ON THE USE OF CROSS PRODUCT TERMS IN THE PERFORMANCE INDEX TO ELIMINATE SPECIFIED FEEDBACKS FROM A LINEAR OPTIMAL CONTROL

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Summary

It is known that, in general, restricting the feedback structure causes the optimal feedback control of a linear system to depend on the initial condition of the plant. It is shown that it is, however, possible to alter the performance index in such a way that specified feedbacks are eliminated from a control, which minimizes the new performance index for all initial conditions. Analysis of a class of composite systems, in which a main plant is driven by an auxiliary plant, suggests that, for this class of systems, the change in the performance index, required to eliminate feedbacks from the auxiliary plant, should not lead to a large change in the characteristics of the resulting system.

Introduction

One of the most effective approaches to the design of controls for complex multi-input multi-output systems is to minimize a quadratic performance index. It is well known [1] that this leads to a control which is a linear function of the state, and which can thus be realized by feedbacks from each state variable. The need for a measure of every state variable is to be expected, because this is the minimum information required to integrate the equations of motion and predict the path. Measurements of some of the state variables may, however, be relatively unimportant, or it may be inconvenient to mechanize certain feedbacks. If, for example, the main plant is controlled through actuators, and the equations of motion of the actuators are included in the mathematical model, the feedbacks from state variables representing the actuators probably do not convey much information about the motions of the main plant, and they may represent a demand for an increase in actuator performance which is not available.

It is known that if certain feedbacks are eliminated, and the remainder are then chosen to minimize a quadratic performance index, the optimum gains depend on the initial condition of the plant [2]. It will be shown, however, that if a suitable cross product term between the control and state vectors is added to the performance index, then it is possible to construct a control from which specified feedbacks are eliminated, and which minimizes the modified performance index for all initial conditions of the plant. Since the performance index is usually introduced simply as a convenient method of obtaining some desired characteristics in the system, it is not unreasonable to modify the performance index, as long as the resulting system still retains similar characteristics. One might guess that if the feedbacks which it is desired to eliminate are relatively unimportant, the required modification of the performance index should have a correspondingly insignificant effect on the characteristics of the resulting optimal system. This question is investigated for the case of a main plant driven by an auxiliary plant, for which it is desired to restrict the feedbacks to the main plant. It is shown that if the auxiliary plant has a much faster response than the main plant, and the original performance index contains only the state vector of the main plant, then

(a) the unrestricted optimal control with feedbacks from both plants approaches the control which would have been obtained if the main plant were optimized by itself.

(b) a restricted optimal control, with feedbacks from the main plant only, can be constructed which has the same property.

Construction of an optimal control with specified feedbacks eliminated

Consider the linear system

$$\dot{x} = Ax + Bu \tag{1}$$

where x is the state vector, of dimension n, and u is the control vector, of dimension m, and A and B may be time varying matrices. It will be assumed that the columns of B are independent. Let u be chosen to minimize the performance index

$$J = \int_0^T (x^T Qx + u^T Ru)\, dt \tag{2}$$

where Q is non-negative definite and R is positive definite. The optimal control is then [1]

$$u = -R^{-1} B^T Px \tag{3}$$

where P is determined from the matrix Riccati equation

$$-\dot{P} = Q + A^T P + PA - PBR^{-1}B^T P \tag{4a}$$

$$P(T) = 0 \tag{4b}$$

Suppose that the performance index (2) is replaced by the modified index

$$M = \int_0^T (x^T Qx + 2u^T B^T Kx + u^T Ru)\, dt \tag{5}$$

which contains an additional cross product term between x and u. Let

$$u = \hat{u} - R^{-1}B^T Kx \tag{6}$$

Then expressed in terms of $\hat{u}$ the performance index becomes

$$M = \int_0^T \left\{ x^T (Q-K^T BR^{-1}B^T K)\, x + \hat{u}^T R\, \hat{u} \right\} dt \tag{7}$$

