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ON THE DESIGN OF A COMPENSATOR WITH DYNAMIC ELEMENTS TO REALIZE THE OPTIMAL CONTROL AND REDUCE SENSITIVITY TO PARAMETER VARIATIONS

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Summary

It is shown that, given a complete measurement of the state, it is possible to realize the optimal control of a linear system by means of a compensator which incorporates dynamic elements. In certain circumstances these dynamic elements can be chosen in such a way that the trajectory deviations, which result from arbitrary small variations in the parameters of plant, are smaller than they would be using the state feedback realization of the optimal control. In the general case a parameter optimization procedure could be used to design a dynamic control equivalent to the optimal state feedback control, which would reduce trajectory deviations resulting from particular changes in the parameters.

Introduction

One of the major problems encountered by anyone trying to design a control system is that the parameters of a physical plant are often not accurately known, or else subject to variations in the course of normal operation. This is particularly the case with an aircraft, the characteristics of which may change drastically over the flight envelope. It is generally desired to produce satisfactory behaviour throughout the flight envelope,while restricting the complexity of the control system.

An effective approach to the design of controls for complex multi-input multi-output linear systems has been found to be optimization of a quadratic performance index. Consider the constant linear system

$$\dot{x} = Ax + Bu \tag{1}$$

where x is the state vector, of dimension n, and u is the control, of dimension n. If the control is chosen to minimize the performance index

$$J = \int_0^\infty (x^T Qx + u^T Ru)\, dt \tag{2}$$

where Q is non-negative definite and R is positive definite, it is well known [1] that the optimal control is a linear function of the state

$$u = Dx \tag{3}$$

where

$$D = -R^{-1}B^T P$$

and P is the unique positive definite solution of the matrix quadratic equation

$$Q + A^T P + PA - PBR^{-1}B^T P = 0 \tag{4}$$

Solutions of this type can be readily computed.

One might be tempted to conclude that, assuming measurements of the complete state vector x are available, there is no profit in considering any other class of control. For a plant which varies, however, the strict realization of a control which is optimal for all conditions of the plant would require a system with feedback gains scheduled as functions of the variable parameters. To limit the complexity, one might prefer to implement a system which is optimal at a number of design points, and fix the gains for a range of parameter values close to each design point. In this case, if the control could be constructed in such a way as to reduce the sensitivity of the system to parameter variations, it should be possible to extend the range of variations over which the gains are fixed, and to reduce the number of design points. Viewed in this light, it will be shown that there may in certain circumstances be an advantage in considering a wider class of control systems, in which the control is realized with the aid of dynamic elements, so that the form of the control is

$$u = Hx + Kz \tag{5}$$

where

$$\dot{z} = Fz + Gx + Eu \tag{6}$$

Such controls can be designed in such a way that for the nominal parameter values they are equivalent to the state feedback control (3), while permitting some additional degrees of freedom in the design, which may be used to try to limit the sensitivity. In this respect, it is worth noting that one of the reasons for using feedback controls themselves is in order to reduce sensitivity to parameter variations, and it has been shown that optimal linear feedback control systems do in a certain sense reduce the sensitivity of the system, compared with open loop controls which would give the same trajectory for the nominal parameter values [2].

Equivalent dynamic and state feedback controls

By a suitable choice of the matrices F, G, and E, which define the auxiliary system (6) used in the realization of the dynamic control (5), it is possible to make z approach Wx, where W may be chosen in any desired manner. Let

$$v = z - Wx \tag{7}$$

Then

$$\dot{v} = \dot{z} - W\dot{x}$$

$$= F(v + Wx) + Gx + Eu - WAx - WBu$$

Now let G and E be chosen to be

$$G = WA - FW \tag{8}$$

$$E = WB \tag{9}$$

It follows that

$$\dot{v} = Fv \tag{10}$$

and if F is stable, $v \to o$ as $t \to \infty$, so that $z \to Wx$. Also

$$u = (H + KW)x + Kv$$

Now let H be chosen to be

$$H = D - KW \tag{11}$$

Then

$$u = Dx + Kv \tag{12}$$

and if $z(0)$ is set equal to $Wx(0)$, it follows from (10) that the dynamic control (5) is identical to the state feedback control (3) for all t.

