PRELIMINARY INVESTIGATION OF THE LIFT OF A WING IN AN ELLIPTIC SLIPSTREAM

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SUMMARY

If an aircraft has two or more closely spaced propellers on each wing semi-span, their slipstreams may merge to form a single wide slipstream on each side. Using an elliptic jet as a model of a wide slipstream, the interference potential is determined for wings of high aspect ratio by lifting line theory, and for wings of low aspect ratio by slender body theory. When the wing exactly spans the foci of the ellipse, the formulas reduce to a very simple form, representing a uniform increase in the induced downwash across the span.

1. Introduction

With the advent of STOL aircraft such as the Brequet 941 (McDonnell Douglas 188), which use the slipstream behind the propellers to augment the lift of the wing at low speeds, it is becoming increasingly necessary to find a method of accurately predicting the lift of a wing in a slipstream. There have been numerous studies of a wing in a circular slipstream (ref. 1-4). With the exception of De Young's treatment of a rectangular slipstream (ref. 5), the case of a wing in a wide slipstream, as might be obtained when the slipstreams from two or more propellers merge, seems to have been largely neglected. It was shown in an earlier note (ref. 6) that De Young's solution satisfies the boundary conditions of the problem only when the aircraft has no forward speed.

When the aircraft has forward speed, it is necessary to solve Laplace's equation both inside and outside the slipstream boundary, and to match the two solutions in the correct fashion at the boundary. In this case, the rectangular slipstream is not a very convenient case to treat, because the outside of a rectangle is not amenable to any simple mathematical procedures. A shape that lends itself to comparatively easy treatment both inside and outside is the ellipse. It appears, moreover, that the shape of a wide slipstream might well be more closely approximated by an ellipse than by a rectangle. In this note, a preliminary study is made of the properties of a wing in an elliptic slipstream. Section 2 treats a lifting line lying completely inside the slipstream. Section 3 gives a slender body solution for a wing piercing the slipstream when the immersed portion of the wing has a small aspect ratio. Section 4 treats a wing which exactly spans the foci of the elliptic cross section of the slipstream. The results for this particular arrangement are remarkably simple.

To facilitate the analysis, the following simplifying assumptions are made:

(1) The fluid is inviscid and incompressible.
(2) Before it is influenced by the wing, the slipstream is a uniform jet moving with a velocity V_j different from the velocity V_o of the external stream; transverse velocities and variations of the axial velocity induced by the propellers are neglected.
(3) The jet boundary extends back parallel to the free stream; deflection of the slipstream by the wing is neglected.

Under the first two assumptions, the perturbation velocity due to the wing can be represented both inside and outside the slipstream as the gradient of a velocity potential which satisfies Laplace's equation; and, according to the third assumption, the location of the boundary between the two regions is known. Let p_j and φ_j be the pressure and potential inside the slipstream, and p_j and φ_j the pressure and potential in the external flow. At the boundary, both the pressure and normal flow angle must be continuous; that is

$$p_j = p_o$$

$$\frac{1}{V_j} \frac{\partial \varphi_j}{\partial n} = \frac{1}{V_o} \frac{\partial \varphi_o}{\partial n}$$

where $\frac{\partial}{\partial n}$ represents differentiation in the normal direction. Now if the perturbation velocities are small compared with V_j and V_o, then, neglecting the squares of the perturbation velocities in Bernoulli's equation, the pressure changes inside and outside the slipstream are proportional to $V_j \frac{\partial \varphi_j}{\partial x}$ and $V_o \frac{\partial \varphi_o}{\partial x}$.

Since these must be equal along the whole length of the boundary, the boundary conditions can be expressed as

$$\varphi_j = \mu \varphi_o \tag{1.1}$$

$$\mu \frac{\partial \varphi_j}{\partial n} = \frac{\partial \varphi_o}{\partial n} \tag{1.2}$$

where μ is the velocity ratio

$$\mu = \frac{V_o}{V_j} \tag{1.3}$$

2. Solution for a lifting line inside an elliptic slipstream

If the wing is represented by a lifting line, the downwash in the plane of the wing is generated solely by the trailing vortices. It follows also by consideration of symmetry that the downwash in the plane of the wing is half the downwash far downstream, which may be calculated from the two dimensional potential of the trailing vortex sheet when it is regarded as being infinitely long. Consider therefore the flow in the crossplane due to a symmetric pair of trailing line vortices lying inside the slipstream on the major axis of the ellipse (Fig. 1). Introduce elliptic cylinder coordinates ξ and η by the transformation.

