1 Introduction

The optimization of linear systems will be considered, subject to restrictions on the form of the control. If $x$ is the state vector and $u$ the control vector the system can be described by

$$\dot{x} = Ax + Bu, \quad x(0) = x_0$$

(1.1)

We are generally interested in controlling an output vector of the form

$$y = Cx$$

(1.2)

and it is convenient to measure the performance by a quadratic cost function

$$J = \int_0^t (y^T Q y + u^T R u) \, ds$$

(1.3)

We seek to minimize $J$. A penalty on $u$ is included to limit its magnitude which might otherwise approach infinity at a minimum of $J$. It is often desirable to use a feedback control of the form

$$u = Dx$$

(1.4)

Then

$$\dot{x} = Fx, \quad x(0) = 0$$

(1.5a)

$$\dot{J} = x^T S x, \quad J(0) = 0$$

(1.5b)

where

$$F = A + BD$$

(1.6a)

$$S = C^T QC + D^T RD$$

(1.6b)

$D$ may be constant or may be allowed to vary with time. For the sake of engineering simplicity we may also wish to restrict the form of $D$. If no feedback is allowed from the $j^{th}$ state variable to the $i^{th}$ control, then

$$D_{ij} = 0$$
2 Properties of the transition matrix

If the system satisfies (1.5a), the principle of superposition may be applied to solutions for different initial conditions,

\[ x(t) = \phi(t, s)x(s) \] (2.1)

where for all \( t, s, \tau \)

\[ \phi(t, t) = I \]

\[ \phi(t, t) \phi(\tau, s) = \phi(t, s) \] (2.2)

and for arbitrary \( x(s) \)

\[ \frac{d}{dt}\phi(t, s)x(s) = F(t)\phi(t, s)x(s) \]

whence

\[ \frac{d}{dt}\phi(t, s) = F(t)\phi(t, s) \] (2.3)

Also

\[ \frac{d}{ds}x(t) = 0 = \left[ \frac{d}{ds}\phi(t, s) \right] x(s) + \phi(t, s)F(s)x(s) \]

whence

\[ \frac{d}{ds}\phi(t, s) = -\phi(t, s)F(s) \] (2.4)

if \( F \) is constant \( \phi \) depends only in the difference \( t - s \).

Then (2.3) and (2.4) yield

\[ F\phi = \phi F \] (2.5)

If there is a forcing function \( y(t) \) such that

\[ \dot{x} = Fx + y \]

then it is easy to verify by differentiation that the solution is

\[ x(t) = \phi(t, s)x(s) + \int_s^t \phi(t, \tau)y(\tau)d\tau \] (2.6a)

In the case of a backward integration it is more convenient to write

\[ x(s) = \phi(s, t)x(t) - \int_s^t \phi(s, \tau)y(\tau)d\tau \] (2.6b)
Gradient of a function with respect to time-varying and fixed parameters

It is convenient to give a parallel treatment of optimization with respect to time-varying and fixed parameters by introducing the concept of the gradient in function space for time-varying parameters. Consider a linear space of vector functions on the interval $(0, t)$ for which the inner product is defined as

$$\langle x, y \rangle = \int_0^t \sum_i x_i y_i ds$$

If $f$ depends on the function $v$, and a small variation $\delta v$ in $v$ causes $f$ to change by

$$\delta f = \left\langle \frac{\partial f}{\partial v}, \delta v \right\rangle$$

in the sense that

$$\lim_{\epsilon \to 0} \frac{f(v + \epsilon h) - f(v)}{\epsilon} = \left\langle \frac{\partial f}{\partial v} \right\rangle$$

then $\frac{\partial f}{\partial v}$ is called the gradient (weak derivation) of $f$ with respect to $v$. If we wish to minimize a function $J$ of the final state $x(t)$,

\[
\dot{x}_i = f_i(x, v, t) \tag{3.1}
\]

and $v$ is a time-varying vector parameter, then for a small change $\delta v$ in $v$,

\[
\begin{align*}
\delta \dot{x}_i &= \sum_k \frac{\partial f_i}{\partial x_k} \delta x_k + \sum_j \frac{\partial f_i}{\partial v_j} \delta v_j, \quad \delta x_i(0) = 0 \tag{3.2a} \\
\delta J &= \sum_i \frac{\partial J}{\partial x_i} \delta x_i(t) \tag{3.2b}
\end{align*}
\]

