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EFFECTIVENESS OF A LARGE SALVO.

by

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S U M M A R Y

A possible method of defence against an intruder is to saturate an area through which he is expected to pass. In this note the distribution of a salvo which will maximize the chance of a hit is determined. The chance of a hit is also determined for a normally distributed salvo, such as might be obtained without any special measures to control the distribution. It is found that if the dispersion of a normally distributed salvo is correctly chosen, it is almost as effective as the optimum salvo.

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LIST OF CONTENTS

1. INTRODUCTION

A possible method of defence against an intruder is to try to saturate an area through which he is expected to pass with a large salvo of unguided missiles.

This raises the question of how to determine the effectiveness of a large salvo. The error of any individual member of the salvo will arise from two sources. First, the members of the salvo will spread out from the point of aim in a manner which will be given by some probability distribution. Second, the point of aim itself will differ from the actual position of the target at the time it crosses the salvo, because of the error in pointing the launcher, and because the target may manoeuvre after the salvo has been fired. The aiming error can be specified as a probability distribution of the target's position relative to the point of aim. From the two probability distributions it is then possible to determine the chance of a given number of hits by the salvo. It is not possible to calculate this according to the simple law for the combination of independent probabilities, because all the members of the salvo have the same aiming error.

In this note the probability distribution of the shots is determined for a salvo which will maximize the chance of a hit, given the probability distribution of the target's position. Such a salvo might be approximated by splaying the barrels of a multiple launcher. Then the chance of a hit is determined for a normally distributed salvo, such as might be obtained by random disturbances of the individual shots, without any special measures to control the distribution. In this case it is assumed that the probability distribution of the target's position is a circular normal distribution. It appears that if the dispersion of a normally distributed salvo is correctly chosen, it is almost as effective as the optimum salvo. Finally an alternative model is examined, in which it is assumed that the vulnerability of the target varies with the distance from the centre of the target according to the law

$$e^{-\frac{r^2}{\sigma_0^2}} .$$

Most of the results require the assumptions that the number of shots is large and that the target is small compared with the dispersion of the salvo. This enables an analytic solution to be found. Terry (Ref. 1 and 2) has developed computer programmes for the exact evaluation of the chance of k hits with n shots when the probability distribution of the aiming error is a circular normal distribution.

2. OPTIMUM DISTRIBUTION OF A SALVO

Suppose that a salvo is fired at a target of area S, and that the probability distribution of the target's position is $f_1(r)$. Suppose also that the probability distribution of the position of a missile in the salvo is $f_2(r)$. The conditional probability of k hits with n shots, given that the target is at a radius r, is

$$C(k,\ n)\Big|_r = \binom{n}{k} p^k q^{n-k}$$

where if the dispersion of the salvo $\gg$ S

$$p \doteq Sf_2(r),\ q \doteq 1 - Sf_2(r)$$

When n is large the binomial distribution approaches a Poisson distribution, so that $C(k,\ n)\Big|_r$ approaches

$$C(k)\Big|_r = e^{-a}\ \frac{a^m}{m!}$$

where

$$a = nSf_2(r)$$

The probability that the target lies at a radius between r and r + dr is

$$2\pi r f_1(r)dr$$

so the probability of k hits approaches

$$C(k) = 2\pi \int_0^{\infty} f_1\ e^{-nSf_2}\ \frac{(nSf_2)^k}{k!}\ rdr$$

when n is large.

The probability of at least one hit is then approximately

$$P(1) = 2\pi \int_0^{\infty} f_1(1 - e^{-nSf_2})\ rdr$$

Taking this as the criterion of performance, it should be maximized subject to

$$2\pi \int_0^\infty f_2 \, r dr = 1, \quad f_2 \geqslant 0.$$

To ensure the second condition put $f_2 = \phi^2$. Then the Euler equation which ϕ must satisfy is

$$\lambda\phi - f_1 nS\phi \, e^{-nS\phi^2} = 0$$

where λ is an undetermined multiplier. Thus either

$$\phi = 0, \; f_2 = 0$$

or

$$nSf_2 = \log \frac{f_1}{c}$$

where $c = \dfrac{\lambda}{nS}$ is an undetermined constant.

