ARTIFICIAL DIFFUSION AND UPWINDING FOR SYSTEMS OF CONSERVATION LAWS

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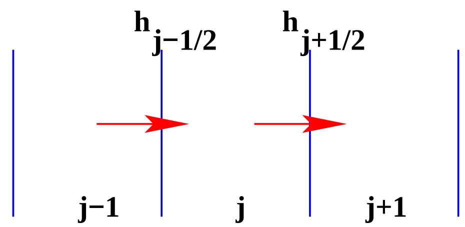
☞ UPWIND BIASING FOR A SYSTEM AND FLUX DIFFERENCE SPLITTING



Suppose that a system of conservation laws

$$\frac{\partial w}{\partial t} + \frac{\partial}{\partial x} f(w) = 0$$

is approximated by the semi-discrete scheme

$$\Delta x \frac{dw_j}{dt} + h_{j+\frac{1}{2}} - h_{j-\frac{1}{2}} = 0$$

where the numerical flux is

$$h_{j+\frac{1}{2}} = \frac{1}{2}\left(f_{j+1} + f_j\right) - d_{j+\frac{1}{2}}$$

and $d_{j+\frac{1}{2}}$ is a diffusive flux . Represent the mean value Jacobian matrix $A(w) = \frac{\partial f}{\partial w}$ by $A_{j+\frac{1}{2}}$ such that

$$A_{j+\frac{1}{2}}\left(w_{j+1} - w_j\right) = f_{j+1} - f_j$$

following the definition of Roe. Set

$$A_{j+\frac{1}{2}} = T \Lambda T^{-1}$$

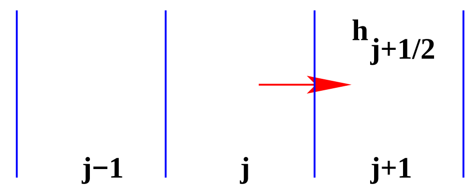
where the eigenvectors of A are columns of T and Λ contains the eigenvalues. Now set

$$d_{j+\frac{1}{2}} = \frac{1}{2}\left|A_{j+\frac{1}{2}}\right|\left(w_{j+1} - w_j\right)$$

where

$$\left|A_{j+\frac{1}{2}}\right| = T\left|\Lambda\right| T^{-1}$$

☞ GENERAL MODEL FOR NUMERICAL FLUX AND ARTIFICIAL DIFFUSION



The numerical flux is

$$h_{j+\frac{1}{2}} = \frac{1}{2}\left(f_{j+1} + f_j\right) - d_{j+\frac{1}{2}}$$

Introduce Roe's linearization

$$f_{j+1} - f_j = A_{j+\frac{1}{2}}\left(w_{j+1} - w_j\right)$$

Define the diffusive flux as

$$d_{j+\frac{1}{2}} = \frac{1}{2} B_{j+\frac{1}{2}}\left(w_{j+1} - w_j\right)$$

where B is a function of A.

Expand B as a power series

$$B = \alpha_0 I + \alpha_1 A + \alpha_2 A^2$$

where higher powers are eliminated by the Cayley Hamilton theorem .

☞ GENERAL MODEL FOR NUMERICAL FLUX AND ARTIFICIAL DIFFUSION (continued)

Three main classes of schemes can thus be identified depending on the number of terms in the expansion

1 term : scalar diffusion

$$d_{j+\frac{1}{2}} = \alpha_0 \Delta w_{j+\frac{1}{2}}$$

2 terms : CUSP, HLL, AUSM

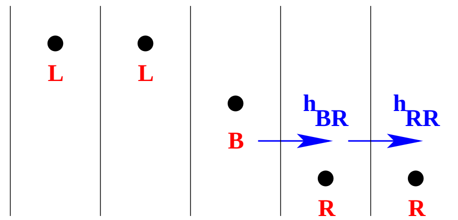
$$\begin{aligned} d_{j+\frac{1}{2}} &= \left(\alpha_0 I + \alpha_1 A\right) \Delta w_{j+\frac{1}{2}} \\ &= \alpha_0 \Delta w_{j+\frac{1}{2}} + \alpha_1 \Delta f_{j+\frac{1}{2}} \end{aligned}$$

3 terms : α_0, α_1, and α_2 can be chosen so that $B = |A|$ (characteristic splitting) by solving

$$\alpha_0 + \alpha_1 \lambda_k + \alpha_2 \lambda_k^2 = |\lambda_k|, \qquad k = 1, 2, 3$$

Criteria for choosing coefficients for these schemes can be based on positivity and numerical shock structure .