We introduce a set of 'costate' functions $\psi$, satisfying the 'adjoint' equations

\[
\dot{\psi}_i = -\sum_k \frac{\partial f_k}{\partial x_i} \psi_k, \quad \psi_i(t) = \frac{\partial J}{\partial x_i}
\]
Then
\[
\frac{d}{dt} \left( \sum_i \psi_i \delta x_i \right) = \sum_j \psi_i \frac{\partial f_i}{\partial v_j} \delta v_j
\]
\[
\delta J = \int_0^t \sum_i \sum_j \psi_i \frac{\partial f_i}{\partial v_j} \delta v_j ds = \langle G, \delta v \rangle
\]

Where
\[
G_j(s) = \sum_i \psi_i \frac{\partial f_i}{\partial v_j}
\]

(3.3)

$G$ may thus be identified as the gradient in function space. Evidently if $J$ reaches a minimum it is necessary that $G$ should vanish throughout the interval: otherwise one could find a $\delta v$ such that $\delta J < 0$.

If $v$ is a fixed vector the development is similar.

Denote $\frac{\partial x_i}{\partial v_j}$ by $\sigma_{ij}$. Then
\[
\sigma_{ij} = \sum_k \frac{\partial f_i}{\partial x_k} \sigma_{ij} + \frac{\partial f_i}{\partial v_j} \quad , \quad \sigma_{ij}(0) = 0
\]
\[
\frac{\partial J}{\partial v_j} = \sum_i \frac{\partial J}{\partial x_i} \sigma_{ij}(t)
\]

Again introducing the costate variable $\psi$ :
\[
\frac{d}{dt} \left( \sum_i \psi_i \sigma_{ij} \right) = \sum_i \psi_i \frac{\partial f_i}{\partial v_j}
\]

and denoting the gradient by $G$ ,
\[
G_j = \frac{\partial J}{\partial v_j} = \int_0^t \sum_i \psi_i \frac{\partial f_i}{\partial v_j} ds
\]

(3.4)

For a fixed interval the gradient with respect to a fixed parameter is thus the integral of the gradient with respect to the same parameter when it is allowed to vary with time.

If the cost function is an integral
\[
J = \int_0^t h(x, v, s) ds
\]
then (3.2b) is replaced by

\[ \delta J = \sum_i \frac{\partial h}{\partial x_i} \delta x_i + \sum_j \frac{\partial h}{\partial v_j} \delta v_j \] (3.5)

It is convenient to identify \( J \) with an additional variable \( x_{n+1} \) satisfying

\[ \dot{x}_{n+1} = h(x, v, t) \]

Since \( x_{n+1} \) does not appear in any of the \( f_i \)

\[ \dot{\psi}_{n+1} = 0, \quad \psi_{n+1}(t) = \frac{\partial J}{\partial x_{n+1}} = 1 \]

whence

\[ \psi_{n+1} = 1 \]

Also

\[ \psi_i(t) = \frac{\partial J}{\partial x_i} = 0 \], \( i = 1, n \)

Thus the gradient with respect to a time-varying parameter is

\[ G_j(s) = \sum_i \psi_i \frac{\partial f_i}{\partial v_j} + \frac{\partial h}{\partial v_j} \] (3.6)

where

\[ \dot{\psi}_i = -\sum_k \frac{\partial f_k}{\partial x_i} \psi_k - \frac{\partial h}{\partial x_i} \], \( \psi_i(t) = 0 \) (3.7)

and the gradient with respect to a fixed parameter is

\[ G_j = \int_0^t \left[ \sum_i \psi_i \frac{\partial f_i}{\partial v_j} + \frac{\partial h}{\partial v_j} \right] ds \]
4 Evaluation of the cost function

If (1.5) holds then according to (2.1)

\[ J(t) - J(s) = \int_s^t x^T(\tau) S(\tau) x(\tau) \, d\tau \]
\[ = x^T(s) P(t, s) x(s) \quad (4.1) \]

where

\[ P(t, s) = \int_s^t \phi^T(\tau, s) S(\tau) \phi(\tau, s) \, d\tau \quad (4.3) \]