Then if U is the set over which $f_2 > 0$,

$$P(1) = 2\pi \int_U (f_1 - c) \, r dr$$

where

$$nS = 2\pi \int_U \log \left(\frac{f_1}{c} \right) r dr$$

Suppose that U is composed of intervals one of which is (a, b), and consider the effect of varying b. Then the last condition requires that

$$b \log \frac{f_1(b)}{c} - \int_a^b \frac{1}{c} \frac{dc}{db} \, r dr = 0$$

and therefore

$$\frac{1}{2\pi cb}\frac{dP(1)}{db} = \frac{f_1(b)}{c} - 1 - \log\frac{f_1(b)}{c}$$

$$= \log\left[\frac{f_1(b)}{c} + \frac{1}{2!}\left(\frac{f_1(b)}{c} - 1\right)^2 + \dots\right] - \log\frac{f_1(b)}{c}$$

But $\frac{f_1(b)}{c} \geqslant 1$ since $f_2 \geqslant 0$, so $\frac{dP(1)}{db} > 0$ unless $f_1(b) = c$.

Similar arguments apply for other end points, so these should all be chosen so that $f_1 = c$. To maximize $P(1)$ one should therefore choose

$$f_2(r) = \begin{cases} \frac{1}{nS}\log\frac{f_1(r)}{c}, & f_1(r) \geqslant c, \\ 0, & f_1(r) < c \end{cases}$$

where c is chosen so that on the set V for which $f_1 \geqslant c$

$$nS = 2\pi\int_V \log\left(\frac{f_1}{c}\right) r\,dr$$

Then

$$P(1) = 2\pi\int_V (f_1 - c)\,r\,dr$$

The probability of at least k hits is approximately

$$P(k) = 2\pi\int_0^\infty f_1(r)\left\{1 - e^{-nSf_2}\left[1 + nSf_2 \dots + \frac{(nSf_1)^{k-1}}{(k-1)!}\right]\right\} r\,dr$$

Thus if f_2 is chosen to maximize P(1),

$$P(2) = 2\pi \int_V \left\{ f_1 - c \left[1 + \log \frac{f_1}{c} + \dots \frac{\left(\log \frac{f_1}{c} \right)^{k-1}}{(k-1)!} \right] \right\} rdr$$

On the other hand it may be preferred to maximize P(2), for example. Then with $f_2 = \phi^2$ the Euler equation is

$$\lambda\phi - f_1 \, e^{-nS\phi^2} \left[nS\phi(1 + nS\phi^2) - nS\phi \right] = 0 \quad .$$

and either

$$\phi = 0, \quad f_2 = 0$$

or

$$nSf_2 \, e^{-nSf_2} = \frac{\lambda}{nSf_1}$$

Consider now the case where the probability distribution of the target's position is a circular normal distribution

$$f_1(r) = \frac{e^{-\frac{r^2}{2\sigma_1{}^2}}}{2\pi\sigma_1{}^2}$$

Then the distribution of the salvo which maximizes the chance P(1) of at least one hit is a parabolic distribution

$$f_2(r) = \begin{cases} \dfrac{a^2 - r^2}{2\sigma_1{}^2 nS} , & r \leqslant a, \\ 0 , & r > 0 \end{cases}$$

where

$$nS = \frac{\pi}{\sigma_1{}^2} \int_0^a (a^2 - r^2) \, rdr = \frac{\pi a^4}{4\sigma_1{}^2}$$

or

$$\frac{a^2}{2\sigma_1{}^2} = \left(\frac{nS}{\pi\sigma_1{}^2} \right)^{\frac{1}{2}}$$

Also with this distribution

$$P(k) = \int_{0}^{\frac{a^2}{2\sigma_1^2}} \left\{ e^{-\frac{r^2}{2\sigma_1^2}} - e^{-\frac{a^2}{2\sigma_1^2}} \left[1 + \frac{a^2 - r^2}{2\sigma_1^2} \cdots + \frac{1}{(k-1)!} \left(\frac{a^2 - r^2}{2\sigma_1^2} \right)^{k-1} \right] \right\} d \left(\frac{r^2}{2\sigma_1^2} \right)$$

$$= 1 - e^{-\frac{a^2}{2\sigma_1^2}} \left[1 + \frac{a^2}{2\sigma_1^2} + \frac{1}{2!} \left(\frac{a^2}{2\sigma_1^2} \right)^2 \cdots + \frac{1}{k!} \left(\frac{a^2}{2\sigma_1^2} \right)^k \right]$$

Figure 1 shows the variation of P(1) and P(2) with $\frac{nS}{2\pi\sigma_1^2}$ for a salvo which maximizes P(1).