$$y + i z = a \cosh(\xi + i\eta), \quad y = a \cosh\xi \cos\eta, \quad z = a \sinh\xi \sin\eta \tag{2.1}$$

The lines of constant ξ are confocal ellipses with foci at $y = \pm a$, and the lines of constant η are hyperbolas. The line $\xi = 0$ is a slit between the foci, and the slipstream boundary is at $\xi = \xi_o$. Denote the potential of the vortex pair in the absence of a slipstream boundary by φ_v, and let the potential inside and outside the slipstream be

$$\varphi_j = \varphi_v + \Delta\varphi_j \tag{2.2}$$

$$\varphi_o = \varphi_v + \Delta\varphi_o \tag{2.3}$$

The boundary conditions (1.1) and (1.2) then require that at $\xi = \xi_o$

$$\Delta\varphi_j = \mu\,\Delta\varphi_o - (1-\mu)\;\varphi_v \tag{2.4}$$

$$\mu \frac{\partial}{\partial\xi}\Delta\varphi_j = \frac{\partial}{\partial\xi}\Delta\varphi_o + (1-\mu)\frac{\partial}{\partial\xi}\varphi_v \tag{2.5}$$

Laplace's equation remains unchanged in the elliptic coordinates as

$$\frac{\partial^2\varphi}{\partial\xi^2} + \frac{\partial^2\varphi}{\partial\eta^2} = 0$$

Let

$$\varphi = Y(\xi)\ Z(\eta)$$

Then

$$\frac{Y''}{Y} = \frac{-Z''}{Z} = n^2 \text{ say}$$

so that the basic separated solutions are

$$e^{n\xi} \cos n\eta,\ e^{n\xi} \sin n\eta.\ e^{-n\xi} \cos n\eta,\ e^{-n\xi} \sin n\eta\ .$$

where n must be an integer to preserve continuity between $\eta = 0$ and $\eta = 2\pi$. The only combinations of these solutions which are continuous and have continuous first derivatives across the line $\xi = 0$ between the foci are (ref. 8, p. 536)

$$\cosh n\xi \cos n\eta\ ,\ \sinh n\xi \sin n\eta$$

Now the disturbance potentials must have the same symmetry as φ_v, that is they must be antisymmetric in z and symmetric in y. Also, $\Delta\varphi_j$ must be continuous across the line $\xi = 0$, and $\Delta\varphi_o$ must vanish at infinity. Thus, they can be represented as

$$\Delta\varphi_j = \sum_{n=1,3,5...}^{\infty} B_n \sinh n\xi \quad \sin n\eta \tag{2.6}$$

$$\Delta\varphi_o = \sum_{n=1,3,5...}^{\infty} C_n\ e^{-n\xi} \quad \sin n\eta \tag{2.7}$$

Also, Tani and Sanuki have shown (ref. 9) that the stream function of a vortex pair with coordinates ξ_1, η_1, and ξ_1, $\pi - \eta_1$ can be represented outside any ellipse enclosing the pair, both inside and outside the slipstream boundary, as

$$\psi = \sum_{n=1,3,5\ldots}^{\infty} A_n \, e^{-n\xi} \cos n\eta \tag{2.8}$$

where

$$A_n = \frac{2}{\pi n} \cosh n\xi_1 \cos n\eta_1 \tag{2.9}$$

The corresponding potential is

$$\varphi_v = \sum_{n=1,3,5\ldots}^{\infty} A_n \, e^{-n\xi} \sin n\eta \tag{2.10}$$

On substituting the series for φ_v, $\Delta\varphi$ and $\Delta\varphi_o$ in the boundary conditions (2.4) and (2.5), it follows that

$$\sum B_n \sinh n\,\xi_o \sin n\eta = \sum \left[\mu C_n - (1-\mu) A_n\right] e^{-n\xi_o} \sin n\eta$$

$$\sum \mu n\, B_n \cosh n\xi_o \sin n\eta = -\sum \left[C_n + (1-\mu) A_n\right] n e^{-n\xi_o} \sin n\eta$$