and the system equation (1) becomes

$$\dot{x} = (A-BR^{-1}B^T K)\, x + B\hat{u} \tag{8}$$

These equations are in the same form as (1) and (2). Thus the optimal control $\hat{u}$ is

$$\hat{u} = -R^{-1}B^T Px \tag{9}$$

where

$$-\dot{P} = Q - K^T BR^{-1}B^T K + (A-BR^{-1}B^T K)^T P$$
$$+ (A-BR^{-1}B^T K) P - PBR^{-1}B^T P \qquad (10a)$$

$$P(T) = 0 \qquad (10b)$$

It follows from (6) that the control u which minimizes the performance index (5) is

$$u = Dx \qquad (11)$$

where

$$D = -R^{-1}B^T(P+K) \qquad (12)$$

and P satisfies equation (10), which may be rearranged as

$$-\dot{P} = Q + A^T P + PA - D^T RD \qquad (13a)$$

$$P(T) = 0 \qquad (13b)$$

The matrix K in the modified performance index M has not yet been specified. Let it now be chosen in such a way that certain specified feedbacks are eliminated. If, for example, D_{rs} is to be eliminated, then according to (12) K must satisfy

$$\sum_{i=1}^{n} U_{ri} K_{is} = - \sum_{i=1}^{m} U_{ri} P_{is} \qquad (14)$$

where

$$U = R^{-1}B^T \qquad (15)$$

and the assumptions that R is positive definite, and that B has independent columns, ensure that the rows of U are independent. Provided that the control u is not of greater dimension than the state x, it then follows that a solution to equation (14) always exists, and is in general not unique. A case of particular interest is when it is desired to use feedbacks only from a specified output vector of dimension p,

$$y = Cx \qquad (16)$$

where C is a given p x n matrix. Then (14) is satisfied by setting

$$K = P\ (EC-I) \tag{17}$$

where E is an arbitrary n x p matrix. The optimal control becomes

$$u = -R^{-1}B^{T}PEy \tag{18}$$

Since it is proposed to modify the performance index by a term which depends on the equations for determining the control, it needs to be verified that if the control and the performance index are jointly constructed in this manner, the control actually minimizes the resulting performance index. This will now be proved.

Theorem 1

Given the system (1), let P satisfy (13), and let the control law be defined by (11) and (12), where K is chosen to satisfy (14), so that specified feedbacks are eliminated. Then the performance index (5) is minimized.

Proof: Let u be an arbitrarily chosen control.

Using (13) and (1)

$$x^{T}Qx = x^{T}\dot{P}x - (\dot{x} - Bu)^{T}Px - x^{T}P(\dot{x} - Bu) + x^{T}D^{T}RDx$$

Thus

$$x^{T}Qx + 2u^{T}B^{T}Kx + u^{T}Ru$$

$$= -\frac{d}{dt}(x^{T}Px) + u^{T}Ru + u^{T}B^{T}(P+K)\,x + x^{T}(P+K)^{T}Bu + x^{T}D^{T}RDx$$

or, introducing (12),

$$x^{T}Qx + 2u^{T}B^{T}Kx + u^{T}Ru = -\frac{d}{dt}(x^{T}Px) + (u-Dx)^{T}R(u-Dx)$$

Since P(T) = 0, integration of this equation from 0 to T yields

$$M = x^{T}(0)\,P(0)\,x(0) + \int_{0}^{T} (u-Dx)^{T}R\,(u-Dx)\,dt$$

and, since R is positive definite, M assumes the minimum value $x^{T}(0)\,P(0)\,x(0)$ when $u = Dx$.

It is clear, moreover, that numerical integration of equation (13) provides a means of constructing the desired performance index and control. If the system is constant, and (13) approaches a steady state as $T \to \infty$, a solution with constant feedbacks is obtained, which minimizes the performance index M defined over an infinite time interval. Such a convergence is not assured, however, because the elimination of certain feedbacks may make it impossible to stabilize the system.