3. NORMALLY DISTRIBUTED SALVO

Consider again the case where the probability distribution of the position of a target of area S is a circular normal distribution

$$f_1 = \frac{e^{-\frac{r^2}{2\sigma_1^2}}}{2\pi\sigma_1^2}$$

But suppose now that the salvo is normally distributed, so that the probability distribution for the position of a missile in the salvo is

$$f_2 = \frac{e^{-\frac{r^2}{2\sigma_2^2}}}{2\pi\sigma_2^2}$$

where it is assumed that the dispersion of the salvo is large compared with the target area,

$$\sigma_2^2 \gg S .$$

Then the conditional probability of k hits with n shots, given that the target is at a radius r, is

$$C(k, n)\Big|_r = \binom{n}{k} p^k \left(1 - p \right)^{n-k}$$

$$= \binom{n}{k} \left[p^k - (n - k)\, p^{k+1} + \binom{n-k}{2} p^{k+2} \dots + (-1)^{n-k} p^n \right]$$

where

$$p \doteq \frac{S}{2\pi\sigma_2^2} e^{-\frac{r^2}{2\sigma_2^2}}$$

The probability that the target lies between r and r + dr is

$$\frac{r}{\sigma_1^2} e^{-\frac{r_1^2}{2\sigma_1^2}} dr$$

The probability of k hits with n shots is therefore

$$C(k, n) = \frac{\binom{n}{k}}{\sigma_1^2} \int_0^\infty \left[\left(\frac{S}{2\pi\sigma_2^2} \right)^k e^{-\frac{r^2}{2}\left(\frac{1}{\sigma_1^2} + \frac{k}{\sigma_2^2} \right)} \right.$$

$$\left. - (n-k) \left(\frac{S}{2\pi\sigma_2^2} \right)^{k+1} e^{-\frac{r^2}{2}\left(\frac{1}{\sigma_1^2} + \frac{k+1}{\sigma_2^2} \right)} \cdots \right] r dr$$

$$= \frac{\binom{n}{k}}{\sigma_1^2} \left[\left(\frac{S}{2\pi\sigma_2^2} \right)^k \frac{1}{\frac{1}{\sigma_1^2} + \frac{k}{\sigma_2^2}} \right.$$

$$\left. - (n-k) \left(\frac{S}{2\pi\sigma_2^2} \right)^{k+1} \frac{1}{\frac{1}{\sigma_1^2} + \frac{k+1}{\sigma_2^2}} \cdots \right]$$

$$= \binom{n}{k} \left[\frac{1}{Y + kX} \frac{1}{Y^{k-1}} - \frac{n-k}{Y + (k+1)X} \frac{1}{Y^k} \cdots \right.$$

$$\left. + \frac{(-1)^{n-k}}{Y + nX} \frac{1}{Y^{n-1}} \right]$$

where

$$X = \frac{2\pi\sigma_1^2}{S} \gg 1, \quad Y = \frac{2\pi\sigma_2^2}{S}$$

Writing $s = \frac{Y}{X}$

$$C(k, n) = \binom{n}{k}\left[\frac{s}{s+k}\left(\frac{1}{Y}\right)^{k} - \frac{(n-k)s}{s+k+1}\left(\frac{1}{Y}\right)^{k+1}\right.$$

$$\left. + \frac{\binom{n-k}{2}s}{s+k+2}\left(\frac{1}{Y}\right)^{k+2} \ldots + \frac{(-1)^{n-k}s}{s+n}\left(\frac{1}{Y}\right)^{n}\right]$$

$$= \binom{n}{k} sY^{s}\left[\frac{1}{s+k}\left(\frac{1}{Y}\right)^{s+k} - \frac{n-k}{s+k+1}\left(\frac{1}{Y}\right)^{s+k+1} \ldots\right.$$

$$\left. + \frac{(-1)^{n-k}}{s+n}\left(\frac{1}{Y}\right)^{s+n}\right]$$

$$= \binom{n}{k} sY^{s}\int_{0}^{\frac{1}{Y}} u^{s+k-1}\left[1 - (n-k)u + \binom{n-k}{2}u^{2} \ldots\right.$$

$$\left. + (-1)^{n-k} u^{n-k} \, du\right]$$

$$= \binom{n}{k} sY^{s}\int_{0}^{\frac{1}{Y}} u^{s+k-1}(1-u)^{n-k}\, du$$

$$= \binom{n}{k} sY^{s} B_{\frac{1}{Y}}(n-k+1, s+k)$$

where B_x is the incomplete beta function.