These are satisfied if

$$B_n \frac{e^{2n\xi_o} - 1}{2} = \mu C_n - (1-\mu) A_n$$

$$\mu B_n \frac{e^{2n\xi_o} + 1}{2} = -C_n - (1-\mu) A_n$$

whence

$$B_n = -\frac{1-\mu^2}{1+\mu^2 \coth n\xi_o} \quad \frac{2\,A_n}{e^{2n\xi_o}-1} \tag{2.11}$$

$$C_n = -\frac{(1-\mu)\,(1-\mu\, \coth n\xi_o)A_n}{1+\mu^2\, \coth n\xi_o} \tag{2.12}$$

The ratio of the width to the height of the slipstream is

$$\lambda = \coth \xi_o \tag{2.13}$$

and

$$e^{2\xi_o} = \frac{\lambda+1}{\lambda-1}$$

Also let $F_n(\lambda)$ be defined as

$$F_n(\lambda) = \coth n\xi_o = \frac{\left(\frac{\lambda+1}{\lambda-1}\right)^n + 1}{\left(\frac{\lambda+1}{\lambda-1}\right)^n - 1} \tag{2.14}$$

The complete solution for $\Delta\varphi_j$ and $\Delta\varphi_o$ is thus given by (2.6) and (2.7) where

$$B_n = -\frac{1-\mu^2}{1+\mu^2 F_n(\lambda)} \quad \frac{2\,A_n}{\left(\frac{\lambda+1}{\lambda-1}\right)^n - 1} \tag{2.15}$$

$$C_n = -\frac{(1-\mu)\,(1-\mu\, F_n(\lambda))\,A_n}{1+\mu^2\, F_n(\lambda)} \tag{2.16}$$

The variation of the interference potential inside the slipstream with forward speed is determined by the factor $\frac{1-\mu^2}{1+\mu^2 F_n(\lambda)}$ in B_n. Since this factor varies from term to term, the dependence of the potential on forward speed varies at different points in space. Note that

$$\lambda > F_n(\lambda) > 1$$

Also, if λ is not very large, $F_n(\lambda)$ approaches 1 rather rapidly. For example, if $\lambda = 2$, then

$$F_3(\lambda) = \frac{14}{13}, \quad F_5(\lambda) = \frac{122}{121}, \quad \ldots$$

The downwash due to a vortex pair at a given location is determined as the vertical derivative of the total potential $\varphi_v + \Delta\varphi_j$. The downwash due to a distribution of circulation across the span is then obtained by integrating with respect to the vortex coordinates. If it is assumed that the local lift coefficient and circulation are proportional to the local angle of attack allowing for the downwash, substitution of the expression for the downwash leads to an integral equation for the circulation. The solution of this equation yields the wing properties as in standard wing theory (ref. 7).

3. Slender body solution for a wing piercing an elliptic slipstream

The lifting line theory is likely to be reasonably accurate for a wing of high aspect ratio lying completely inside the slipstream. If the wing pierces the slipstream and the immersed portion is of low aspect ratio (Fig. 2), it is more realistic to apply slender body theory, as was done by Graham et al (ref. 2) for the circular jet. In this theory, the streamwise variation of the flow is considered to be unimportant compared with the variation in the cross planes, so that it may be neglected. Then, only the two dimensional Laplace's equation need be satisfied in each cross plane. Let φ_w be the potential of the wing in the absence of a slipstream. Assuming that the wing span is much longer than the width of the slipstream, the further idealization is made that in the region of the slipstream, φ_w can be sufficiently accurately represented as the potential of an infinite flat plate moving in the cross plane; that is

$$\varphi_w = -V_o \alpha z + \text{constant} \tag{3.1}$$

Let the potential inside and outside the slipstream be

$$\varphi_j = \mu \varphi_w + \Delta \varphi_j \tag{3.2}$$

$$\varphi_o = \varphi_w + \Delta \varphi_o \tag{3.3}$$

The boundary condition at the wing surface z=0 is then

$$\frac{\partial}{\partial z} \Delta \varphi_j = \frac{-V_o \alpha}{\mu} - \mu \frac{\partial \varphi_w}{\partial z} = -V_o \alpha \left(\frac{1}{\mu} - \mu \right) \tag{3.4}$$