☞ DISCRETE SHOCK STRUCTURE WITH CUSP SCHEMES



The numerical flux is defined as

$$h_{j+\frac{1}{2}} = \frac{1}{2}\left(f_{j+1} + f_j\right) - d_{j+\frac{1}{2}}$$

where the diffusive flux is

$$d_{j+\frac{1}{2}} = \frac{1}{2}\alpha^* c \Delta w_{j+\frac{1}{2}} + \frac{1}{2}\beta \Delta f_{j+\frac{1}{2}}$$

and in supersonic flow the scheme is upwind

$$\alpha^* = 0 \quad \text{and} \quad \beta = sign(M) \qquad \text{when } |M| \geq 1$$

At the entrance to the shock

$$h_{LB} = f_L \quad \text{because} \quad M_{LB} > 1$$

maintaining equilibrium. At the downstream side

$$\begin{aligned} h_{RR} &= f_R \qquad \text{(since } w \text{ and } f \text{ are constant)} \\ h_{BR} &= \frac{1}{2}\left(f_R + f_B\right) - \frac{1}{2}\beta\left(f_R - f_B\right) - \frac{1}{2}\alpha^* c\left(w_R - w_B\right) \end{aligned}$$

Equating these

$$(1+\beta)\left(f_R - f_B\right) + \alpha^* c\left(w_R - w_B\right) = 0$$

This is the Hugoniot condition for a shock speed $\dfrac{-\alpha^* c}{1+\beta}$

☞ DISCRETE SHOCK STRUCTURE WITH CUSP SCHEMES (continued)

Since

$$(f_R - f_B) = A_{BR}\left(w_R - w_B\right),$$

this is also an eigenvalue problem ,

$$A_{BR}(w_R - w_B) + \frac{\alpha^* c}{1+\beta}(w_R - w_B) = 0$$

For a solution $\dfrac{-\alpha^* c}{1+\beta}$ must be an eigenvalue of A_{BR}.

When $u > 0$ the only negative eigenvalue is $u - c$.

Hence

$$\frac{\alpha^* c}{1+\beta} = c - u$$

giving a 1 parameter family of solutions where α^* determines β.

☞ E-CUSP SCHEME

(Int J. of CFD, 5, 1998, 1-38)

The diffusive flux formed as a combination of differences of the state and flux vectors

$$d_{j+\frac{1}{2}} = \frac{1}{2}\alpha^*_{j+\frac{1}{2}} c \left(w_{j+1} - w_j\right) + \frac{1}{2}\beta_{j+\frac{1}{2}} \left(f_{j+1} - f_j\right)$$

Full upwinding in supersonic flow is obtained by setting

$$\alpha^*_{j+\frac{1}{2}} = 0, \quad \beta_{j+\frac{1}{2}} = sign(M) \qquad \text{when } |M| > 1$$

To support a stationary discrete shock structure with a single interior point α^* and β cannot be chosen independently.

To satisfy equilibrium at the shock exit

$$\alpha^* = (1+\beta)(1-M) \qquad \text{when } M > 0$$

The choice $\beta = M$ when $|M| < 1$ corresponds to the HLL scheme , which is very diffusive.

☞ E-CUSP SCHEME (continued)

To produce a less diffusive scheme split the flux vectors as

$$f = uw + f_p, \quad f_p = \begin{pmatrix} 0 \\ p \\ up \end{pmatrix}$$

Then

$$f_{j+1} - f_j = \bar{u}\left(w_{j+1} - w_j\right) + \bar{w}\left(u_{j+1} - u_j\right) + f_{p_{j+1}} - f_{p_j}$$

where $\bar{u}$ and $\bar{w}$ are arithmetic averages.