If $n = mX$, $Y = \frac{ns}{m}$

$$C(k, n) = \binom{n}{k} s\left(\frac{ns}{m}\right)^s \int_0^{\frac{m}{ns}} u^{s+k-1}(1-u)^{n-k}\, du$$

$$= \frac{n(n-1)\ \ldots\ (n-k+1)}{n^k} \frac{s}{k!}\left(\frac{s}{m}\right)^s \times$$

$$\int_0^{\frac{m}{s}} t^{s+k-1}\left(1-\frac{t}{n}\right)^{n-k} dt$$

Then for fixed k, as n becomes very large C(k, n) approaches

$$C(k) = \frac{s}{k!}\left(\frac{s}{m}\right)^s \int_0^{\frac{m}{s}} t^{s+k-1}\ e^{-t}\, dt$$

$$= \frac{s}{k!}\left(\frac{s}{m}\right)^s \Gamma_{\frac{m}{s}}(s+k)$$

where Γ_x is the incomplete gamma function. The chance of at least k hits is then

$$P(k) = 1 - \sum_{1}^{k-1} C(i)$$

Figure 2 shows the way in which the chance of at least 1 hit varies with the dispersion of the salvo for different values of the parameter $\frac{nS}{2\pi\sigma_1{}^2}$. One can choose the dispersion to maximize this chance, and figure 3 shows the chance of at least 1 hit which can be obtained with the optimum normally distributed salvo and with the optimum salvo. It can be seen that the optimum normally distributed salvo is almost as effective as the optimum salvo.

4. MODEL WITH A VULNERABILITY DISTRIBUTION

In an alternative model it may be imagined that the vulnerability of the target varies with the distance from its centre. Suppose that the chance of a 'kill' varies according to the law

$$e^{-\frac{r^2}{\sigma_0^2}},$$

where σ_0 is a measure of the size of the vulnerable area. The effectiveness of a salvo will now be measured by the total chance of a kill instead of the chance of scoring a given number of hits.

Now if the probability distribution of the target's position is

$$f_1(r) = \frac{e^{-\frac{r^2}{2\sigma_1^2}}}{2\pi\sigma_1^2}$$

and the probability distribution of the position of the missiles is

$$f_2(r) = \frac{e^{-\frac{r^2}{2\sigma_2^2}}}{2\pi\sigma_2^2}$$

the conditional probability of a kill with 1 shot, given that the target is at (x, y) is

$$K(1)\Big|_{(x,\, y)} = \frac{1}{\sigma_2^2}\int_0^\infty e^{-\frac{(x-\xi)^2}{2\sigma_0^2}}\, e^{-\frac{-\xi^2}{2\sigma_2^2}}\, d\xi \;\times$$

$$\int_0^\infty e^{-\frac{(y-\eta)^2}{2\sigma_2^2}}\, e^{-\frac{\eta^2}{2\sigma_2^2}}\, d\eta$$

$$= \frac{\sigma_0^2}{\sigma^2}\, e^{-\frac{x^2+y^2}{2\sigma^2}}$$

where

$$\sigma^2 = \sigma_0^2 + \sigma_2^2$$

The conditional probability of a kill with n shots is

$$K(n)\Big|_{(x,\ y)} = 1 - \left[1 - K(1)\Big|_{(x,\ y)}\right]^n$$

The probability that the target is near (x, y) is $f_1(x, y)\, dx\, dy$, so the probability of a kill with n shots is

$$K(n) = 2\pi \int_0^\infty f_1\, K(n)\Big|_{(x,\ y)} \, rdr$$

$$= \frac{1}{\sigma_1^2} \int_0^\infty \left[n \frac{\sigma_0^2}{\sigma^2} e^{-\frac{r^2}{2}\left(\frac{1}{\sigma_1^2} + \frac{1}{\sigma^2}\right)} \right.$$

$$\left. - \binom{n}{2} \left(\frac{\sigma_0^2}{\sigma^2}\right)^2 e^{-\frac{r^2}{2}\left(\frac{1}{\sigma_1^2} + \frac{2}{\sigma^2}\right)} \cdots \right] rdr$$