$$\frac{\partial}{\partial z} \Delta \varphi_j = 0 \tag{3.5}$$

Also, introducing elliptic coordinates ξ and η as before, the boundary conditions (1.1) and (1.2) require that at $\xi = \xi_o$

$$\Delta \varphi_j = \mu \Delta \varphi_o \tag{3.6}$$

$$\mu \frac{\partial}{\partial \xi} \Delta \varphi_j = \frac{\partial}{\partial \xi} \Delta \varphi_o + (1-\mu^2) \frac{\partial}{\partial \xi} \varphi_w \tag{3.7}$$

To facilitate the treatment of the boundary conditions, it is desirable to represent the potentials in terms of the basic solutions

$$e^{n\xi} \cos n\eta , \; e^{n\xi} \sin \eta , \; e^{-n\xi} \cos n\eta , \; e^{-n\xi} \sin n\eta$$

of Laplace's equation in elliptic coordinates. Consider the potential

$$\varphi = \sinh n \xi \sin n \eta$$

On the line z = 0 between the foci $\xi = 0$ and

$$\frac{\partial \varphi}{\partial z} = \frac{1}{a \sin \eta} \frac{\partial \varphi}{\partial \xi} = \frac{n \sin n\eta}{a \sin \eta}$$

On the line z = 0 outside the foci $\eta = 0$ and

$$\frac{\partial \varphi}{\partial z} = \frac{1}{a \sinh \xi} \frac{\partial \varphi}{\partial \eta} = \frac{n \sinh n\xi}{a \sinh \xi}$$

When n = 1, $\frac{\partial \varphi}{\partial z} = \frac{1}{a}$ everywhere on the line z = 0. Thus, the wing boundary condition (3.4) is satisfied by the term

$$-V_o \alpha a \left(\frac{1}{\mu} - \mu \right) \sinh \xi \sin \eta$$

Also, the potential

$$\varphi = \cosh n\xi \cos n\eta$$

has a vanishing derivative $\dfrac{\partial \varphi}{\partial z}$ everywhere on the z axis. whereas all other combinations of the basic solutions have nonvanishing derivatives with respect to z on the line z = 0 either inside or outside the foci. Thus the most general representation in the upper half plane of the interior perturbation potential $\Delta\varphi_j$, such that the wing boundary condition is satisfied, is

$$\Delta\varphi_j = A_o + \sum_{n=1}^{\infty} A_n \cosh n\xi \cos n\eta - V_o \alpha a\left(\frac{1}{\mu} - \mu\right)\sinh\xi \sin\eta \tag{3.8}$$

The exterior perturbation potential $\Delta\varphi_o$ vanishes at infinity, and according to the wing boundary condition (3.5) its derivative with respect to z vanishes at z = 0. Thus its most general representation in the upper half plane is

$$\Delta\varphi_o = \sum_{n=1}^{\infty} B_n \, e^{-n\xi} \cos n\eta \tag{3.9}$$

Since the potentials are antisymmetric their representation in the lower half plane must be the same with the signs of A_n and B_n reversed.

It remains to satisfy the slipstream boundary conditions (3.6) and (3.7). Now according to (3.1)

$$\frac{\partial \varphi_w}{\partial \xi} = -V_o \alpha \frac{\partial z}{\partial \xi} = -V_o \alpha a \cosh\xi \sin\eta$$

Thus, substituting (3.8) and (3.9) in (3.7),

$$\mu\left[\sum_{n=1}^{\infty} n A_n \sinh n\xi_o \cos n\eta - V_o \alpha a\left(\frac{1}{\mu} - \mu\right)\cosh\xi_o \sin\eta\right]$$

$$= -\sum_{n=1}^{\infty} n B_n e^{-n\xi_o} \cos n\eta - (1-\mu^2) V_o \alpha a \cosh\xi_o \sin\eta$$

whence

$$B_n = -\mu \frac{e^{2n\xi_o}-1}{2} A_n$$

Finally (3.6) then requires that

$$A_o + \sum_{n=1}^{\infty} A_n \cosh n\,\xi_o \cos n\,\eta - V_o \alpha\, a\left(\frac{1}{\mu} - \mu\right) \sinh \xi_o \sin \eta$$

$$= -\mu^2 \sum_{n=1}^{\infty} A_n \sinh n\xi_o \cos n\eta$$

or

$$V_o \alpha\, a\left(\frac{1}{\mu} - \mu\right) \sinh \xi_o \sin \eta$$

$$= A_o + \sum_{n=1}^{\infty} A_n \left(\cosh n\xi_o + \mu^2 \sinh n\xi\right) \cos n\eta$$

