DEVELOPMENT OF COMPUTATIONAL TECHNIQUES FOR TRANSONIC FLOWS: AN HISTORICAL PERSPECTIVE

David A. Caughey
Sibley School of Mechanical and Aerospace Engineering
Cornell University
Ithaca, New York 14853-7501 U S A

Antony Jameson
Department of Aeronautics and Astronautics
Stanford University
Stanford, California 94305-4035 USA

1. Introduction

The history of computational fluid dynamics (CFD), especially as driven by the requirements of transonic flow prediction, has been largely coincident with the history of these Symposia. The first Symposium Transsonicum [48], held in Aachen in 1962, was held in the early days of commercial jet aviation at a time of intense interest in the transonic drag rise phenomenon. That first symposium undoubtedly contributed to the growing interest in the development of theoretical (and computational) techniques for predicting transonic flows of practical interest. The analytical papers in the first symposium, however, largely were limited to the hodograph method, although there also were some highly approximate techniques to deal with the essential non-linearity of the transonic problem.

Symposium Transsonicum II was held in Göttingen in 1975 [49]. This followed on the heels of the first AIAA CFD Conference [1], held in Palm Springs, California, just two years earlier. Both these conferences were dominated by papers dealing with the numerical solution of the potential equation for transonic flows, often in the small-disturbance approximation. By the time of Symposium Transsonicum III in 1988, [63] also held in Göttingen, the focus of computational work had shifted

Title	Authors	Year
Relaxation methods in fluid mechanics	Lomax & Steger	1975
Finite-element methods in fluid mechanics	S. F. Shen	1977
Numerical methods for compressible, viscous flows	MacCormack & Lomax	1979
Computation of transonic potential flows	Caughey	1982
Grid generation for fluid mechanics computations	Eiseman	1985
Characteristics based schemes for the Euler equations	Roe	1986
Computation of flows with shocks	Moretti	1987
New directions in fluid mechanics	Boris	1989
Finite-element methods for Navier-Stokes Equations	Glowinski & Pironneau	1992
Unstructured grid techniques	Mavriplis	1997
CFD of whole-body aircraft	Agarwal	1999
Preconditioning techniques in fluid mechanics	Turkel	1999

Table 1. Articles relevant to transonic CFD appearing in Annual Reviews of Fluid Mechanics

to solution of the Euler and Navier-Stokes equations, primarily for steady flows. And the present Symposium Transsonicum IV Göttingen sees a continued evolution of interest to unsteady flows, and to the problems of design and optimization.

This history also is seen in the articles relevant to transonic CFD appearing in the Annual Review of Fluid Mechanics, which are summarized in Table 1.

The present article attempts to trace some of the important developments in CFD, as applied to transonic flow problems. An article of this length cannot hope to cover all important topics, even in this restricted fluid mechanics domain, nor can it treat any of them with sufficient depth; in particular, we will limit our summary to the prediction of steady flows. Within these limitations, we do hope to capture the flavor of the developments, and give an overview of the scope of the achievements. Some of these topics were treated in greater depth and detail by Jameson in the 1997 Theodorsen Lecture [27].

2. Early History

The history of numerical techniques for fluid mechanics has followed the trend of developing methods for solving increasingly accurate representations of the physics, for a given level of geometrical complexity, in the hierarchy of mathematical models. These models range from the small-disturbance potential equation to direct numerical simulation (DNS) of turbulent flows. The early history of important ideas and

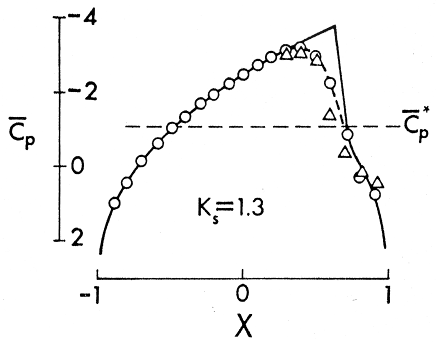


Figure 1. Scaled pressure coefficient on surface of a thin, circular-arc airfoil in transonic flow, compared with experimental data; from Murman & Cole [45].