$$= n \frac{\sigma_0^2}{\sigma^2 + \sigma_1^2} - \binom{n}{2} \frac{\sigma_0^2}{\sigma^2} \frac{\sigma_0^2}{\sigma^2 + 2\sigma_1^2} \cdots\cdots\cdots$$

$$+ (-1)^n \left(\frac{\sigma_0^2}{\sigma^2}\right)^{n-1} \frac{\sigma_0^2}{\sigma^2 + n\sigma_1^2}$$

$$= \frac{n}{Y + X} - \binom{n}{2} \frac{1}{Y + 2X} \frac{1}{Y} \cdots\cdots + (-1)^n \frac{1}{Y + nX} \frac{1}{Y^{n-1}}$$

where

$$X = \frac{\sigma_1^2}{\sigma_0^2}, \quad Y = \frac{\sigma_0^2 + \sigma_2^2}{\sigma_0^2}$$

This is just the expression for the chance $P(1, n)$ of at least 1 hit obtained in the previous section.

LIST OF REFERENCES

REFERENCE No.	AUTHOR	TITLE
1	E.R. Terry	Multiple Hit Frequencies in Fully Correlated Fire. Nature 13th February, 1965.
2	E.R. Terry.	Fundamental Equations Governing the Hit Probabilities Associated with the Release of a Pattern or Group of Weapons Using a Single Aim Point. Nature 12th June, 1965.

LIST OF FIGURES

CHANCE OF SUCCESS WITH A SALVO WHICH MAXIMISES P(1)