Main plant driven by an auxiliary plant

Large vehicles, such as ships, or long range aircraft, generally require actuators to deflect their control surfaces. If the dynamic equations of the actuators are included in the mathematical model, the control variables become the input signals to the actuators. This is a typical example of a main plant driven by an auxiliary plant. the equation of the main plant is

$$\dot{x}_1 = A_1 x_1 + B_1 y \tag{19}$$

where the control y is the output of the auxiliary plant. The equations of the auxiliary plant are

$$y = C_2 x_2 \tag{20}$$

$$\dot{x}_2 = A_2 x_2 + B_2 u \tag{21}$$

where u is the control input of the whole system.

These equations may be written in combined form as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_1 & B_1 C_2 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ B_2 \end{bmatrix} u \tag{22}$$

With a plant of this type, it is likely that it would be desired to limit the complexity of the control system by excluding feedbacks from the auxiliary plant. This class of plants is a natural candidate for investigation of the characteristics of control systems designed by the method of modifying the performance index to eliminate feedbacks. Attention will be restricted to time invariant systems.

The characteristics of the conventional linear optimal control system will first be examined. Let the performance index be

$$J = \int_0^\infty (x_1^T Q x_1 + u^T R u)\, dt \tag{23}$$

Let the matrix P of the Riccati equation for the combined system be divided into partitions corresponding to x_1 and x_2 as

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$$

where $P_{21} = P_{12}^T$. Then the optimal control is

$$u = -R^{-1}B_2^TP_{21}x_1 \quad -R^{-1}B_2^TP_{22}x_2 \tag{24}$$

where

$$0 = Q + A_1^TP_{11} + P_{11}A_1 - P_{12}B_2R^{-1}B_2^TP_{21} \tag{25a}$$

$$0 = A_1^TP_{12} + P_{11}B_1C_2 + P_{12}A_2 - P_{12}B_2R^{-1}B_2P_{22} \tag{25b}$$

$$0 = C_2^TB_1^TP_{12} + P_{21}B_1C_2 + A_2^TP_{22} + P_{22}A_2 - P_{22}B_2R^{-1}B_2^TP_{22} \tag{25c}$$

If the input y to the main plant could be generated directly, minimization of the same performance index would result in an optimal input

$$y = -R^{-1}B_1^TPx_1 \tag{26}$$

where

$$0 = Q + A_1^TP + PA_1 - PB_1R^{-1}B_1^TP \tag{27}$$

Suppose now that

$$A_2 = \alpha F \ , \ B_2 = \alpha G \ , \ C_2 = H \tag{28}$$

where

$$HF^{-1}G = -I \tag{29}$$

so that the output y of the auxiliary plant tends to follow the input u. Consider the effect of increasing α, which corresponds to increasing the speed of the actuators. Then (25c) becomes

$$0 = H^TB_1^TP_{12} + P_{21}B_1H + F^T\alpha P_{22} + \alpha P_{22}F - \alpha P_{22}GR^{-1}G^T\alpha P_{22}$$

whence as $\alpha \to \infty$

$$\| P_{22} \| \sim \frac{\| P_{12} \|}{\alpha}$$

Also (25b) becomes

$$0 = A_1^TP_{12} + P_{11}B_1H + \alpha P_{12} (F - GR^{-1}G^T \alpha P_{22})$$

and as $\alpha \to \infty$ the first term becomes negligible compared with the third. Then, using (29),

$$P_{11}B_1 \to \alpha P_{12}G (I - R^{-1}G^T\alpha P_{22}F^{-1}G) \tag{30}$$

whence

$$\| P_{12} \| \sim \frac{\| P_{11} \|}{\alpha}$$

But then (25a) indicates that

$$\| P_{11} \| \sim 1$$

so that

$$\| P_{12} \| \sim \frac{1}{\alpha} , \quad \| P_{22} \| \sim \frac{1}{\alpha^2} \tag{31}$$

and

$$\| R^{-1} B_2^T P_{22} \| = \| R^{-1} \alpha G^T P_{22} \| \sim \frac{1}{\alpha} \tag{32}$$