achievements, dating from before Symposium Transsonicum II includes the development of Riemann-based schemes for gas dynamics by Godunov [12]. This work was published by Godunov in the Soviet Union as early as 1959, but was not appreciated in the West for more than a decade. The multigrid method suffered a similar fate. After being developed for the Laplace equation in 1964 by Fedorenko [11], it lay undiscovered in the West until the early 1970s. Research at the Courant Institute on hyperbolic systems of conservation laws led to the development, in 1960, of the first second-order accurate dissipative methods for these problems by Lax & Wendroff [39]. This class of methods forms the basis of the efficient explicit method developed in 1969 for the Navier-Stokes equations by MacCormack [41]. At the Douglas Aircraft Company, the aerodynamic research group, led by A. M. O. Smith, developed the first panel method for three-dimensional, linear, potential flows (Hess & Smith [14]), and the first implementation of Keller's box method for turbulent boundary layer flows (Keller & Cebeci [36]).

3. The (Nonlinear) Potential Revolution

A major breakthrough was accomplished by Murman & Cole [45] with their development of type-dependent differencing in 1970. They obtained stable solutions by simply switching from central differencing in the subsonic zone to upwind differencing in the supersonic zone, and using a line-implicit relaxation scheme. Their discovery provided major impetus for the further development of CFD by demonstrating that solutions for steady transonic flows could be computed economically. Figure 1, taken from their landmark paper, illustrates the scaled pressure distribution on the surface of a symmetric airfoil.

At nearly the same time, the complex characteristics method, applied to airfoil design by Garabedian & Korn [2], demonstrated the feasibility of designing airfoils for shock-free flow in the transonic regime. Their method was formulated in the hodograph plane, and it required great skill to obtain solutions corresponding to physically realizable shapes. However, the success of this and other hodograph methods (Boerstoel [4]), and also Sobieczky's fictitious gas method [56], in producing shock free flows finally resolved a long standing controversy about the existence of shock free transonic flows. At the same time the development of effective numerical methods for direct calculations enabled the verification of Morawetz' theorem [43] that shock free solutions are isolated points.

The Murman-Cole scheme was extended to the full, nonlinear potential equation with the rotated difference scheme of Jameson [18]. One of the most successful, and widely-used, implementations of that scheme was the Flo-22 code for calculating the transonic potential flow past three-dimensional, swept wings [29]. In spite of the non-conservative formulation of Jameson's scheme used in Flo-22, many aircraft companies have calibrated its behavior and the code still is used today in several companies for preliminary design. It is useful in this role, as it is capable of computing three-dimensional flow fields on grids containing about 150,000 cells in less than 15 seconds on a current laptop computer (having a 1 GHz processor speed).

Murman & Cole's original scheme also was not fully conservative, with the result that it did not enforce correct jump conditions across shocks. In fact, in one-dimensional flow the scheme allows a range of solutions. Murman soon devised a fully conservative version of the scheme [44]. In a one-dimensional model, this could still allow (multiple equilibrium points with combinations of compression and) entropy-violating expansion shocks. However, with a judicious choice of the iterative method, these prove to be unstable equilibrium points that would not be obtained in practice. Stable, fully-conservative schemes for the full potential equation were subsequently devised through the interpretation of upwinding as a controlled form of numerical diffusion; see, e.g., Jameson [19] and Holst [16]. Jameson's fully conservative scheme for the full potential equation was also presented in Göttingen at the second Symposium Transsonicum [20].

Although the type-dependent schemes resulted in a major reduction in computer time required for steady solutions of transonic flow problems, considerable further efficiency was possible through use of the multigrid algorithm. After Fedorenko's development of the method in the 1960s, it was discovered, further developed, and popularized by Brandt in the 1970s [7]. Multigrid was applied to the transonic small-disturbance equation by South & Brandt [57], and to the full potential equation by Jameson [21].

