Control of a string of vehicles

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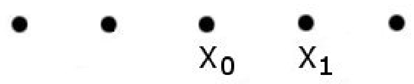
It has been established by wind tunnel testing that the vehicles in a closely spaced string experience a drag reduction. This is true even for the lead vehicle. Moreover, the volume of traffic on a freeway will be increased if the vehicles are more closely spaced. This motivates the study of how to control the spacing of a sting of vehicles. In a finite string the optimal control law would vary with the position of the vehicle in the string. The problem is simplified, however, if one considers the control of an infinite string of vehicles. The situation observed by every vehicle is then the same, with the consequence that the optimal control law must also be the same for every vehicle. The following analysis examines a simplified abstract model of an infinite string of vehicles in order to discover how rapidly the optimal feedback from the other vehicles decays with the separation from a given vehicle.

To simplify the analysis it is assumed that the velocity of each vehicle can be directly controlled. Suppose that the entire string of vehicles is traveling at a speed v, and that d is the desired spacing between vehicles. Then we can represent the absolute position of the j^{th} vehicles as



$$x_j = vt + x_j + jd$$

where x_j is the displacement of the vehicle from its desired position. Then

$$\dot{x}_j = v + \dot{x}_j$$

and we shall suppose that if u_j is the control for the j^{th} vehicle, then

$$\dot{x}_j = u_j$$

or

$$\dot{x}_j = v + u_j$$

In order to measure the performance of the control system we introduce a cost function

$$J = \int^{\infty} \left(M + r \sum u_j^2 \right) dt$$

where

$$\begin{aligned}
M &= \sum_{j=-\infty}^{\infty} \left\{ (x_j - x_{j-1} - d)^2 + \alpha (x_j - vt - jd)^2 \right\} \\
&= \sum_{j=-\infty}^{\infty} \left\{ (x_j + jd - x_{j-1} - jd + d - d)^2 + \alpha x_j^2 \right\} \\
&= \sum_{j=-\infty}^{\infty} \left\{ ((x_j - x_{j-1})^2 + \alpha x_j^2 \right\} \\
&= \cdots x_{-1}^2 - 2x_{-1}x_0 + x_0^2 + \alpha x_{-1}^2 + x_0^2 - 2x_0x_1 + x_1^2 + \alpha x_0^2 + x_1^2 - 2x_1x_2 + x_2^2 + \alpha x_1^2 \cdots \\
&= x^T Q x
\end{aligned}$$

where Q is a Toeplitz matrix with the form

$$Q = \begin{pmatrix} \cdot & \cdot & & & \\ \cdot & \cdot & \cdot & & \\ & -1 & 2+\alpha & -1 & \\ & & \cdot & \cdot & \cdot \\ & & & \cdot & \cdot \end{pmatrix}$$

The elements in the diagonals are

$$\begin{aligned} \dot{q}(0) &= 2+\alpha \\ q(1) &= q(-1) = -1 \\ q(k) &= 0, \quad |k| > 1. \end{aligned}$$

The control problem can now be written in vector matrix notation with infinite vectors and Toeplitz matrices as

$$\dot{x} = Ax + Bu$$

where

$$A = 0, \quad B = I$$

with the cost function

$$J = \int_0^\infty \left(x^T Q x + u^T R u \right) dt$$

where

$$R = rI$$

The optimal control is

$$u = -R^{-1} B^T P x$$

where P satisfies the limiting solution as $t \to \infty$ of the matrix Ricatti equation

$$\dot{P} + A^T P + PA + Q - PBR^{-1}B^T P = 0$$

In this case the equation reduces simply to

$$u = -\frac{1}{r} P x$$

where P is a Toeplitz matrix satisfying

$$P^2 = Q$$

In terms of the diagonal elements $p(k)$ of P, and using indices in brackets instead of subscripts to denote the position in the string,

$$u(j) - \frac{1}{r} \sum_{k=-\infty}^{\infty} p(k)\ x(j+k)$$

with a feedback $\frac{1}{r} p(k)$ from a vehicle separated by k spacings. Also

$$\sum_{k=-\infty}^{\infty} p(k)\ p(j-k) = q(j)$$

This may be conveniently solved by introducing the discrete Fourier transforms

$$\hat{q} = \sum_{k=-\infty}^{\infty} q(k) e^{i\omega k}, \quad \hat{p} = \sum_{k=-\infty}^{\infty} p(k) e^{i\omega k}.$$

Then

$$\begin{aligned}
\hat{q} &= \sum_{j=-\infty}^{\infty} e^{i\omega j} \sum_{k=-\infty}^{\infty} p(k)\ p(j-k) \\
&= \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} e^{i\omega(j-k)} e^{i\omega k} p(k)\ p(j-k) \\
&= \hat{p}^2
\end{aligned}$$

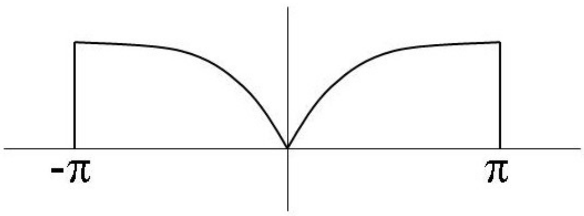
Also

$$\begin{aligned}
\hat{q} &= 2 + \alpha - e^{-i\omega} - e^{i\omega} \\
&= 2(1 - cos\ \omega) + \alpha \\
&= 4sin^2\ \frac{1}{2}\omega + \alpha.
\end{aligned}$$

In the case that $\alpha = 0$, so that there is no penalty on the individual position of each vehicle,

$$\hat{p} = 2\left|sin\ \frac{\omega}{2}\right|.$$

In this case the elements $p(k)$ are simply the coefficients of the Fourier series representing $2\left|sin\ \frac{\omega}{2}\right|$.