But in the upper half plane

$$\sin \eta = \frac{2}{\pi} - \frac{4}{\pi} \sum_{n=2,4,6\ldots}^{\infty} \frac{\cos n\eta}{n^2-1}$$

Thus (3.6) is satisfied if

$$A_o = \frac{2}{\pi} V_o \alpha \; a \sinh \xi_o \left(\frac{1}{\mu} - \mu \right)$$

$$A_n = -\frac{4}{\pi} \frac{V_o \alpha \; a \sinh \xi_o}{\cosh n \xi_o + \mu^2 \sinh n \xi_o} \left(\frac{1}{\mu} - \mu \right), \quad n=2,4,6\ldots$$

$$0 \qquad , \quad n=1,3,5\ldots$$

The height of the slipstream is

$$H = a \sinh \xi_o \tag{3.10}$$

The potentials in the upper half plane can thus finally be expressed as

$$\Delta \varphi_j = V_j \alpha H (1-\mu^2) \left[\frac{2}{\pi} - \sinh \xi \sin \eta - \sum_{n=2,4,6\ldots}^{\infty} \frac{4}{\pi} \frac{\cosh n \xi \cos n \eta}{\cosh n \xi_o + \mu^2 \sinh n \xi_o} \right] \tag{3.11}$$

$$\Delta \varphi_o = V_o \alpha H (1-\mu^2) \left[\sum_{n=2,4,6\ldots}^{\infty} \frac{4}{\pi} \frac{\sinh n \xi_o \, e^{-n(\xi - \xi_o)}}{\cosh n \xi_o + \mu^2 \sinh n \xi_o} \right] \tag{3.12}$$

The influence of forward speed varies from term to term through the factor

$$\frac{1 - \mu^2}{1 + \mu^2 \tanh n \xi_o}$$

This can be expressed in terms of the ratio of the slipstream width to height as

$$\frac{1-\mu^2}{1+\dfrac{\mu^2}{F_n(\lambda)}}$$

where $F_n(\lambda)$ is defined by (2.14)

Since the potential is antisymmetric, the pressure difference between the upper and lower surface of the wing is found, on applying Bernoulli's theorem, to be $2\rho V_j \dfrac{\partial \varphi_j}{\partial x}$ inside the slipstream and $2\rho V_o \dfrac{\partial \varphi_o}{\partial x}$ outside it, where ρ is the density of the air. Integrating in the streamwise direction, the change inside and outside the slipstream in the local lift coefficient referred to the freestream velocity is found to be

$$\Delta C\ell_j = \frac{4}{\mu V_o c} \Delta\varphi_j \Big|_{z=0}$$

$$= \begin{cases} 4\alpha \dfrac{H}{c} \dfrac{1-\mu^2}{\mu^2} \left[\dfrac{2}{\pi} - \displaystyle\sum_{n=2,4,6\ldots}^{\infty} \dfrac{4}{\pi} \dfrac{\cos n\eta}{\cosh n\xi_o + \mu^2 \sinh n\xi_o} \right], & \xi = 0 \\ 4\alpha \dfrac{H}{c} \dfrac{1-\mu^2}{\mu^2} \left[\dfrac{2}{\pi} - \displaystyle\sum_{n=2,4,6\ldots}^{\infty} \dfrac{4}{\pi} \dfrac{\cosh n\xi}{\cosh n\xi_o + \mu^2 \sinh n\xi_o} \right], & \xi > 0 \end{cases}$$

(3.13)

$$\Delta C\ell_o = \frac{4}{V_o c} \Delta\varphi_o \Big|_{z=0}$$

$$= 4\alpha \frac{H}{c} (1-\mu^2) \left[\sum_{n=2,4,6\ldots}^{\infty} \frac{4}{\pi} \frac{\sinh n\xi_o \, e^{-n(\xi-\xi_o)}}{\cosh n\xi_o + \mu^2 \sinh n\xi_o} \right]$$