Just as solution of the full potential equation for transonic flows with shock waves was becoming relatively efficient, it was discovered by Steinhoff & Jameson [34] that solutions to the steady potential problem for certain flows containing zones of supersonic flow terminated by shock waves exhibited non-uniqueness. They actually obtained a triple solution for a symmetric Joukowsky airfoil at zero angle of attack, corresponding to a reversal of the lift slope C_{L_α} in a critical Mach number range around $\mathbf{M}_\infty = 0.80$, and a pitchfork bifurcation. This provided further impetus for the development of techniques to solve the Euler and Navier-Stokes equations for these flows. By this time, full potential solutions for three-dimensional wing geometries had become fairly routine, and the solution for a complete Falcon 50 business jet configuration, including horizontal and vertical tails and engine nacelles, was presented by Bristeau, Glowinski, Periaux, Perrier, Pironneau, & Poirier [9] using a finite-element formulation.

4. Solution of Euler Equations

The solution of the Euler equations became a central focus of CFD research in the 1980s. Most of the early solvers tended to exhibit pre- or post-shock oscillations. Also, in a workshop held in Stockholm in 1979 [53] it was apparent that none of the existing schemes converged to a steady state. The Jameson-Schmidt-Turkel (JST) scheme [33], which used Runge-Kutta time stepping and a blend of second- and fourth-differences (both to control oscillations and to provide background dissipation), consistently demonstrated convergence to a steady state, with the consequence that it became one of the most widely used methods.

The issues of oscillation control and positivity had already been addressed by Godunov in his pioneering work in the 1950s (translated into English in 1959). He had introduced the concept of representing the flow as piecewise constant in each computational cell, and solving a Riemann problem at each interface, thus obtaining a first-order accurate solution that avoids non-physical features such as expansion shocks. When this work was eventually recognized in the West, it became very influential. It was also widely recognized that numerical schemes might benefit from distinguishing the various wave speeds, and this motivated the development of characteristics-based schemes.

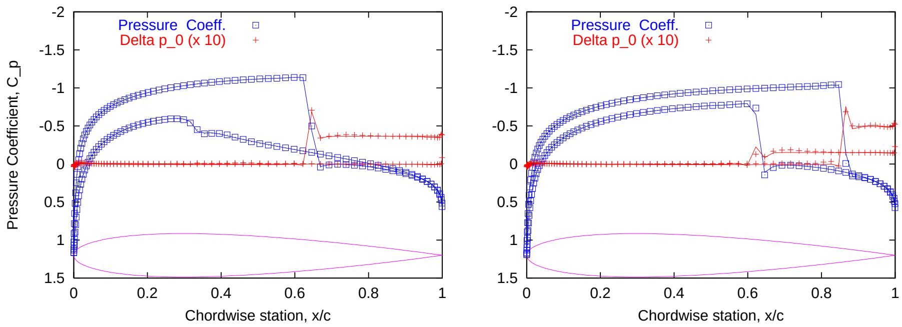
The earliest higher-order characteristics-based methods used flux-vector splitting [58], but suffered from oscillations near discontinuities similar to those of central-difference schemes in the absence of numerical dissipation. The Monotone Upwind Scheme for Conservation Laws (*MUSCL*) of Van Leer [60] extended the monotonicity-preserving behavior of Godunov's scheme to higher order through the use of limiters. A general framework for oscillation control in the solution of non-linear problems is provided by Harten's demonstration that Total Variation Diminishing (*TVD*) schemes are monotonicity-preserving [13].

Roe's introduction of the concept of locally linearizing the equations through a mean value Jacobian [54] had a major impact. It provided valuable insight into the nature of the wave motions and also enabled the efficient implementation of Godunov-type schemes using approximate Riemann solutions. Roe's flux-difference splitting scheme has the additional benefit that it yields a single-point numerical shock structure for stationary normal shocks. Roe's and other approximate Riemann solutions, such as that due to Osher, have been incorporated in a variety of schemes of Godunov type, including the Essentially Non-Oscillatory (*ENO*) schemes of Harten, Engquist, Osher, & Chakravarthy [10].