Since $\left|sin\ \frac{\omega}{2}\right|$ is even, only cosine terms appear,

$$2\left|sin\ \frac{\omega}{2}\right| = a(0) + \sum_{k=1}^{\infty} a(k)\ cos\ \omega k$$

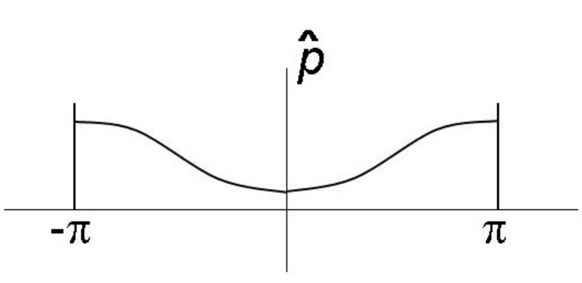
where

$$a(0) = \frac{1}{\pi}\int_0^{\pi} 2sin\frac{\omega}{2}d\omega = -\frac{1}{\pi}\left[4cos\frac{\omega}{2}\right]_0^{\pi} = \frac{4}{\pi}$$

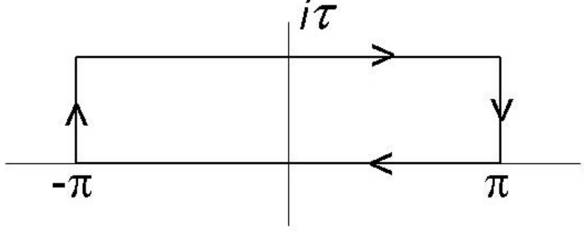
and for $k > 0$

$$\begin{aligned} a(k) &= \frac{2}{\pi}\int_0^{\pi} 2sin\frac{\omega}{2}\ cos\ k\omega d\omega \\ &= \frac{2}{\pi}\int_0^{\pi}\left[sin\left(k+\frac{1}{2}\right)\omega - sin\left(k-\frac{1}{2}\right)\omega\right]d\omega \\ &= -\frac{2}{\pi}\left[\frac{cos\left(k+\frac{1}{2}\right)\omega}{k+\frac{1}{2}} - \frac{cos\left(k-\frac{1}{2}\right)\omega}{k-\frac{1}{2}}\right]_0^{\pi}. \\ &= -\frac{2}{\pi}\left(\frac{1}{k+\frac{1}{2}} - \frac{1}{k-\frac{1}{2}}\right) \\ &= -\frac{2}{\pi}\frac{1}{k^2-\frac{1}{4}}. \end{aligned}$$

Here the coefficients decay algebraically with the separation distance. This is because $\hat{p}(\omega)$ is discontinuous at $\omega = 0$. When $\alpha > 0$, however, $\hat{p}(\omega)$ is continuous.



It is then possible to estimate the decay of the coefficients using contour integration because $\hat{p}(\omega)$ is analytic in a neighborhood of the origin. It turns out that the decay is exponential as shown below.



To evaluate

$$a_k = \frac{1}{\pi}\int_0^{\pi} \sqrt{\alpha + 2(1 - cos\ \omega)} cos\ k\omega d\omega,$$

note this is even and is the real part of

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ik\omega}\sqrt{\alpha + 2(1 - cos\ \omega)} d\omega.$$

Extending the integrand to the complex plane as

$$e^{ik(\omega+i\tau)}\sqrt{\alpha + 2(1 - cos\ (\omega + i\tau))},$$

it is analytic for small enough τ. Hence the integral around the contour in the diagram is zero, and since contributions on the vertical segments cancel, we can evaluate a_n as the real part of

$$c_k = e^{-k\tau}\int_{-\pi}^{\pi} e^{ik\omega}\sqrt{\alpha + 2(1 - cos\ (\omega + i\tau))} d\omega.$$

A bound on τ can be estimated by finding the range of the imaginary axis for which

$$\alpha + 2(1 - cos\ i\tau) > 0$$

or

$$cosh\ \tau < 1 + \frac{1}{2}\alpha.$$

Since when $\tau > 0$,

$$cosh\ \tau = \frac{1}{2}(e^{\tau} + e^{-\tau}) < e^{\tau}.$$

This is satisfied if

$$e^{\tau} < 1 + \frac{1}{2}\alpha$$

or

$$\tau < log(1 + \frac{1}{2}\alpha).$$

Also

$$\int_{-\pi}^{\pi} e^{ik\omega}\sqrt{\alpha + 2(1 - cos\ (\omega + i\tau))}d\omega = \int_{-\pi}^{\pi} e^{ik\omega}\sqrt{\alpha + 2 - e^{i\omega - \tau} - e^{-i\omega + \tau}}d\omega$$

and this has an absolute value bounded by

$$\int_{-\pi}^{\pi} \sqrt{\alpha + 2 + e^{-\tau} + e^{\tau}}d\omega = 2\pi\sqrt{\alpha + 2 + 2cosh\ \tau}.$$

Hence

$$|a_k| \leq e^{-k\tau}\sqrt{\alpha + 2 + 2cosh\ \tau}.$$

Substituting $e^\tau < 1 + \frac{1}{2}\alpha$,

$$|a_k| < \frac{2\sqrt{1 + \frac{1}{2}\alpha}}{(1 + \frac{1}{2}\alpha)^k}.$$

2 Conclusions

3 Acknowledgements

References