity vector $\mathbf{q}$ is the gradient of a potential $\phi$. In the presence of weak shock waves this remains a fairly good approximation. Let $p$, $\rho$, $c$, and $M$ be the pressure, density, speed-of-sound, and Mach number $q/c$. Then the potential flow equation is

$$\nabla \cdot (\rho \nabla \phi) = 0,$$

where the density is given by

$$\rho = \left\{ 1 + \frac{\gamma - 1}{2} M_\infty^2 \left( 1 - q^2 \right) \right\} ^{\frac{1}{(\gamma - 1)}},$$

while

$$p = \frac{\rho^\gamma}{\gamma M_\infty^2}, \quad c^2 = \frac{\gamma p}{\rho}.$$ 

Here $M_\infty$ is the Mach number in the free stream, and the units have been chosen so that $p$ and $q$ have a value of unity in the far field.
Suppose that the domain $D$ exterior to the profile $C$ in the $z$-plane is conformally mapped on to the domain exterior to a unit circle in the $\sigma$-plane. Let $R$ and $\theta$ be polar coordinates in the $\sigma$-plane, and let $r$ be the inverted radial coordinate $\frac{1}{R}$. Also let $h$ be the modulus of the derivative of the mapping function

$$h = \left| \frac{dz}{d\sigma} \right|. \quad (22)$$

Now the potential flow equation becomes

$$\frac{\partial}{\partial \theta} \left( \rho \phi_{\theta} \right) + r \frac{\partial}{\partial r} \left( r \rho \phi_{r} \right) = 0 \text{ in } D, \quad (23)$$

where the density is given by equation (20), and the circumferential and radial velocity components are

$$u = \frac{r \phi_{\theta}}{h}, \quad v = \frac{r^2 \phi_{r}}{h}, \quad (24)$$

while

$$q^2 = u^2 + v^2. \quad (25)$$
The condition of flow tangency leads to the Neumann boundary condition

\[ v = \frac{1}{h} \frac{\partial \phi}{\partial r} = 0 \text{ on } C. \]  

(26)

In the far field, the potential is given by an asymptotic estimate, leading to a Dirichlet boundary condition at \( r = 0 \).

Suppose that it is desired to achieve a specified velocity distribution \( q_d \) on \( C \). Introduce the cost function

\[ I = \frac{1}{2} \int_C (q - q_d)^2 \, d\theta, \]

\[ (26) \]
The design problem is now treated as a control problem where the control function is the mapping modulus $h$, which is to be chosen to minimize $I$ subject to the constraints defined by the flow equations (19–26).

A modification $\delta h$ to the mapping modulus will result in variations $\delta \phi$, $\delta u$, $\delta v$, and $\delta \rho$ to the potential, velocity components, and density. The resulting variation in the cost will be

$$\delta I = \int_C (q - q_d) \, \delta q \, d\theta,$$

(27)

where, on $C$, $q = u$. Also,

$$\delta u = r \frac{\delta \phi \theta}{h} - u \frac{\delta h}{h}, \quad \delta v = r^2 \frac{\delta \phi r}{h} - v \frac{\delta h}{h},$$

while according to equation (20)

$$\frac{\partial \rho}{\partial u} = -\frac{\rho u}{c^2}, \quad \frac{\partial \rho}{\partial v} = -\frac{\rho v}{c^2}.$$
It follows that $\delta \phi$ satisfies

$$
L \delta \phi = - \frac{\partial}{\partial \theta} \left( \rho M^2 \phi \frac{\delta h}{h} \right) - r \frac{\partial}{\partial r} \left( \rho M^2 r \phi \frac{\delta h}{h} \right)
$$

where

$$
L \equiv \frac{\partial}{\partial \theta} \left\{ \rho \left( 1 - \frac{u^2}{c^2} \right) \frac{\partial}{\partial \theta} - \frac{\rho uv}{c^2} r \frac{\partial}{\partial r} \right\} + r \frac{\partial}{\partial r} \left\{ \rho \left( 1 - \frac{v^2}{c^2} \right) r \frac{\partial}{\partial r} - \frac{\rho uv}{c^2} \frac{\partial}{\partial \theta} \right\}.
$$

(28)

Then, if $\psi$ is any periodic differentiable function which vanishes in the far field,

$$
\int_D \frac{\psi}{r^2} L \delta \phi \, dS = \int_D \rho M^2 \nabla \phi \cdot \nabla \frac{\delta h}{h} \, dS,
$$

(29)

where $dS$ is the area element $r \, dr \, d\theta$, and the right hand side has been integrated by parts.
Now we can augment equation (27) by subtracting the constraint (29). The auxiliary function $\psi$ then plays the role of a Lagrange multiplier. Thus,

$$
\delta I = \int_C (q - q_d) \frac{\delta h}{h} \, d\theta - \int_C \delta \phi \frac{\partial}{\partial \theta} \left( \frac{q - q_d}{h} \right) \, d\theta - \int_D \frac{\psi}{r^2} L \delta \phi \, dS + \int_D \rho M^2 \nabla \phi \cdot \nabla \psi \frac{\delta h}{h} \, dS.
$$

Now suppose that $\psi$ satisfies the adjoint equation

$$
L \psi = 0 \quad \text{in } D \tag{30}
$$

with the boundary condition

$$
\frac{\partial \psi}{\partial r} = \frac{1}{\rho} \frac{\partial}{\partial \theta} \left( \frac{q - q_d}{h} \right) \quad \text{on } C. \tag{31}
$$

Then, integrating by parts,

$$
\delta I = - \int_C (q - q_d) \frac{\delta h}{h} \, d\theta + \int_D \rho M^2 \nabla \phi \cdot \nabla \psi \frac{\delta h}{h} \, dS. \tag{32}
$$

Here the first term represents the direct effect of the change in the metric, while the area integral represents a correction for the effect of compressibility. When the second term is deleted the method reduces to a variation of Lighthill's method.
Equation (32) can be further simplified to represent $\delta I$ purely as a boundary integral because the mapping function is fully determined by the value of its modulus on the boundary. Set

$$\log \frac{dz}{d\sigma} = \mathcal{F} + i\beta,$$

where

$$\mathcal{F} = \log \left| \frac{dz}{d\sigma} \right| = \log h,$$

and

$$\delta \mathcal{F} = \frac{\delta h}{h}.$$  

Then $\mathcal{F}$ satisfies Laplace’s equation

$$\Delta \mathcal{F} = 0 \quad \text{in} \quad D,$$

and if there is no stretching in the far field, $\mathcal{F} \to 0$. Introduce another auxiliary function $P$ which satisfies

$$\Delta P = \rho M^2 \nabla \psi \cdot \nabla \psi \quad \text{in} \quad D,$$

and

$$P = 0 \quad \text{on} \quad C.$$
Then after integrating by parts we find that

$$\delta I = \int_C G \delta F_c \, d\theta,$$

where $F_c$ is the boundary value of $F$, and

$$G = \frac{\partial P}{\partial r} - (q - q_d) q.$$

(34)

Thus we can attain an improvement by a modification

$$\delta F_c = -\lambda \tilde{G}$$

in the modulus of the mapping function on the boundary, which in turn defines the computed mapping function since $F$ satisfies Laplace's equation. In this way the Lighthill method is extended to transonic flow.