It also follows from (31) that the second term in the bracket in (30) is negligible. Thus

$$\alpha P_{12} G \rightarrow P_{11} B_1 \tag{33}$$

and making this substitution for $\alpha P_{21}^T G$ in (25a), it can be seen that the resulting equation for P_{11} approaches the same form as (27). Examination of (24) in the light of (32) and (33) leads to the following conclusion:

<u>Convergence property of optimal controls for composite plants:</u>

Let the composite system defined by equations (19), (20), and (21), where the matrices of the auxiliary system are of the form described by equations (28) and (29), be optimized for a performance index of the form (23), containing only the state vector of the main plant. Then if the speed of the auxiliary system is increased by increasing the scalar factor α, the optimal control u of the composite system approaches the optimal input y, which would have been obtained by optimizing the main plant alone.

The foregoing result suggests that the feedbacks from the auxiliary plant are relatively unimportant. The effect of such feedbacks, moreover, may well be to call for an increase in actuator performance. In order to find an optimal control for the combined system, while eliminating these feedbacks, let the performance index (23) be replaced by

$$M = \int_0^\infty (x_1^T Q x_1 + 2u^T B^T K x_2 + u^T R u)\, dt \tag{34}$$

Choosing K so that

$$B^T(P_{22} + K) = 0 \tag{35}$$

the optimal control becomes

$$u = -R^{-1}B^T P_{21} x_1 \tag{36}$$

where

$$0 = Q + A_1^T P_{11} + P_{11}A_1 - P_{12}B_2R^{-1}B_2^T P_{21} \tag{37a}$$

$$0 = A_1^T P_{12} + P_{11}B_1C_2 + P_{12}A_2 \tag{37b}$$

$$0 = C_2^T B_1^T P_{12} + P_{21}B_1C_2 + A_2^T P_{22} + P_{22}A_2 \tag{37c}$$

Compared with equations (25) the quadratic terms are eliminated from the last two of these equations, and it is only necessary to solve the first two in order to determine the control. They might be solved by adding differential terms $\dot{P}_{11}$ and $\dot{P}_{12}$ to the left hand side, and integrating them until they reach a steady state.

Consider now the effect of increasing the speed of the auxiliary plant. As before, let the auxiliary plant be defined by equations (28) and (29). Equation (37b) becomes

$$0 = A_1^T P_{12} + P_{11}B_1H + \alpha P_{12}F = 0$$

As $\alpha \to \infty$ the first term becomes negligible compared with the third, and, using (29),

$$\alpha P_{12}G \to P_{11}B_1 \tag{38}$$

With this substitution equation (37a) reduces to the same form as (27). Then (38) and (36) show that a result similar to that already obtained for the unrestricted optimal control holds for a restricted optimal control constructed in this manner:

Convergence property of restricted optimal controls for composite plants: Let the composite system defined by equations (19), (20), (21), (28), and (29) be optimized for the performance index (34), with K chosen to satisfy (35), so that only feedbacks from x_1 are used. Then if the speed of the auxiliary plant is increased by increasing the scalor factor α, the restricted feedback control approaches the optimal input y, which would have been obtained by optimizing the main plant alone for the performance index (23).

Conclusions

It has been shown that if a cross product term between the control and state vectors is added to the performance index, it is possible to compute the matrix of this additional term jointly with the control, in such a way that the modified performance index is minimized by the control for all initial conditions of the plant, and specified feedbacks are eliminated from the control. If this method is used to eliminate feedbacks from an auxiliary plant which drives a main plant, the convergence property proved in the last section suggests that, at least for this class of systems, the modification of the performance index need not lead to a significant change in the characteristics of the resulting system.

References

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2. A. Jameson, Optimization of Linear Systems of Constrained Configuration, Int. J. Control, Vol. 11, No. 3, March 1970, pp. 409-421.