The effective coefficient of convective diffusion is thus

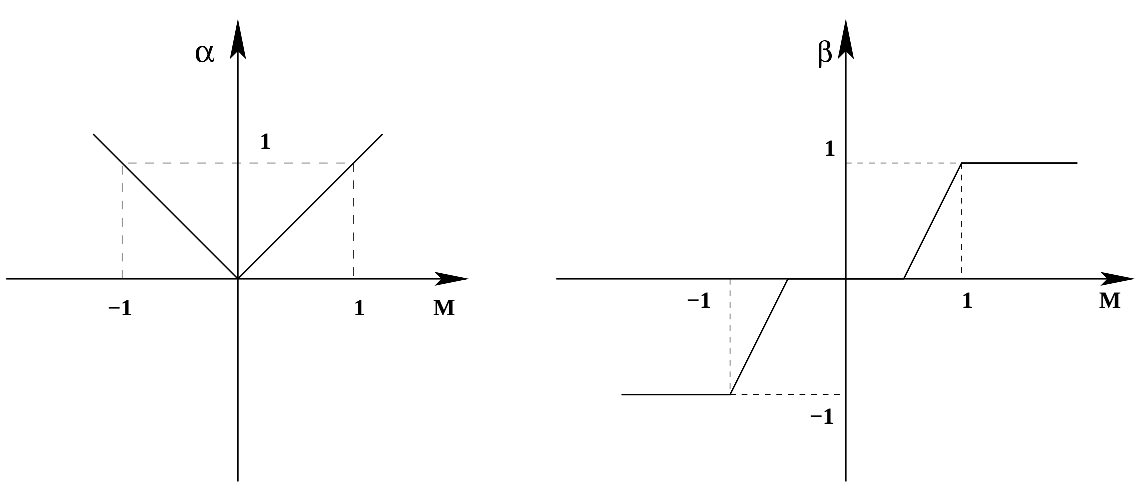
$$\alpha c = \alpha^* c + \beta \bar{u}$$

Take

$$\alpha = |M|$$

Then

$$\beta = \begin{cases} sign(M) & , \quad |M| \geq 1 \\ max(2M - 1, 0) & , \quad 0 \leq M \leq 1 \\ min(2M + 1, 0) & , \ -1 \leq M \leq 0 \end{cases}$$



☞ CHARACTERISTIC DECOMPOSITION OF THE E-CUSP SCHEME

The diffusive flux is

$$\begin{aligned} d_{j+\frac{1}{2}} &= \frac{1}{2}\alpha^* c \Delta w_{j+\frac{1}{2}} + \frac{1}{2}\beta \Delta f_{j+\frac{1}{2}} \\ &= \frac{1}{2}\left(\alpha^* c I + \beta A_{j+\frac{1}{2}}\right) \Delta w_{j+\frac{1}{2}} \end{aligned}$$

