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RESEARCH ARTICLE

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On the spectrum of the Steger-Warming flux-vector splitting scheme

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Summary

The flux-vector splitting scheme of Steger and Warming is a popular approach for the Euler equations. In this work, we consider the spectrum of the scheme and show for $1 \le \gamma \le 5/3$, where γ is the ideal gas constant, that the eigenvalues are strictly real and of an appropriate sign.

KEYWORDS

convection, Euler flow, finite difference, finite volume, hyperbolic, subsonic

1 | INTRODUCTION

The Euler equations are a set of hyperbolic equations, which govern the dynamics of inviscid flow. When solving the Euler equations numerically using upwind differences, it is necessary to have a means of determining in which direction the various waves in the domain are travelling in. A popular approach for this is the *flux-vector splitting* scheme of Steger and Warming.[1] Proposed in 1981, the scheme is both efficient, simple to implement, and can be shown to be positivity preserving.[2]

The most detailed analysis to date of the scheme is due to Lerat[3] who proved that the eigenvalues of the Jacobian matrices associated with the split flux have appropriate signs. However, as the article was never translated into English and published in a journal (Journal de Mécanique Théorique et Appliquée) whose archives are yet to be digitized, this result remains little known within the English speaking community of computational fluid dynamicists. In this paper, we provide an alternative proof that the eigenvalues of the Steger and Warming scheme are both real and have appropriate signs.

2 | FORMULATION

The 1-dimensional Euler equations can be written as

$$\frac{\partial w}{\partial t} + \frac{\partial f}{\partial x} = 0, \tag{1}$$

where

$$w = \begin{bmatrix} \rho \\ \rho u \\ \rho E \end{bmatrix} \quad \text{and} \quad f(w) = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho u H \end{bmatrix},$$

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with ρ being the density of the fluid, u the velocity, p the pressure, E the energy, and $H = E + p/\rho$ the total enthalpy. From the ideal gas law, we also have

$$p = \rho(\gamma - 1)\left(E - \frac{u^2}{2}\right),$$

where γ is the ratio of specific heats. Finally, the speed of sound c is given by

$$c = \sqrt{\frac{\gamma p}{\rho}}.$$

The conservation law (1) admits solutions corresponding to wave motion, with wave speeds equal to the eigenvalues of the Jacobian matrix $A = \partial f/\partial w$, namely, u, $u + c$, and $u - c$. Thus, subsonic flows generally contain waves travelling in both directions.

Equation (1) may be approximated on a uniform grid with a mesh interval Δx by the semidiscrete finite volume scheme

$$\Delta x \frac{\mathrm{d}w_j}{\mathrm{d}t} + h_{j+\frac{1}{2}} - h_{j-\frac{1}{2}} = 0,$$

where w_j is the average value of the state in cell j and $h_{j+\frac{1}{2}}$ is the interface flux between cells j and $j + 1$. In order to devise an upwind scheme, the flux-vector splitting method splits the flux vector f as

$$f = f^+ + f^-,$$

where the eigenvalues of $J^+ = \partial f^+/\partial w$ and $J^- = \partial f^-/\partial w$ are positive and negative, respectively, corresponding to right and left running waves. The numerical flux is then defined as

$$h_{j+\frac{1}{2}} = f_j^+ + f_{j+1}^-.$$

Many such splittings are possible. The simplest splitting is

$$f^{\pm} = \frac{1}{2}(f \pm \epsilon w),$$

where $\epsilon \geq \max|\lambda(A)|$. Here, $\lambda(A)$ denotes the spectrum of A. This effectively introduces scalar diffusion terms and corresponds to the Rusanov flux[4] in a flux-difference splitting scheme. In order to produce a scheme with less artificial diffusion, Steger and Warming proposed the following splitting based on characteristics.

An important property of the Euler equations is that they are homogeneous; that is to say

$$f(w) = Aw. \tag{2}$$

Differentiating the aforementioned expression, we find

$$\frac{\partial f}{\partial w} = \frac{\partial A}{\partial w} w + A,$$

and thus conclude that the first term on the right-hand side resolves to zero.