Sensitivity using the state feedback and dynamic controls

The sensitivity of the system to parameter variations will now be considered, when the alternative controls (3) and (5) are used. An important measure of sensitivity is the deviation in the trajectory, which results from parameter variations [3], [4], [5], [6]. Suppose that the matrices A and B in (1) depend on a set of r parameters, which may be regarded as defining a vector μ. Let the state feedback and dynamic controls (3) and (5) be chosen so that they are identical when μ has its nominal value μ^*, and consider the effect of a variation

$$\mu = \mu^* + \Delta\mu$$

where

$$\Delta\mu = \epsilon\delta\mu$$

and ϵ is a scalar. Define

$$\delta x = \lim_{\epsilon \to o} \frac{1}{\epsilon}\left[x(\mu) - x(\mu^*)\right]$$

$$\delta u = \lim_{\epsilon \to o} \frac{1}{\epsilon}\left[u(\mu) - u(\mu^*)\right]$$

Then the variation of (1) leads to

$$\delta\dot{x} = A\delta x + B\delta u + \delta Ax + \delta Bu \tag{13}$$

where

$$\delta A = \sum_{i=1}^{r} \frac{\partial A}{\partial \mu_i}\delta\mu_i\,, \quad \delta B = \sum_{i=1}^{r} \frac{\partial B}{\partial \mu_i}\delta\mu_i \tag{14}$$

Let the subscripts c and f denote the state feedback control (3) and the dynamic control (5). Then, introducing the variation of (3),

$$\delta\dot{x}_c = (A+BD)\;\delta x_c + Mx \tag{15}$$

where

$$M = \delta A + \delta BD \tag{16}$$

and x is the state vector on the nominal trajectory. Also it follows from (12) that

$$\delta u_f = D\delta x_f + K\delta v$$

where from the definition (7)

$$\delta v = \delta z - W\delta x_f$$

Thus using equations (8) and (9)

$$\begin{aligned}
\delta\dot{v} &= \delta\dot{z} - W\delta\dot{x}_f \\
&= F\,(\delta v + W\delta x_f) + G\delta x_f + E\delta u_f \\
&\qquad\qquad - W\,(A\delta x_f + B\delta u_f + \delta A x + \delta B u) \\
&= F\delta v - W\,(\delta A x + \delta B u)
\end{aligned}$$

Remembering that for the nominal trajectory the error v vanishes and the control is identical to Dx, it follows that

$$\delta\dot{x}_f = (A+BD)\;\delta x_f + BK\delta v + Mx \tag{17}$$

$$\delta\dot{v} = F\delta v - WMx \tag{18}$$

Design of the dynamic control to reduce sensitivity

Comparing (15) and (17) it can be seen that the differential equation satisfied by δx_f differs from that satisfied by δx_c by the presence of the additional term $BK\delta v$. Regarding (18) as a system with Mx as its input, and $BK\delta v$ as its output, one might try to choose W, F, and K in such a way that $BK\delta v$ tends to cancel the forcing term Mx in (17). It is clear that in general a complete cancellation may not be possible, because the rank of B is at most m, whereas M may be of rank n, so that if $m < n$ the vector $BK\delta v$ is restricted to a subspace of the space in which Mx may lie.

Consider, however, the case in which M can be expressed as

$$M = BL \tag{19}$$

where L is an m x n matrix. This is possible when B has rank n. It is also possible in the case of the single input system in phase variable canonical form,

$$A=\begin{bmatrix} 0 & 1 & \cdots & & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & & & & \cdot \\ \cdots & & & & \cdot \\ 0 & 0 & \cdots & & 1 \\ a_1 & a_2 & & \cdots & a_n \end{bmatrix} \qquad B=\begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \\ 1 \end{bmatrix}$$

where only the a_k vary so that

$$L = \begin{bmatrix} \delta a_1 & \delta a_2 & \cdots & \delta a_n \end{bmatrix}$$