where

$$A_{j+\frac{1}{2}} \Delta w_{j+\frac{1}{2}} = \Delta f_{j+\frac{1}{2}}$$

with the characteristic decomposition

$$A_{j+\frac{1}{2}} = R \Lambda R^{-1}, \quad \Lambda = diag\,(u, u+c, u-c)$$

Thus

$$d_{j+\frac{1}{2}} = R M R^{-1}, \quad M = diag\,(\mu_1 c, \mu_2 c, \mu_3 c)$$

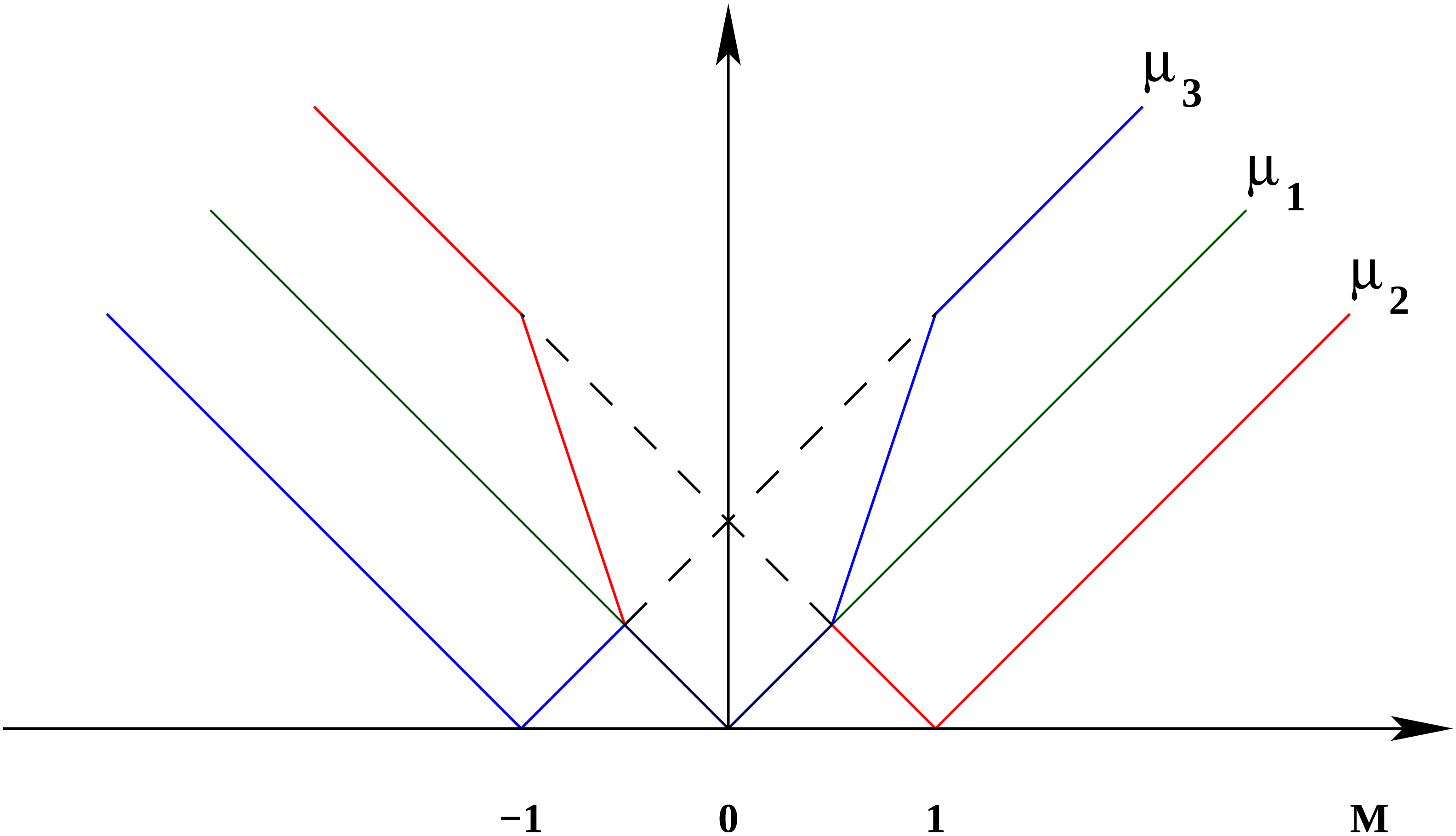
where

$$\begin{aligned} \mu_1 &= \alpha^* + \beta M \\ \mu_2 &= \alpha^* + \beta\,(M+1) \\ \mu_3 &= \alpha^* + \beta\,(M-1) \end{aligned}$$

Also

$$\alpha^* = \alpha - \beta M = |M| - \beta M$$

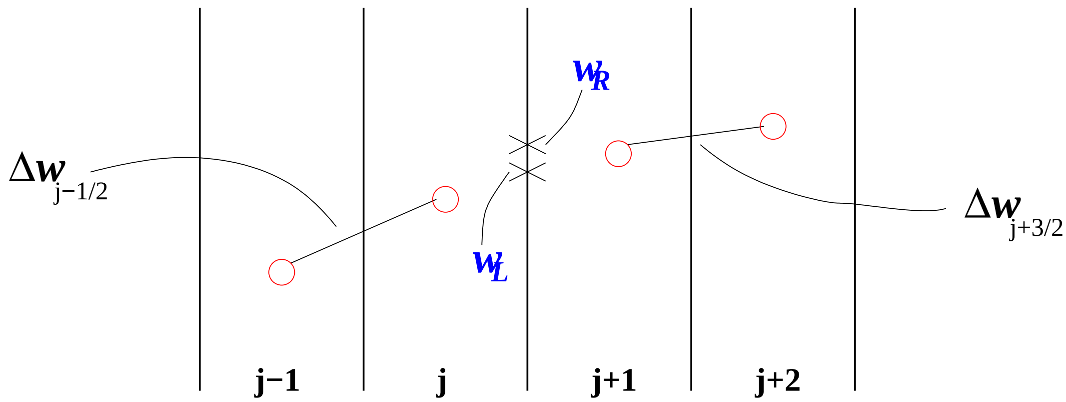
☞ E-CUSP SCHEME



Eigenvalues of the diffusion matrix

The scheme is less diffusive than the characteristic upwind scheme.

☞ HIGHER ORDER CUSP SCHEME



Define left and right states at the interface $\jmath + \frac{1}{2}$

$$
\begin{aligned}
w_L &= w_\jmath + \frac{1}{2} L\left(\Delta w_{\jmath+\frac{3}{2}}, \Delta w_{\jmath-\frac{1}{2}}\right) \\
w_R &= w_{\jmath+1} - \frac{1}{2} L\left(\Delta w_{\jmath+\frac{3}{2}}, \Delta w_{\jmath-\frac{1}{2}}\right) \\
\text{with } f_L &= f\left(w_L\right) \\
f_R &= f\left(w_R\right)
\end{aligned}
$$

Then the numerical flux is

$$
h_{\jmath+\frac{1}{2}} = \frac{1}{2}\left(f_R + f_L\right) - \frac{1}{2}\alpha^* c\left(w_R - w_L\right) - \frac{1}{2}\beta\left(f_R - f_L\right)
$$