This may be differentiated, using the properties of the transition matrix expressed in (2.2) and (2.4), to give

\[ \frac{d}{ds} P(t, s) = -S(s) - F^T(t) P(t, s) - P(t, s) F(s) \quad , \quad P(t, t) = 0 \quad (4.4) \]

Denote the outer product \( xx^T \) by \( X \), and denote the trace \( \sum_i A_{ii} \) of a square matrix \( A \) by \( \text{Tr}(A) \). Note that

\[ \text{Tr}(A^T) = \text{Tr}(A) \]

and that as long as \( AB \) is square

\[ \text{Tr}(AB) = \text{Tr}(BA) \]

even if \( A \) and \( B \) are not square.

Then (1.5b) and (4.1) may be written as

\[ \dot{J} = \text{Tr}(SX) \quad (4.5) \]
\[ J(t) - J(s) = \text{Tr}\left[ P(t, s) X(s) \right] \quad (4.6) \]

where \( P \) is determined from (4.3) and

\[ \dot{X} = FX + XF^T \quad , \quad X(0) = X_0 \quad (4.7) \]

If \( A, B, C, D, Q \) and \( R \) are constant so that \( F \) and \( S \) are constant, \( P(t, s) \) depends only
on $t - s$ and may be written as $P(t - s)$, so that

$$\dot{P} = S + F^T P + PF,$$

so that

$$\dot{P} = S + F^T P + PF, \quad P(0) = 0$$

For a constant system an alternative expression for the cost is

$$J = \text{Tr} (SW)$$

where

$$W = \int_0^t X(s) \, ds$$

and (4.6) may be integrated to give

$$\dot{W} = X_0 + FW + WF^T, \quad W(0) = 0 \quad (4.8)$$
5 Gradient with respect to feedback coefficients

The variational equation corresponding to (1.5) are

\[
\begin{align*}
\delta \dot{x} & = F \delta x + B \delta Dx \\
\delta \dot{J} & = 2x^T S \delta x + 2x^T D^T R \delta D x
\end{align*}
\] (5.1a)

The adjoint equations (3.7) become

\[
\dot{\psi} = -F^T \psi, \quad \psi (t) = 0
\] (5.2)

The gradient in function space with respect to \( D_{qr} \) is thus

\[
G_{qr} (s) = \sum_i \psi_i (s) B_{iq} (s) x_r (s) + \sum_i \sum_j x_i (s) D_{ji} (s) R_{jq} (s) x_r (s)
\]

or using matrix notation and denoting the outer product \( xx^T \) by \( X \),

\[
G (s) = B^T (s) \psi (s) x^T (s) + 2R (s) D (s) X (s)
\] (5.3)

The gradient with respect to fixed gains is therefore

\[
G = \int_0^t \left( B^T (s) \psi (s) x^T (s) + 2R (s) DX (s) \right) ds
\] (5.4)

Let \( \zeta (t, s) \) be the transition matrix of the adjoint equations. Since

\[
\psi (t) = 0
\]

when (2.6b) is applied to (5.1a) it follows that

\[
\psi (s) = 2 \int_s^t \zeta (s, \tau) S (\tau) x (\tau) d\tau
\]

where

\[
\frac{d}{ds} \zeta (s, t) = -F^T (s) \zeta (s, t), \quad \zeta (t, t) = I
\]
But if $\phi$ is the transition matrix for $x$, then according to (2.4)

$$\frac{d}{ds} \zeta(s, t) = -F^T(s) \zeta(s, t)$$

so $\zeta(s, t)$ can be identified as $\phi^T(s, t)$ and

$$\psi(s) = 2 \int_s^t \phi^T(\tau, s) S(\tau) x(\tau) d\tau$$

By comparison with (4.2) it can be seen that

$$\psi(s) = 2P(t, s) x(s) \quad (5.5)$$

where $P$ is the kernel of the quadratic form for the cost.