The use of limiters dates back to the flux-corrected transport (*FCT*) scheme of Boris & Book [6]. They demonstrated essentially perfect propagation of discontinuities at the first AIAA CFD Conference in Palm Springs [5]. The switch in the *JST* scheme [33] can actually be reformulated so that the scheme is local extremum diminishing (*LED*), and may be interpreted as an example of a *symmetric* limited positive scheme [24, 25].

Euler solutions for a complete aircraft configuration, computed on an unstructured, tetrahedral mesh, were presented by Jameson, Baker & Weatherill [28]. This method was essentially equivalent to a Galerkin method with linear elements, stabilized by artificial diffusion to produce an upwind bias. Other finite-element methods also have been developed to solve the Euler equations; an example is the the Streamwise Upwind Petrov-Galerkin (*SUPG*) method (Hughes, Franca, & Mallet [17], Kelly *et al.* [37], Peraire *et al.* [50]; see also Zienkiewicz & Taylor [62]) which automatically incorporates an upwind bias by an appropriate choice of test functions.

Implicit schemes based on the Alternating-Direction Implicit approximate factorization for Euler and/or Navier-Stokes equations were introduced independently by Briley & McDonald [8] and by Beam & Warming [3]. A multiple grid algorithm for the Euler equations was developed by Ni [46], and for the Runge-Kutta algorithm by Jameson [22]. Jameson's scheme was a cell-centered finite-volume scheme that created coarse grids by agglomerating the fine grid cells in groups of four (in two dimensions) or eight (in three dimensions). Subsequently, the



(a) (b)

Figure 2. Surface pressure and entropy distributions for flow past the NACA 0012 airfoil at (a) $\mathbf{M} = 0.80$ and $\alpha = 1.25°$ incidence and (b) $\mathbf{M} = 0.85$ and $\alpha = 1.0°$ incidence. Solutions on 160×32 cell grids after 5 multigrid cycles (symbols) are compared with solutions iteratively converged to machine zero (lines).

idea of agglomeration multigrid has been extended to unstructured grids (Smith [55], Lallemand *et al.* [38], Venkatakrishnan & Mavriplis [61]; see also Mavriplis [42]).

Implicit schemes based on the Lower-Upper (LU) factorization were introduced by Jameson & Turkel [35], and an efficient multigrid implementation of the LU scheme has recently been demonstrated by Jameson & Caughey [30]. The latter scheme seems to hold the record for convergence of solutions to the Euler equations for two-dimensional flows past airfoils, producing results converged to within truncation error in 3 – 5 multigrid cycles for a number of test cases. Figure 2 shows the surface pressure and entropy distributions for the flow past the NACA 0012 airfoil at two transonic conditions. These same conditions were used as standard test cases for the 1979 GAMM Workshop in Stockholm, where up to 5,000 iterations were required for convergence of most algorithms presented [53].

5. Design Methods

The formulation of (inverse) design methods for aerodynamic problems dates to the method based on conformal mapping of Lighthill [40]. For transonic flow problems, the earliest design methods were based on the hodograph method (see, e.g., the numerical hodograph methods of Nieuwland [47] and Boerstoel & Uijlenhoet [4]). The complex characteristics method of Garabedian & Korn [2] and the fictitious gas airfoil design method, inspired by earlier experimental work with a rheoelec-