If (19) holds, then (17) and (18) become

$$\delta \dot{x}_f = (A+BD)\ \delta x_f + B\ (K\delta v + Lx) \tag{20}$$

$$\delta \dot{v} = F\delta v - WBLx \tag{21}$$

Now it is possible to choose W, F and K in such a way that the output $K\delta v$ of the system (21) approaches the negative of its input Lx. The deviation in v is

$$\delta v(t) = \int_0^t \varphi(t,\tau)\, WBLn(\tau)\, d\tau \tag{22}$$

where φ is the transition matrix, which satisfies

$$\frac{d}{dt}\varphi(t,\tau) = F(t)\,\varphi(t,\tau), \quad \frac{d}{d\tau}\varphi(t,\tau) = -\varphi(t,\tau)\,F(\tau)$$

Since F is stable, it is non-singular, so that (22) can be integrated by parts to give

$$\begin{aligned}\delta v(t) &= -F^{-1}WBLx(t) + \varphi(t,0)\,F^{-1}WBLx(0) \\ &\quad + \int_0^\tau \varphi(t,\tau)\,F^{-1}WBL\,\frac{d}{d\tau}x(\tau)\,d\tau \\ &= -F^{-1}WBLx(t) + \varphi(t,0)\,F^{-1}WBLx(0) \\ &\quad + \int_0^t \varphi(t,\tau)\,F^{-1}WBL(A+BD)\,x(\tau)\,d\tau\end{aligned}$$

Repeating this process, it can be seen that

$$\delta v(t) = -Z\,x(t) + \varphi(t,0)\,Z\,x(0) \tag{23}$$

where

$$Z = F^{-1}WBL + F^{-2}WBL\,(A+BD) + F^{-3}WBL\,(A+BD)^2 \cdots \tag{24}$$

If W F and K can be chosen so that

$$K\,Z = -L \tag{25}$$

then

$$K\delta v(t) + Lx(t) = K\varphi(t,0)\,Z\,x(0) \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

Let λ_i and μ_j be the eigenvalues of A+BD and F. Let $\alpha = \max_i |\lambda_i|$ and $\beta = \min_j |\mu_j|$.

If $\frac{\beta}{\alpha} \gg 1$, then the series (24) is rapidly convergent, so that the first term is dominant, and (25) is approximately satisfied if

$$KF^{-1}WB = I \tag{26}$$

Also

$$\| \varphi(t,0) \| \leq e^{-\beta T}$$

so that $K\delta v + Lx$ is very small within a time of order $\frac{1}{\beta}$.

Example

The following simple example shows that this approach can be effective in leading to a reduction in sensitivity. Consider the first order system

$$\dot{x} = -\mu x + u$$

where μ has the nominal value $\mu^* = 1$

Let

$$u = -x$$

which optimizes the performance index

$$J = \int_0^\infty (3x^2 + u^2)\, dt$$

Also let the dynamic control be

$$u = -(1 + k)\, x + kz$$

where

$$\dot{z} = -fz + (f-1)\, x + u \,, \quad z(0) = x(0).$$

This control satisfies equations (8), (9), and (11) with

$$z = x + v$$

and

$$\dot{v} = -fv$$

Suppose that there is a variation in μ to $1 + \epsilon\delta\mu$ where $\delta\mu$ is constant for all t. Then, using the state feedback control,