Thus the gradient with respect to time-varying gains is

$$G(s) = 2 \left[ B^T P(t, s) + RD(s) \right] X(s) \quad (5.6)$$

and the gradient with respect to fixed gains is

$$G = 2 \int_0^t \left[ B^T P(t, s) + RD \right] X(s) ds \quad (5.7)$$

If $D$ is allowed to vary with time and there is no restriction on its form then the gradient vanishes regardless of the state when

$$D(s) = -R^{-1}(s) B^T(s) P(t, s) \quad (5.8)$$

This is a natural optimal solution which does not depend on the initial condition and subsequent path. Substituting (5.4) and (1.6) in (4.3), the equation for $P$ when $D$ is optimal is

$$-\frac{d}{ds} P(t, s) = C^TQC + A^T P + PA - PBR^{-1}B^T P, \quad P(t, t) = 0 \quad (5.9)$$

This is the well known matrix Riccati equation, the properties of which have been thoroughly explored. Its integration yields jointly the optimal $P$ and the optimal $D$. 
If $A, B, C, Q,$ and $R$ are constant and $D$ is restricted to be constant then

$$P = P(t - s)$$

In this case if $t \to \infty$ and the system is stable $P$ approaches a limiting value $P_{\infty}$ so that the gradient vanishes for all $X$ if

$$D = -R^{-1}B^TP_{\infty}$$

The integrand is then everywhere zero, so this is also the optimal solution for an infinite internal when $D$ is allowed to vary with time. For a finite interval, on the other hand, $P$ is not constant and $B^TP + RD$ cannot be made to vanish throughout the interval by choice of a fixed $D$, so the optimal fixed $D$ depends on $X$, and therefore on the initial state. One can then fine the minimum of $J$ by using one of the gradient or conjugate gradient techniques for functions of a finite number of variables. The gradient can be evaluated from (5.2) and (5.4) or (4.3) and (5.7).
6 Gradient with respect to the controls

It is interesting to compare the gradient with respect to the feedback matrix of a closed loop system with the gradient with respect to the control vector $u$ of the corresponding open loop system, (1.1) and (1.5b) yield the variational equations

\[
\begin{align*}
\delta \dot{x} &= A\delta x + B\delta u, \quad \delta x (0) = 0 \\
\delta J &= 2x^T V \delta x + 2u^T P \delta u, \quad \delta J (0) = 0
\end{align*}
\]

where

\[ V = C^T QC \]

The adjoint equations (3.7) now become

\[
\dot{\psi} = -A^T \psi, \quad \psi (t) = 0
\]

and the gradient in function space is

\[
g_j (s) = \sum_i \psi_i (s) B_{ij} (s) + 2 \sum_k u_k (s) R_{kj} (s)
\]

or in vector notation

\[
g (s) = B^T (s) \psi (s) + 2R (s) u (s)
\]

Also we can try to satisfy

\[
g (s) = 0
\]

by setting

\[
u (s) = D (s) x (s)
\]

Then $\psi (s)$ is given by (5.5) and it can be seen that the gradient does in fact vanish if $D$ is given by (5.8). The optimal feedback solution with time-varying gains is thus also the optimal solution for a free choice of $u$. 
7 Gradient in terms of the outer product

The gradient may alternatively be deduced from the equation written in terms of the outer product $X$.

(4.4) and (4.6) yield the variation equations

$$\delta \dot{X} = F \delta X + \delta X F^T + B \delta DX + X \delta D^T B^T, \quad \delta X (0) = 0 \quad (7.1a)$$
$$\delta \dot{J} = \text{Tr} (S \delta X) + 2 \text{Tr} (D^T R \delta DX), \quad \delta J (0) = 0 \quad (7.1b)$$

Let $P$ be a symmetric matrix satisfying

$$\dot{P} = -F^T P - PF - S, \quad P(t) = 0 \quad (7.2)$$

Then after cancelling terms, remembering that

$$\text{Tr} (AB) = \text{Tr} (BA)$$

it can be seen that

$$\frac{d}{dt} \left[ \delta J + \text{Tr} (P \delta X) \right] = \text{Tr} (G \delta D^T)$$

where

$$G = 2 \left( B^T P + RD \right) X \quad (7.3)$$

and since $\text{Tr} (P \delta X)$ vanishes at both boundaries

$$\delta J (t) = \int_0^t \text{Tr} (G \delta D^T) \, ds = \int_0^t \sum_i \sum_j G_{ij} \delta D_{ij} \, ds \quad (7.4)$$