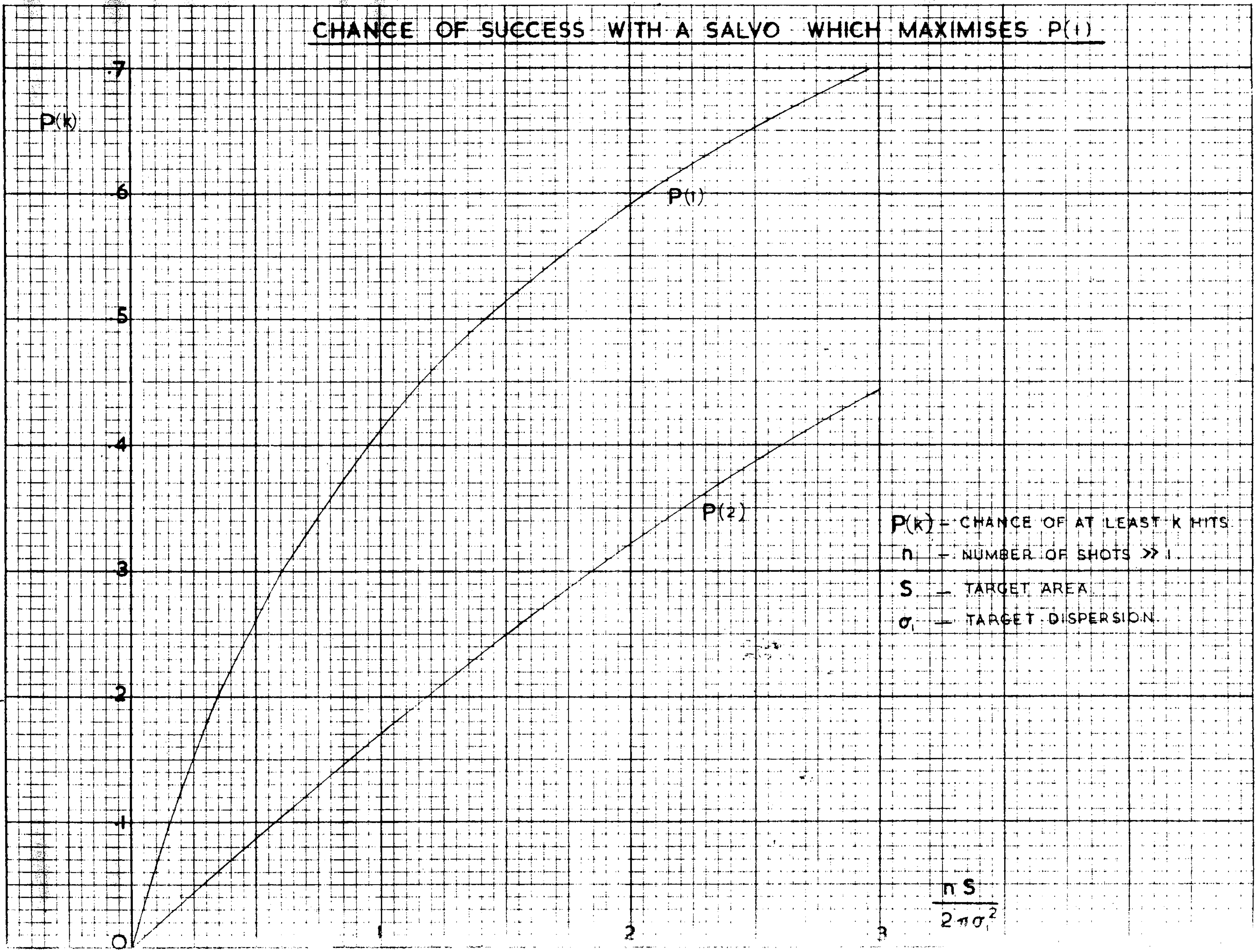

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FIG. No. 1.

AQ. No.