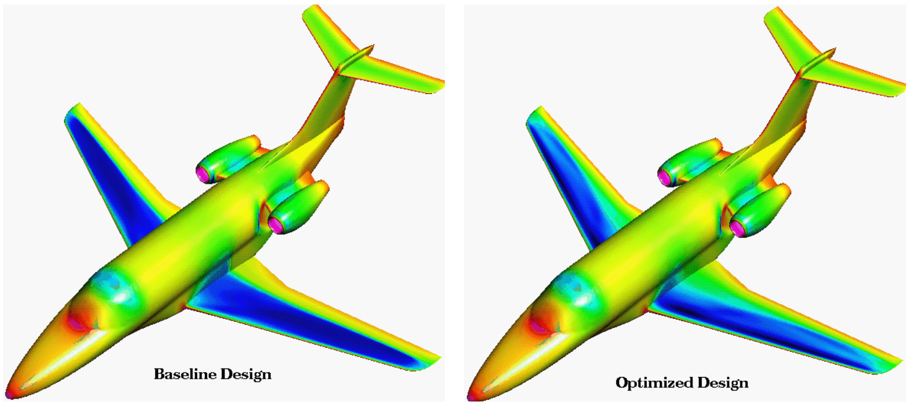


Figure 3. Reduction in shock strength indicated by surface pressure distributions for re-design of wing for complete business jet configuration; wing optimization using adjoint method for the Euler equations.

trical analog approach (Sobieczky *et al.* [56]), have been mentioned in Section 3.

Direct transonic potential flow solvers were coupled with optimization procedures by Hicks & Henne to design transonic airfoils and wings [15]. Takanashi developed an iterative method that used a transonic integral equation to develop "residual corrections" to drive the geometry of a wing in transonic flow to that producing a prescribed pressure distribution [59].

Pironneau has used optimal control techniques for the design of shapes governed by elliptic equations [51], and Jameson has developed the use control theory [23], based on the solution of adjoint problems, and applied it with Reuther to the design of aerodynamic shapes governed by (nonlinear) potential flow [52] and the Euler equations [32], and (with Martinelli and Pierce) to problems governed by the Navier-Stokes equations [26, 31]. Figure 3 illustrates the improvement in pressure distributions for a complete business jet configuration resulting from re-design of the wing, based on solution of the Euler equations.

6. Concluding Remarks

The challenge of calculating transonic flow has been a major driver of the tremendous advances that have been made in computational fluid dynamics over the time span of the Symposia Transsonica, and the use of computational methods has revolutionized aerodynamic design – particularly in the transonic regime. The scope of these advances is illustrated by comparing the complexity of the flows Figs. 1 and 3. Computations that previously took hours on a supercomputer now require seconds of CPU time on a laptop computer. The field has progressed from the *analysis* of two-dimensional shapes in steady flow to: the study of two- and three-dimensional *unsteady* flows; the (optimal?) *design* of complete, complex three-dimensional aircraft configurations; and continued improvement in fidelity of fluid mechanical models – so much so that when numerical and experimental results disagree, often the experimentalist worries more than the computationalist!

Continued advances, in both hardware speed and size, and in algorithm capability and efficiency are almost certain. It is difficult to predict the changes in the aircraft industry that will be necessary to accommodate our predicted ability to performs design optimizations, using the full Navier-Stokes equations for three-dimensional configurations, on wrist-watch-sized computers in the not too distant future.

References

[1] *AIAA Computational Fluid Dynamics Conference (Proceedings)*, Palm Springs, California, July 19-20, 1973.

[2] Bauer, F., Garabedian, P., & Korn, D., 1972. A Theory of Supercritical Wing Sections, with Computer Programs and Examples, *Lecture Notes in Economics and Mathematical Systems 66*, Springer-Verlag, New York.

[3] Beam, R. & Warming, R., 1976. An Implicit Finite Difference Algorithm for Hyperbolic Systems in Conservation Form, *J. Comp. Phys.*, 22:87–109.

[4] Boerstoel, J. W. & Uijlenhoet, R., 1970. Lifting Airfoils with Supercritical Shockless Flow, *ICAS Paper 70-15*.

[5] Book, D. L. & Boris, J. P., 1973. Flux Corrected Transport: A Minimum Error Finite-Difference Technique Designed for Vector Solution of Fluid Equations, *Proc. AIAA Computational Fluid Dynamics Conference*, 182–189.

[6] Boris, J. & Book, D., 1973. Flux Corrected Transport, 1 SHASTA, a Fluid Transport Algorithm that Works, *J. Comp. Phys.*, 11:38–69.