This property is a consequence of the Euler equations being homogeneous and of degree one in the conservative variables. The eigendecomposition of A is computed as

$$A = V \Lambda V^{-1},$$

where V is a matrix whose columns are the eigenvectors of A and Λ is a diagonal matrix containing the corresponding eigenvalues. Denoting the eigenvalues as λ_i, we may go on to define

$$A^+ = V\Lambda^+ V^{-1} \quad \text{and} \quad A^- = V\Lambda^- V^{-1},$$

where Λ^+ is a diagonal matrix with positive entries and Λ^- is a diagonal matrix with negative entries. These matrices are chosen such that $\Lambda = \Lambda^+ + \Lambda^-$, and thus $A = A^+ + A^-$, where

$$\Lambda_{ii}^+ = \max(0, \lambda_i) \quad \text{and} \quad \Lambda_{ii}^- = \min(0, \lambda_i). \tag{3}$$

With this, we may split the flux as

$$f^+ = A^+ w \quad \text{and} \quad f^- = A^- w. \tag{4}$$

However, the utility of this splitting depends on the Jacobian $J^+ = \partial f^+/\partial w$ having purely positive eigenvalues and the Jacobian $J^- = \partial f^-/\partial w$ having only negative eigenvalues. Since, in general $J^+ \neq A^+$ and $J^- \neq A^-$, this property is far from certain.

Theorem 1. *For $1 \le \gamma \le 5/3$ with $0 \le u \le c$, the splitting of Equation (4) yields a Jacobian J^+ whose eigenvalues are real and nonnegative and a Jacobian J^- whose eigenvalues are real nonpositive.*

Proof. It can be readily shown[5] that the eigenvalues of A are u, $u + c$, and $u - c$ with the corresponding eigenvectors, scaled by density, being given as

$$v^{(1)} = \rho \begin{bmatrix} 1 \\ u \\ u^2/2 \end{bmatrix} \quad \text{and} \quad v^{(2)} = \rho \begin{bmatrix} 1 \\ u + c \\ H + uc \end{bmatrix} \quad \text{and} \quad v^{(3)} = \rho \begin{bmatrix} 1 \\ u - c \\ H - uc \end{bmatrix}.$$

The state vector can be expanded as

$$w = \bar{\gamma} v^{(1)} + \frac{1}{2\gamma} v^{(2)} + \frac{1}{2\gamma} v^{(3)},$$

where $\bar{\gamma} = (\gamma - 1)/\gamma$. Since $Av^{(i)} = \lambda_i v^{(i)}$, it follows that the flux vector can be expanded as

$$f(w) = u\bar{\gamma} v^{(1)} + \frac{u + c}{2\gamma} v^{(2)} + \frac{u - c}{2\gamma} v^{(3)}. \tag{5}$$

Hence,

$$f^+ = u\bar{\gamma} v^{(1)} + \frac{u + c}{2\gamma} v^{(2)} \text{ and } f^- = \frac{u - c}{2\gamma} v^{(3)}. \tag{6}$$

Having established the functional forms of $f^\pm$, we may now proceed to compute the Jacobian matrices $J^\pm$. The characteristic polynomials are then given as

$$p^\pm(\lambda) = \det(J^\pm - \lambda I) = \sum_{i=0}^{3} a_i^\pm \lambda^i,$$

with the eigenvalues of $J^\pm$ corresponding to the roots thereof. The coefficients of these polynomials can be readily obtained through either brute force computation or suitable application of a computer algebra system. It is convenient here to introduce the rescaled polynomials

$$q^\pm(\eta) = \frac{p^\pm(c\eta)}{c^3} = \sum_{i=0}^{3} b_i^\pm \eta^i,$$

which have the advantage of enabling the monomial coefficients to be expressed purely in terms of γ and the Mach number $M = u/c$. Specifically, one finds

$$\begin{aligned}
b_3^+ &= -1, \\
b_2^+ &= \frac{10M + 3}{4} - \frac{4M - 5}{4\gamma}, \\
b_1^+ &= \frac{-15M^2 - 9M + 2}{8} + \frac{11M^2 - 9M - 8}{8\gamma} - \frac{2M^2 - 4M + 2}{8\gamma^2}, \\
b_0^+ &= \frac{6M^3 + 6M^2 - 4M}{16} - \frac{5M^3 - 3M^2 - 13M + 3}{16\gamma} + \frac{M^3 - 3M^2 - 3M + 5}{16\gamma^2},
\end{aligned}$$

and

$$\begin{aligned}
b_3^- &= -1, \\
b_2^- &= \frac{2M - 3}{4} + \frac{4M - 5}{4\gamma}, \\
b_1^- &= \frac{M^2 - 3M + 2}{8} - \frac{5M^2 - 13M + 8}{8\gamma} - \frac{2M^2 - 4M + 2}{8\gamma^2}, \\
b_0^- &= \frac{-M^3 + 3M^2 - 3M + 1}{16\gamma} + \frac{3M^3 - 9M^2 + 9M - 3}{16\gamma^2}.
\end{aligned}$$