(3.14)

where c is the chord. The total change in lift coefficient referred to freestream velocity and immersed wing area is then

$$\Delta CL = \frac{2}{B} \int_0^{\frac{B}{2}} \Delta C\ell_j \, dy + \frac{2}{B} \int_{\frac{B}{2}}^{\infty} \Delta C\ell_o \, dy \tag{3.15}$$

where B is the width of the slipstream and

$$dy = -a \sin\eta \, d\eta \ , \ \xi = 0$$

$$a \sinh\xi \, d\xi \ , \ \xi > 0$$

4. Lift of a wing spanning the foci of an elliptic slipstream

A particularly simple situation occurs when the wing exactly spans the line joining the foci, and the lift distribution is such that the downwash is constant across the span (Fig. 3). This was already noted by Glauert (ref. 10) in the case of wind tunnel interference. The crossflow far downstream is like that caused by movement of a flat plate, but subject to the slipstream boundary conditions (1.1) and (1.2). Now in the absence of a boundary ($\mu = 1$), the potential in elliptic coordinates for a flat plate moving vertically is

$$\varphi = A\ e^{-\xi}\ \sin\eta \tag{4.1}$$

It is easily verified that on the line $\xi = 0$ joining the foci, the downwash is

$$\frac{-\partial\varphi}{\partial z} = \frac{-1}{a\sin\eta}\ \frac{\partial\varphi}{\partial\xi} = \frac{-A}{a}$$

The basic solutions of Laplace's equation in elliptic coordinates were shown in section 2 to be

$$e^{n\xi}\ \cos n\eta\ ,\ e^{n\xi}\ \sin n\eta\ ,\ e^{-n\xi}\ \cos n\eta\ ,\ e^{-n\xi}\ \sin n\eta$$

In the presence of a slipstream boundary, the external potential must vanish at infinity. Consider therefore a system for which the internal and external potentials are

$$\varphi_j = (A\cosh\xi + B\sinh\xi)\ \sin\eta$$

$$\varphi_o = C\ e^{-\xi}\ \sin\eta$$

On the line $\xi = 0$

$$\frac{\partial \varphi_j}{\partial z} = \frac{1}{a \sin \eta} \quad \frac{\partial \varphi_j}{\partial \xi} = \frac{B}{a}$$

Also, the slipstream boundary conditions (1.1) and (1.2) are satisfied at $\xi = \xi_o$ if

$$A \cosh \xi_o + B \sinh \xi_o = \mu C e^{-\xi_o}$$

$$\mu A \sinh \xi_o + \mu B \cosh \xi_o = - C e^{-\xi_o}$$

According to (2.13), these equations can be expressed in terms of the ratio λ of the width to the height of the slipstream as

$$\lambda A + B = \mu(\lambda -1)C$$

$$\mu A + \lambda \mu B = - (\lambda - 1)C$$

whence

$$B = - \frac{\lambda + \mu^2}{1 + \lambda \mu^2} A$$

$$C = \mu \frac{\lambda + 1}{1 + \lambda \mu^2} A$$

Thus the potentials

$$\varphi_j = A \left[\cosh \xi - \frac{\lambda + \mu^2}{1 + \mu^2} \sinh \xi \right] \sin \eta$$

$$= A \left[e^{-\xi} -(\lambda - 1) \frac{1-\mu^2}{1+ \lambda \mu^2} \sinh \xi \right] \sin \eta \qquad (4.2)$$

$$\varphi_o = \mu \frac{\lambda + 1}{1 + \lambda\mu^2} \quad A\, e^{-\xi} \quad \sin\eta \tag{4.3}$$

represent a vortex wake with a uniform downwash spanning the line joining the foci of an elliptic slipstream. The vorticity is contributed entirely by the term

$$A \cosh\xi \quad \sin\eta$$

which is discontinuous across the line $\xi = 0$, and represents an elliptic lift distribution as in the case of a wing in a uniform stream. The downwash is contributed entirely by the term

$$-A \frac{\lambda + \mu^2}{1 + \lambda\mu^2} \quad \sinh\xi \quad \sin\eta$$

In the absence of a slipstream boundary, the corresponding potential with the same vorticity in the wake is given by (4.1). Thus the part