[7] Brandt, A., 1977. Multi-level Adaptive Solutions to Boundary Value Problems, *Math. Comp.*, 31:333–390.

[8] Briley, W. R. & McDonald, H., 1975. Solution of the Three-dimensional Navier-Stokes Equations by an Implicit Technique, *Lecture Notes in Physics*, 35:105–110.

[9] Bristeau, M., Glowinski, R., Periaux, J., Perrier, P., Pironneau, O., & Poirier, C., 1985. On the Numerical Solution of Nonlinear Problems in Fluid Dynamics by Least Squares and Finite Element Methods (II), Application to Transonic Flow Simulations, *Comp. Meth. Appl. Mech. and Eng.*, 51:363–394.

[10] Chakravarthy, S., Harten, A., & Osher, S., 1986. Essentially Non-Oscillatory Shock Capturing Schemes of Uniformly Very High Accuracy, *AIAA Paper 86-0339*, AIAA 24th Aerospace Sciences Meeting, Reno, Nevada.

[11] Fedorenko, R., 1964. The speed of convergence of one iterative process, *USSR Comp. Math. and Math. Phys.*, 4:227–235.

[12] Godunov, S., 1959 A difference method for the numerical calculation of discontinuous solutions of hydrodynamic equations, *Mat. Sbornik*, 47:271–306. Translated as JPRS 7225 by U.S. Dept. of Commerce, 1960.

[13] Harten, A., 1983. High Resolution Schemes for Hyperbolic Conservation Laws, *J. Comp. Phys.*, 49:357–393.

[14] Hess, J. & Smith, A. M. O., 1962. Calculation of non-lifting potential flow about arbitrary three-dimensional bodies, *Douglas Aircraft Report ES 40622*.

[15] Hicks, R. M. & Henne, P. A., 1978. Wing Design by Numerical Optimization, *J. Aircraft*, 15:407–412.

[16] Holst, T. L., 1979. Implicit Algorithm for Conservative Transonic Full Potential Equation using an Arbitrary Mesh, *AIAA Journal*, 17:1038–1045.

[17] Hughes, T., Franca, L., & Mallet, M., 1986. A New Finite Element Formulation for Computational Fluid Dynamics, I, Symmetric Forms of the Compressible Euler and Navier-Stokes Equations and the Second Law of Thermodynamics, *Comp. Meth. Appl. Mech. and Eng.*, 59:223–231.

[18] Jameson, Antony, 1974. Iterative Solution of Transonic Flow over Airfoils and Wings, including Flows at Mach 1, *Comm. Pure Appl. Math.*, 27:283–309.

[19] Jameson, Antony, 1975. Transonic Potential Flow Calculations Using Convervation Form, in *Proc. AIAA 2nd Computational Fluid Dynamics Conference*, Hartford, Connecticut, 148–161.

[20] Jameson, Antony, 1975. Numerical Computation of Transonic Flows with Shock Waves, in *Symposium Transsonicum II*, K. Oswatitsch & D. Rues, Eds., Springer-Verlag, Berlin, 384–414.

[21] Jameson, Antony, 1979. Acceleration of Transonic Potential Flow Calculations on Arbitrary Meshes by the Multiple Grid Method, *Proc. AIAA 4th Computational Fluid Dynamics Conference*, Williamsburg, Virginia, 122–146.

[22] Jameson, Antony, 1983. Solution of the Euler Equations by a Multigrid Method, *Appl. Math. and Comp.*, 13:327–356.

[23] Jameson, Antony, 1988. Aerodynamic Design via Control Theory, *J. Sci. Comp.*, 3:233–260.

[24] Jameson, Antony, 1995. Analysis and Design of Numerical Schemes for Gas Dynamics 1; Artificial Diffusion, Upwind Biasing, Limiters, and their Effect on Multigrid Convergence, *Int'l. J. Comp. Fluid Dyn.*, 4:171–218.