Given the bounds on γ and M, we determine the signs of the monomial coefficients to be

$$\begin{aligned} b_3^+ &< 0, & b_3^- &< 0, \\ b_2^+ &> 0, & b_2^- &< 0, \\ b_1^+ &< 0, & b_1^- &\leq 0, \\ b_0^+ &\geq 0, & b_0^- &\leq 0. \end{aligned}$$

Using Descartes' rule of signs, we conclude that $q^+(\eta)$ has no negative roots. Similarly, we also conclude that $q^-(\eta)$ has no positive roots. In the case where $\gamma = 5/3$ and $M = 0$, we find $b_0^+ = 0$, and thus $q^+(\eta)$ admits a zero root. Conversely, when $M = 1$, we find that $b_1^- = b_0^- = 0$, and thus $q^-(\eta)$ admits 2 zero roots.

To complete the proof, we must eliminate the possibility of complex eigenvalues. This can be accomplished by considering the discriminant of $q^\pm(\eta)$, which, for a cubic polynomial, is given by

$$\Delta^\pm = \left(b_2^\pm\right)^2\left(b_1^\pm\right)^2 + 4\left(b_1^\pm\right)^3 - 4\left(b_2^\pm\right)^3 b_0^\pm - 27\left(b_0^\pm\right)^2 - 18 b_2^\pm b_1^\pm b_0^\pm,$$

where we have substituted for $b_3^\pm = -1$. The condition for real roots is that $\Delta^\pm \geq 0$. It can be readily verified graphically that this condition is satisfied along the boundaries of constant γ, as shown in Figure 1, with a similar result holding along boundaries of constant M. Given this, all that remains is to show that $\Delta^\pm$ has no stationary points inside the rectangular region defined by $1 \leq \gamma \leq 5/3$ and $0 \leq M \leq 1$. The stationary points are the roots of

$$\frac{\partial \Delta^\pm}{\partial M} = 0 \quad \text{and} \quad \frac{\partial \Delta^\pm}{\gamma} = 0.$$

These roots can be located by multiplying the derivatives by γ^6 and γ^7, respectively, which results in both expressions taking the form of bivariate polynomials. Computer methods for determining the simultaneous roots of pairs of bivariate polynomials are well established.[6] Although the resulting system is somewhat ill-conditioned using a computer algebra package, one can verify that all of stationary points are either outside of the region or on the boundary. It therefore follows that all of the eigenvalues must be real, as claimed. □

Remark 1. The upper bound on γ is a consequence of the fact that, when $M = 0$, we have $b_0^+ = (5 - 3\gamma)/(16\gamma^2)$, and thus, when $\gamma > 5/3$, the sign flips from positive to negative. This, in turn, enables $q^+(\eta)$ to admit a negative root.

Remark 2. As a point of comparison, the aforementioned proof of Lerat begins by deriving the matrices

$$E^- = \frac{1}{8\gamma}\begin{bmatrix} (1+5\gamma)u - (3+7\gamma)c & 2(\gamma-1)(u+c) & (1+\gamma)u + (5-3\gamma)c \\ -4(u-c) & 4(u-c) & 0 \\ -(5-3\gamma)(u-c) & -2(\gamma-1)(u-c) & (3-\gamma)(u-c) \end{bmatrix},$$

and

$$E^+ = \frac{1}{8\gamma}\begin{bmatrix} (3\gamma-1)u + (3-\gamma)c & -2(\gamma-1)(u+c) & -(1+\gamma)u - (5-3\gamma)c \\ 4(u-c) & 4(2\gamma-1)u + 4c & 0 \\ (5-3\gamma)(u-c) & 2(\gamma-1)(u-c) & 3(3\gamma-1)u + (3+7\gamma)c \end{bmatrix},$$

which are similar to J^- and J^+, respectively. Next, Lerat argues for $0 \leq u < c$ that $E_{ii}^- < 0$ and $E_{ii}^+ > 0$. He then proceeds to demonstrate that both matrices are diagonally dominant. By Gershgorin's circle theorem, it then follows that the signs of the real parts of the eigenvalues of $E^\pm$ are as claimed.

One limitation of this proof is that it does not explicitly exclude the possibility that 2 of the eigenvalues may be complex. In order to show that the eigenvalues are necessarily real, Lerat relies on unspecified numerical computations. In contrast, our proof contains a somewhat more detailed verification thereof, which is based around the considering the stationary points of the discriminant.