$$-A\,(\lambda - 1) \frac{1 - \mu^2}{1 + \lambda\mu^2} \quad \sinh\xi \quad \sin\eta$$

can be regarded as the interference potential. At every point in space its strength varies with forward speed according to the factor

$$\frac{1 - \mu^2}{1 + \lambda\mu^2}$$

Also, the downwash for a given vorticity is increased in the ratio

$$\frac{\lambda + \mu^2}{1 + \lambda\mu^2}$$

In the absence of a slipstream boundary, the induced downwash angle far downstream due to a wing with lift coefficient C_L and aspect ratio AR is (ref. 7)

$$\alpha_i = \frac{2CL}{\pi AR}$$

In the slipstream, the induced downwash angle is thus

$$\alpha_i = \frac{2CL}{\pi AR} \frac{\lambda + \mu^2}{1 + \lambda \mu^2} \tag{4.4}$$

where the lift coefficient is referred to the slipstream velocity V_j. The effect of the slipstream is thus a reduction in the effective aspect ratio from AR to AR_μ, where

$$AR_\mu = AR \frac{1 + \lambda \mu^2}{\lambda + \mu^2} \tag{4.5}$$

If the wing is slender, the downwash angle should already approach its final downstream value within the length of the wing chord, so that the condition to be satisfied is that the downwash angle, due to the fully infinite vortex wake, equals the wing trailing edge angle (ref. 11). The lift is thus

$$CL = \frac{\pi AR_\mu}{2} \alpha \tag{4.6}$$

It can be seen from (4.5) that, in the static case, the reduction in the mass flow influenced by the wing causes a reduction in the lift coefficient to the fraction $\frac{1}{\lambda}$ of its freestream value. If the wing is of high aspect ratio so that it can be treated as a lifting line, the downwash angle in the plane of the wing is that for a semi-infinite wake, or half the downwash angle developed far downstream. Thus

$$\alpha_i = \frac{CL}{\pi AR_\mu}$$

If the section lift slope is a_o, and the wing is untwisted, then substituting for CL,

$$\alpha_i = \frac{a_o}{\pi AR_\mu} (\alpha - \alpha_i)$$

whence

$$\alpha_i = \frac{\dfrac{a_o}{\pi AR_\mu}}{1 + \dfrac{a_o}{\pi AR_\mu}} \alpha$$

and

$$CL = \frac{a_o}{1 + \dfrac{a_o}{\pi AR_\mu}} \alpha \qquad (4.7)$$

Finally, for wings of intermediate aspect ratio, it may be expected that the approximation given by Lowry and Polhamus (ref. 12) will hold, with AR replaced by AR_μ ; that is

$$CL = \frac{a_o}{\dfrac{a_o}{\pi AR_\mu} + \sqrt{1 + \left(\dfrac{a_o}{\pi AR_\mu}\right)^2}} \alpha \qquad (4.8)$$

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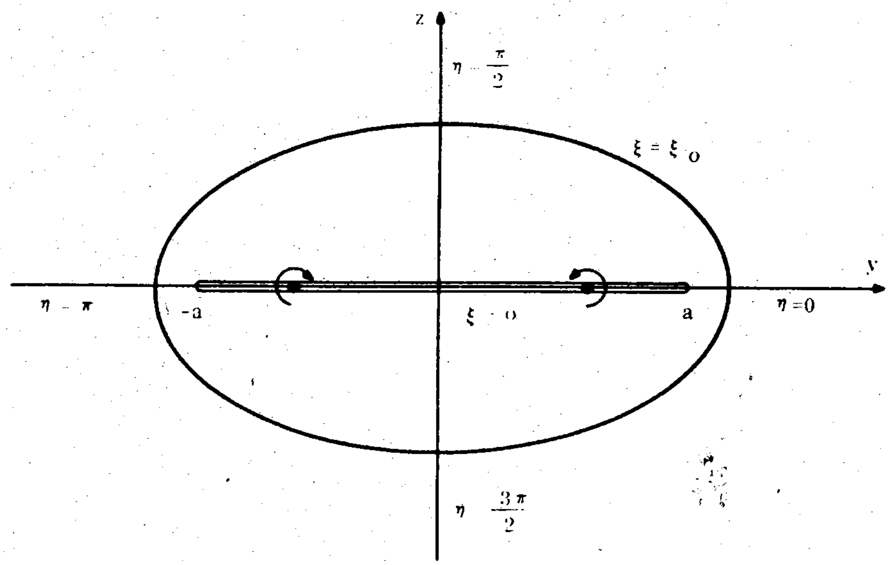