[25] Jameson, Antony, 1995. Analysis and Design of Numerical Schemes for Gas Dynamics 2; Artificial Diffusion and Discrete Shock Structure, *Int'l. J. Comp. Fluid Dyn.*, 5:1–38.

[26] Jameson, Antony, 1995. Optimum Aerodynamic Design using Control Theory, *CFD Review*, 495-528.

[27] Jameson, Antony, 1999. Essential Elements of Computational Algorithms for Aerodynamic Analysis and Design, 1997 Theodorsen Lecture, NASA Langley Research Center, April 10, 1997. Published as *ICASE Report 97-68* and as *NASA CR-97-206268*.

[28] Jameson, A., Baker, T., & Weatherill, N., 1986. Calculation of Inviscid Transonic Flow over a Complete Aircraft, *AIAA Paper 86-0103*, AIAA 24th Aerospace Sciences Meeting, Reno, Nevada.

[29] Jameson, Antony & Caughey, D. A., 1977. Numerical Calculation of the Flow past a Swept Wing, *ERDA Research and Development Report COO-3077-140*, Courant Institute of Mathematical Sciences, New York University.

[30] Jameson, Antony & Caughey, D. A., 2001. How Many Steps are Required to Solve the Euler Equations of Steady, Compressible Flow: In Search of a Fast Solution Algorithm, *AIAA Paper 2001-2673*, 15th Computational Fluid Dynamics Conference, June 11-14, 2001, Anaheim, California.

[31] Jameson, Antony, Martinelli, L., & Pierce, N. A., 1998. Optimum Aerodynamic Design using the Navier-Stokes Equations, *Theor. and Comp. Fluid Dyn.*, 10:213–237.

[32] Jameson, A. & Reuther, J., 1994. Control Theory Based Airfoil Design using the Euler Equations, *AIAA Paper 94-4272*, AIAA-NASA-USAF-SSMO Symposium on Multidisciplinary Analysis and Optimization, Panama City, Florida.

[33] Jameson, Antony, Schmidt, W., & Turkel, E., 1981. Numerical Solutions of the Euler Equations by Finite Volume Methods using Runge-Kutta Time-Stepping Schemes, *AIAA Paper 81-1259*, AIAA 14th Fluid and Plasma Dynamics Conference, Palo Alto, California.

[34] Jameson, Antony & Steinhoff, J., 1981. Non-Uniqueness of FCPOT Numerical Solution, *Proc. GAMM Workshop on Numerical Methods for the Computation of Inviscid Transonic Flows with Shock Waves*, Stockholm, 264–266.

[35] Jameson, A. & Turkel, E., 1981. Implicit schemes and LU decompositions, *Math. Comp.*, 37:385–397.

[36] Keller, H. B. & Cebeci, T. 1972 Accurate numerical methods for boundary layer flows. II: Two-dimensional turbulent flows, *AIAA Journal*, 10:1193–99.

[37] Kelly, D. W., Nakazawa, S., & Zienkiewicz, O. C., 1980. A Note on Anisotropic Balancing Dissipation in the Finite-Element Method Approximation to Convective Diffusion Problems, *Int'l. J. Num. Meth. Engr.*, 15:1705–1711.

[38] Lallemand, M., Steve, H., & Dervieux, A., 1992. Unstructured Multigridding by Volume Agglomeration: Current Status, *Computers & Fluids*, 21:397–433.

[39] Lax, P. & Wendroff, B., 1960. Systems of conservation laws, *Comm. Pure. Appl. Math.*, 13:217–237.

[40] Lighthill, M. J., 1945. A New Method of Two-dimensional Aerodynamic Design, *R & M 1111*, Aeronautical Research Council.

[41] MacCormack, R. W., 1969. The Effect of Viscosity in Hypervelocity Impact Cratering, *AIAA Paper 69-354*, AIAA Hypervelocity Impact Conference, Cincinnati, Ohio, April 30 – May 2, 1969.

[42] Mavriplis, D., 1997. Unstructured Grid Techniques, *Ann. Rev. Fluid Mech.*, 29:473–514.