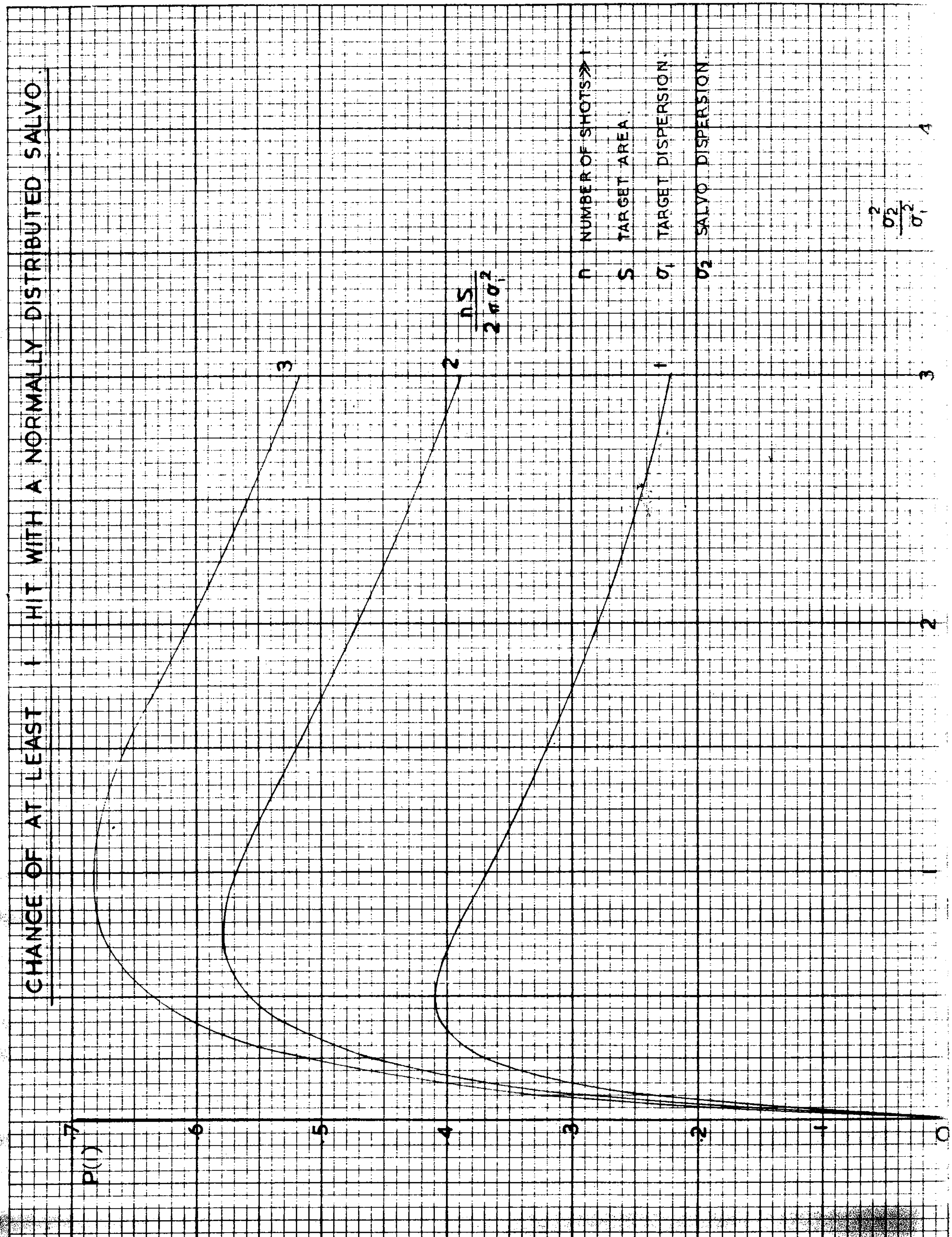

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FIG. No. 2

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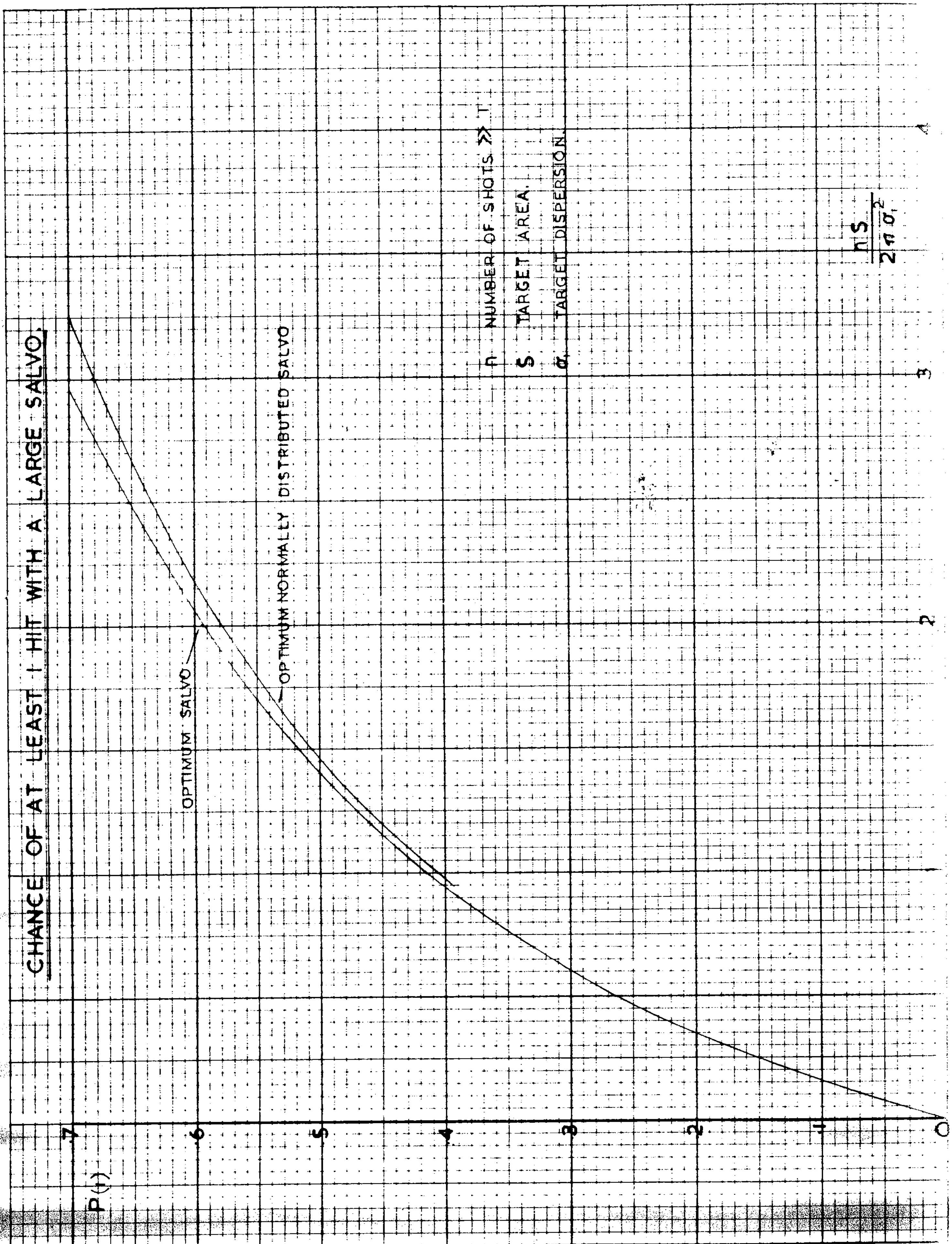

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FIG. No. 3.

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