Figure 1. Vortex Pair in Slipstream

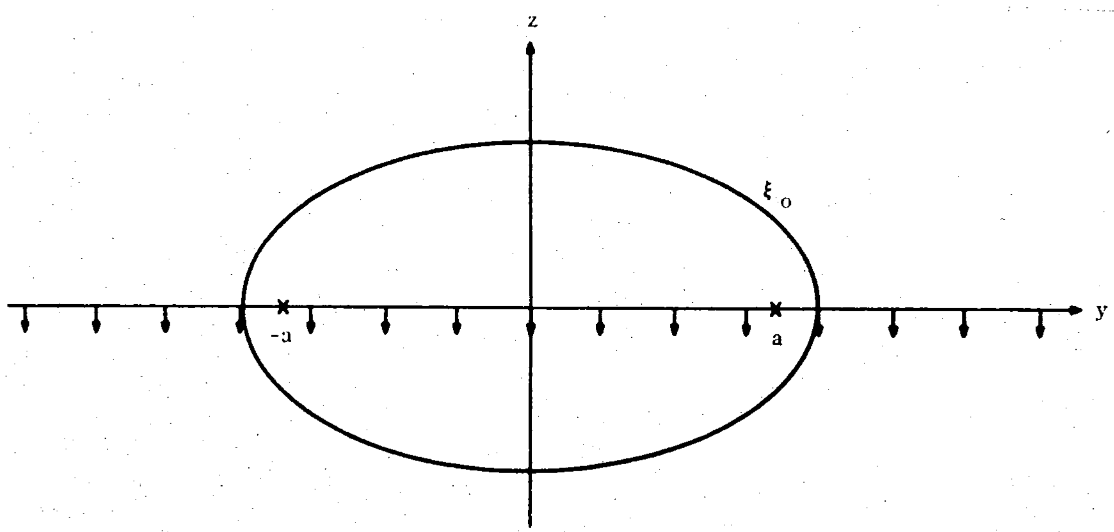

Figure 2. Wing Piercing Slipstream with Immersed Portion of Low Aspect Ratio

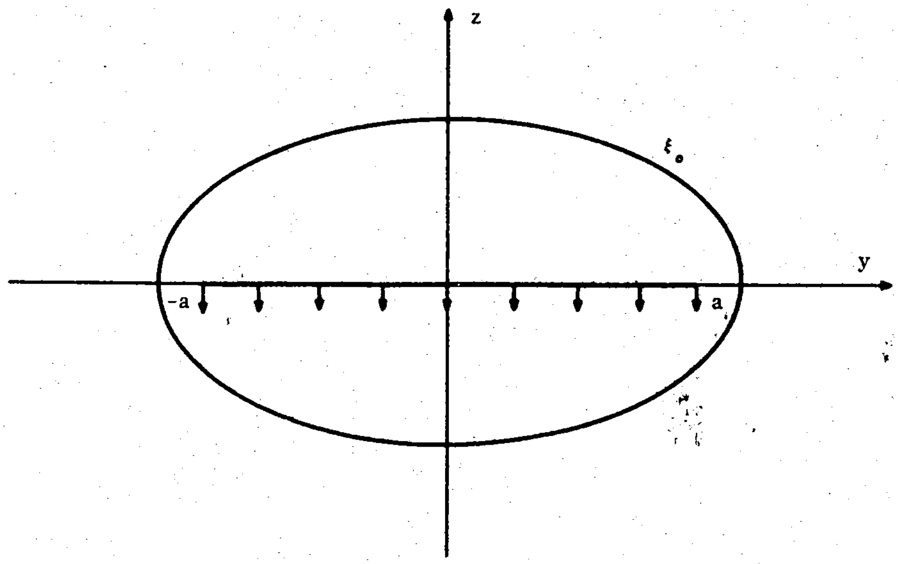

Figure 3. Wing Spanning Foci of Slipstream