Remark 3. It may be noted that each of the eigenvectors $v^{(1)}$, $v^{(2)}$, and $v^{(3)}$ correspond to physically realizable states with nonnegative pressure and positive density. With a forward Euler time discretisation, the finite volume scheme generates a new state w_j after one time step, which, provided the time step is sufficiently small, is a convex combination

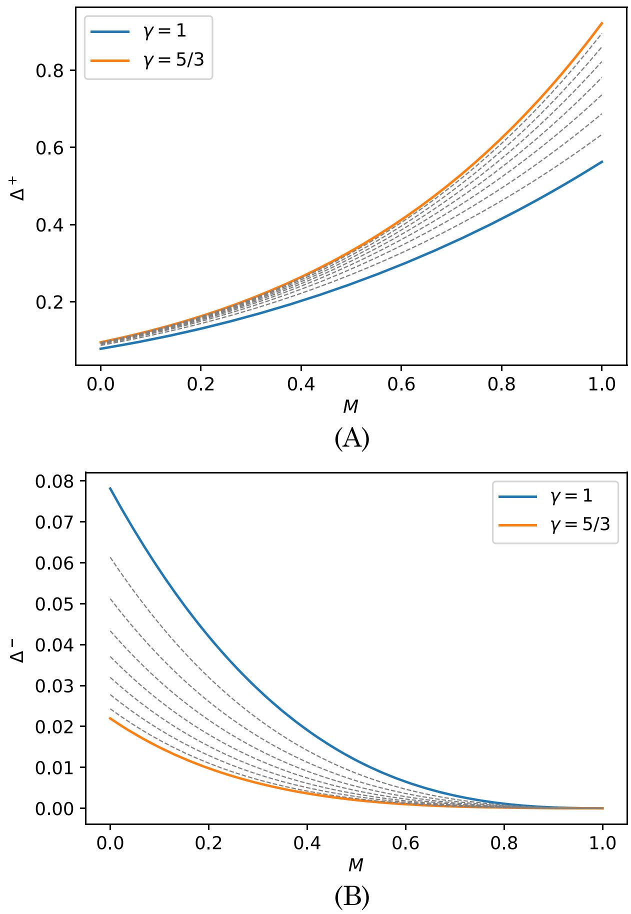

FIGURE 1 Plots of Δ^+ and Δ^- against M for $\gamma = 1$ and $\gamma = 5/3$. We remark here that, in all cases, $\Delta^{\pm} \geq 0$, as required [Colour figure can be viewed at wileyonlinelibrary.com]

of the eigenvectors for the states in cells $j-1$, j, and $j+1$. This leads to the positivity preserving property mentioned in the introduction.

As a concrete example, Figure 2 shows the eigenvalues of $J^{\pm}$ against Mach number in the case where $\gamma = 7/5 = 1.4$. Looking at the figure, we observe that all of the eigenvalues of J^+ are always positive, whilst those of J^- are always negative, as claimed.

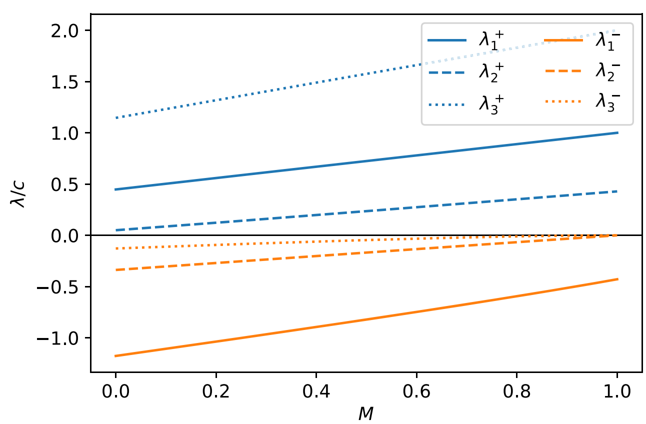

FIGURE 2 Eigenvalues $\lambda^{\pm}$ of $J^{\pm}$ against Mach number when $\gamma = 7/5$ [Colour figure can be viewed at wileyonlinelibrary.com]

3 | CONCLUSION

In this short note, we have provided an alternative proof that the spectrum of the Jacobian matrices J^+ and J^- associated with the flux-vector splitting scheme of Steger and Warming are always nonnegative and nonpositive, respectively. This is shown to hold in the regime of $1 \leq \gamma \leq 5/3$ and thus includes the common case of $\gamma = 7/4 = 1.4$.

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