[43] Morawetz, C. S., 1956. On the Non-existence of Continuous Transonic Flows Past Profiles, Part I, *Comm. Pure Appl. Math.*, 9:45–68.

[44] Murman, E. M., 1974. Analysis of Embedded Shocks Waves Calculated by Relaxation Methods, *AIAA Journal*, 12:626–633.

[45] Murman, E. M. & Cole, J., 1971. Calculation of Plane, Steady, Transonic Flows, *AIAA Journal*, 9:114-121.

[46] Ni, R., 1982. A multiple grid scheme for solving the Euler equations, *AIAA Journal*, 20:1565–1571.

[47] Nieuwland, G. Y., 1967. Transonic Potential Flow around a Family of Quasi-elliptical Airfoil Sections, *Tech. Rept. T 172*, NLR, Netherlands.

[48] Oswatitsch, K., Ed., 1962. *Symposium Transsonicum*, Springer-Verlag.

[49] Oswatitsch, K. & Rues, D., Eds., 1975. *Symposium Transsonicum II*, Springer-Verlag.

[50] Peraire, J., Peiro, J., & Morgan, K., 1993. Multigrid Solution of the Three-dimensional Compressible Euler Equations on Unstructured, Tetrahedral Grids, *Int'l. J. Num. Meth. Engr.*, 36:1029–1044.

[51] Pironneau, O., 1984. *Optimal Shape Design for Elliptic Systems*, Springer-Verlag, New York.

[52] Reuther, J. & Jameson, A., 1994. Control Theory Based Airfoil Design for Potential Flow and a Finite-Volume Discretization. *AIAA Paper 94-499*, 32nd Aerospace Sciences Meeting, Reno, Nevada.

[53] Rizzi, A. & Viviand, H., Eds., 1979. *Numerical Methods for the Computation of Inviscid Transonic Flows with Shock Waves: A GAMM Workshop*, Vieweg & Sohn, Braunschweig.

[54] Roe, P., 1981. Approximate Riemann Solvers, Parameter Vectors, and Difference Schemes, *J. Comp. Phys.*, 43:357–372.

[55] Smith, W. A., 1990. Multigrid Solutions of Transonic Flow on Unstructured Grids, in *Recent Advances and Applications in Computational Fluid Dynamics*, Proc. ASME Winter Annual Meeting, O. Baysal, Ed., 93–103.

[56] Sobieczky, H., Yu, N. J., Fung, K-Y., & Seebass, A. R., 1979. New Method for Designing Shock-free Transonic Configurations, *AIAA Journal*, 17:722–29.

[57] South, Jerry C., Jr. & Brandt, Achi, 1977. Application of a Multi-level Grid Method to Transonic Flow Calculations, in *Transonic Flow Problems in Turbomachinery*, T. C. Adamson, Jr. and M. F. Platzer, Eds., Hemisphere, Washington, 180–207.

[58] Steger, J. & Warming, R., 1981. Flux Vector Splitting of the Inviscid Gas Dynamic Equations with Applications to Finite Difference Methods, *J. Comp. Phys.*, 40:263–293.

[59] Takanashi, S., 1985. Iterative Three-Dimensional Transonic Wing Design using Integral Equations, *J. Aircraft*, 22:655–660.

[60] Van Leer, B., 1974. Towards the Ultimate Conservative Difference Scheme. II: Monotonicity and Conservation combined in a Second-order Scheme, *J. Comp. Phys.*, 14:361–70.

[61] Venkatakrishnan, V. & Mavriplis, D., 1995. Agglomeration Multigrid for the Three-dimensional Euler Equations, *AIAA Journal*, 33:633–640.

[62] Zienkiewicz, O. C. & Taylor, R. L., 2000. *Finite Element Method: Volume 3 – Fluid Dynamics*, Butterworth Heinemann, London.

[63] Zierep, J. & Oertel, H., Eds., 1988. *Symposium Transsonicum III*, Springer-Verlag.