$$\delta\dot{x}_c = -2\delta x_c - \delta\mu x, \quad \delta x_c(0) = 0$$

On the nominal trajectory

$$\dot{x} = -2x$$

so that

$$x(t) = e^{-2t}\, x(0)$$

Thus the solution for the trajectory deviation is

$$\delta x_c(t) \;=\; -\,t\,e^{-2t}\;\delta\mu x(0)$$

Also using the dynamic control,

$$\delta\dot{x}_f \;=\; -\,2\delta x_f \;+\; k\delta v \;-\; \delta\mu x$$

where

$$\delta\dot{v} \;=\; -\,f\delta v \;+\; \delta\mu x$$

Substituting the solution for x, δv is found to be

$$\delta v(t) \;=\; \frac{e^{-2t} - e^{-ft}}{f-2}\; x(0)\;\delta\mu$$

$$\;=\; \frac{\delta\mu}{f-2}\left[x(t) - e^{-ft}\,x(0)\right]$$

In this case condition (25) can be exactly satisfied by setting

$$k \;=\; f-2$$

Then

$$k\delta v(t) \;-\; \delta\mu x(t) \;=\; -\,e^{ft}\;x(0)\;\delta\mu$$

and the trajectory deviation is found to be

$$\delta x_f(t) \;=\; \frac{e^{-ft} - e^{-2t}}{f-2}\;\delta\mu x(0)$$

The ratio of the trajectory deviations with the alternative controls is

$$\frac{\delta x_f}{\delta x_c} \;=\; \frac{1 - e^{-(f-2)t}}{(f-2)t}$$

But if $f > 2$, then for all $t > 0$

$$1 > e^{-(f-2)t} > 1 - (f-2)t$$

It follows that for all $t > 0$

$$0 \;<\; \frac{\delta x_f}{\delta x_c} \;<\; 1$$

It may be noted that the approximate condition (26) is satisfied by setting

$$k \;=\; f$$

Then

$$k\delta v(t) - \delta\mu x(t) = \frac{2}{f-2} \delta\mu x(t) - \frac{f}{f-2} e^{-ft} \delta\mu x(0)$$

and

$$\delta\mu_f(t) = \left[\frac{2te^{-2t}}{f-2} - \frac{f}{f-2} \frac{e^{-ft} e^{-2t}}{(f-2)t}\right] \delta\mu x(0).$$

Thus

$$\frac{\delta x_f}{\delta x_c} = \delta - \frac{2}{f-2} (1-\delta)$$

where

$$\delta = \frac{1 - e^{-(f-2)t}}{(f-2)t}$$

Since δ lies between 0 and 1 when $t > 0$ and $f-2 > 0$, it follows that in this case

$$\left| \frac{\delta x_f}{\delta x_c} \right| < 1$$

for all $t > 0$, provided that $f > 4$.

Conclusions

It is clear from the example that it is possible, at least in particular cases, to construct a dynamic control which is identical to the optimal state feedback control when the parameters have their nominal values, and which leads to a reduction in trajectory sensitivity compared with the state feedback control. The example may be contrasted with an example given by Kreindler [4], in which he showed that the trajectory sensitivity of a first order system could be reduced by replacing the optimal state feedback control by a dynamic control, which was close, but not identical, to the state feedback control on the nominal trajectory. In the general case, when condition (19) is not satisfied, so that it is not possible to design the auxiliary system to produce a complete cancellation of the forcing term in (17), it is possible to define some measure of sensitivity such as

$$V \int_0^\infty \delta x^T Z \ \delta x \ dt$$

where Z is positive definite, and to use a parameter optimization procedure to determine the matrices K, F and W in equations (17) and (18) which minimize V, subject to some constraints on the magnitude of their elements. Then the matrices G, E, and H in equations (5) and (6) may be constructed using equations (8), (9), and (11). It is evident that this procedure can at worst result in a system with the same sensitivity as the optimal state feedback system, because this system is a member of the class under consideration, for which $K = 0$. It will in general lead to a solution which is dependent on the nature of the parameter deviations.

References

1. R. E. Kalman, Contributions to the Theory of Optimal Control, Bol. Soc. Mat. Mexicana, 1960, pp. 102-119.
2. E. Kreindler, Closed Loop Sensitivity Reduction of Linear Optimal Control Systems, IEEE Trans. on Automatic Control, Vol. AC-13, No. 3, June 1968, pp. 254-262.
3. E. Kreindler, On the Definition and Application of the Sensitivity Function, J. Franklin Inst., Vol. 285, No. 1, Jan. 1968, pp. 26-36.
4. E. Kreindler, Synthesis of Flight Control Systems Subject to Vehicle Parameter Variations, Air Force Flight Dynamics Laboratory Report AFFDL-TR-66-209, 1966.
5. J. B. Cruz and W. R. Perkins, A New Approach to the Sensitivity Problem in Mult Variable Feedback System Design, IEEE Trans. on Automatic Control, Vol. AC-9, No. 3, June 1964, pp. 216-223.
6. W. R. Perkins and J. B. Cruz, The Parameter Variation Problem in State Feedback Control Systems, Trans. A.S.M.E. Series D, Vol. 87, March 1965, pp. 120-124.