Thus $G_{ij}$ is the gradient with respect to $D_{ij}$. By comparing (4.3) and (7.2) $P$ may be identified with the kernal $P(t, s)$ of the quadratic form for the cost. This kernal is thus seen to be precisely the costate variable for the outer product.
8 The statistical case

Let

$$
\dot{x} = Ax + Bu + Gv
$$

(8.1)

where $v$ is a white noise vector with zero mean and correlation matrix $V(t) \delta(t-s)$. Let $\bar{x}$ be the mean of $x$, and let $X$ denote the expectation $E\{xx^T\}$. The covariance matrix $E\{(x-\bar{x})(x-\bar{x})^T\}$, the mean $\bar{x}$, and $X$ are related by the equation

$$
X = E\{(x-\bar{x})(x-\bar{x})^T\} + \bar{x}\bar{x}^T
$$

If

$$
u = Dx
$$

then $\bar{x}$ and $X$ now satisfy the separate equations

$$
\dot{\bar{x}} = F\bar{x}
$$

(8.2)

$$
\dot{X} = FX + XF^T + W
$$

(8.3)

where $F$ is defined by (1.6a) and

$$
W = GV^T
$$

(8.4)

Also if for the performance index we now take the expectation

$$
J = E\left\{\int_0^t (y^TQy + u^TRu) \, ds\right\} = E\left\{\int_0^t x^T Sx \, ds\right\}
$$

where $S$ is defined by (1.6b) then

$$
\dot{J} = Tr(SX)
$$

It can easily be verified by differentiation that

$$
X(t) = \phi(t, s)X(s)\phi^T(t, s) + \int_s^t \phi(t, \tau)W(\tau)\phi^T(t, \tau) \, d\tau
$$
Thus

\[
\dot{J} = \text{Tr} \left[ S(t) \phi(t, s) X(s) \phi^T(t, s) \right] + \text{Tr} \left[ S(t) \int_s^t \phi(t, \tau) W(\tau) \phi^T(t, \tau) d\tau \right]
\]

\[
= \text{Tr} \left[ \phi^T(t, s) S(t) \phi(t, s) X(s) \right] + \text{Tr} \left[ \int_s^t \phi^T(t, \tau) S(t) \phi(t, \tau) W(\tau) d\tau \right]
\]

and

\[
J(t) - J(s) = \text{Tr} \left[ \int_s^t \phi(\tau, s) S(\tau) \phi(\tau, s) X(\tau) d\tau \right]
\]

\[
+ \text{Tr} \left[ \int_s^t d\tau \int_s^t \phi(\tau, \rho) S(\tau) \phi(\tau, \rho) W(\rho) d\rho \right]
\]

The first term is \( \text{Tr} \left[ P(t, s) X(s) \right] \) where \( P \) has been defined in (4.2), and the second term is

\[
\text{Tr} \left[ \int_s^t W(\rho) d\rho \int_s^t \phi^T(\tau, \rho) S(\tau) \phi(\tau, \rho) d\tau \right] = \int_s^t \text{Tr} \left[ W(\rho) P(t, \rho) \right] d\rho
\]

Thus

\[
J(t) - J(s) = \text{Tr} \left[ P(t, s) X(s) \right] + Y(t, s)
\]

where

\[
\frac{d}{ds} Y(t, s) = -\text{Tr} \left[ P(t, s) W(s) \right] , \quad Y(t, s) = 0
\]

The variational equations

\[
\delta \dot{X} = F \delta X + \delta XF^T + B \delta DX + X \delta D^T B^T
\]

\[
\delta \dot{J} = \text{Tr} \left( S \delta X \right) + 2\text{Tr} \left( D^T R \delta DX \right)
\]

are identical to (7.1) with the new definition of \( X \). Thus in the absence of any restriction of the feedback paths the optimal time-varying gains are unchanged by the presence of white noise and may be determined by integrating the matrix Riccati equation (5.9) The cost is increased by the term \( Y(t, s) \).