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SHOCKS AND KINKS IN STRINGS.

by

ANTONY JAMESON

S U M M A R Y

A simple theory of finite disturbances in strings is developed. It is found that two types of finite disturbance may occur, 'shocks', across which there is a change in the stress, and 'kinks', across which the string is bent. Shocks and kinks travel at different speeds. The theory is used to predict the motion of a string struck by a bullet, both immediately after the impact, and after the waves travelling along the string have been reflected from its ends. Photographs of bullets hitting strings show that the effects predicted by the theory do in fact occur.

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1. INTRODUCTION

The behaviour of a string under impact has been investigated by Ringleb (Ref. 1). Ringleb formulated the problem in terms of the differential equations of the string. The resulting motion of the string, however, is discontinuous, and it seems worth while to focus attention on the properties of finite disturbances. In this paper a simple theory of finite disturbances is developed, using arguments similar to the arguments employed in the theory of shock waves in air, or of jumps in a channel of water.

2. SHOCKS

Consider first a discontinuity in stress but not in direction. The principle that mass, momentum and energy are conserved across the discontinuity yields equations corresponding to the Rankine Hugoniot equations for a shock wave in air.

Let S, ρ, σ and e denote the cross-sectional area, density, stress and strain of the string. Suppose that the force is directly proportional to its extension, so that

$$\sigma \frac{S}{S_o} = Ke \qquad (1)$$

where K is a constant, and a suffix o denotes the unstrained condition. Also

$$\rho S(1 + e) = \rho_o S_o \qquad (2)$$

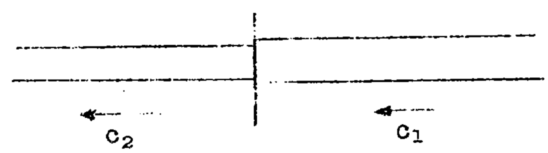


FIG. 2.1

Suppose that the discontinuity is travelling along the string at a steady speed. It is convenient to use axes fixed to the discontinuity, (Fig. 2.1) and to think of the string as approaching at a speed c_1 and leaving at a speed c_2. Then for conservation of mass

$$\rho_1 c_1 S_1 = \rho_2 c_2 S_2$$

or by (2),

$$\frac{c_2}{c_1} = \frac{1 + e_2}{1 + e_1} \qquad (3)$$

The rate of generation of momentum equals the external force, therefore

$$\rho_1 c_1 S_1 (c_2 - c_1) = \sigma_2 S_2 - \sigma_1 S_1$$

or, by (1) (2) and (3),

$$\frac{\rho_0 S_0}{1 + e_1} c_1^2 \left(\frac{1 + e_2}{1 + e_1} - 1 \right) = K(e_2 - e_1)$$

and

$$c_1 = c_0 (1 + e_1) \qquad (4)$$

$$c_2 = c_0 (1 + e_2) \qquad (5)$$

where

$$c_0^2 = \frac{K}{\rho_0}$$

It remains to be checked whether energy is also conserved when these conditions are satisfied. The strain energy per unit of unstrained volume is

$$\frac{Ke^2}{2}.$$

Strain energy is thus generated at a rate

$$S_1 c_1 \frac{\rho_1}{\rho_0} K \frac{e_2^2 - e_1^2}{2}$$

and kinetic energy at a rate

$$S_1 c_1 \rho_1 \frac{c_2^2 - c_1^2}{2}.$$

The total rate of generation of energy is therefore

$$\frac{S_1 c_1 \rho_1 K}{2\rho_0} \left[e_2^2 - e_1^2 + \frac{\rho_0}{K} (c_2^2 - c_1^2) \right] = S_0 K (c_2 e_2 - c_1 e_1),$$

just the work done by the external force. Apparently, therefore, disturbances of this kind, which will be called 'shocks', are possible in a string for which the tension is directly proportional to the extension. It has not been necessary to make any assumptions about the way in which the area of the string varies under stress, provided that the accompanying lateral motion is negligible compared with the longitudinal motion. The shock will usually not be stationary in space. The equations remain valid, however, when an equal speed is added to the shock, and the string on each side of it. Equations (4) and (5) thus correctly give the speed of the shock relative to each part of the string: If the speed changes from v_1 to v_2,

$$v_2 - v_1 = c_0(e_2 - e_1) \tag{7}$$

It is convenient to use the dimensionless notation, $v = Nc_0$, and write

$$N_2 - e_2 = N_1 - e_1 \tag{8}$$

Often the force in the string will not be directly proportional to its extension, so that equation (1) must be replaced by

$$\sigma \frac{S}{S_0} = K f(e) \tag{1a}$$

Then

$$\frac{\rho_0 S_0}{1 + e_1} c_1{}^2 \left(\frac{1 + e_2}{1 + e_1} - 1 \right) = K(f(e_2) - f(e_1))$$

and

$$c_1 = a(1 + e_1) \tag{4a}$$

$$c_2 = a(1 + e_2) \tag{5a}$$

where

$$a^2 = \frac{K}{\rho_0} \frac{f(e_2) - f(e_1)}{e_2 - e_1} \tag{6a}$$

The rate of generation of strain energy is now

$$S_1 c_1 \frac{\rho_1}{\rho_0} K \int_{e_1}^{e_2} f de$$

The total rate of generation of energy is therefore

$$U = \frac{S_1 c_1 \rho_1}{\rho_0} K \left\{ \int_{e_1}^{e_2} f de + \frac{\rho_c}{K} \frac{c_2{}^2 - c_1{}^2}{2} \right\}$$

$$= S_0 Ka \left\{ \int_{e_1}^{e_2} f de + \frac{f(e_2) - f(e_1)}{e_2 - e_1} \quad \frac{(1 + e_2)^2 - (1 + e_1)^2}{2} \right\}$$

$$= S_0 Ka \left\{ \int_{e_1}^{e_2} f de + [f(e_2) - f(e_1)] \left[\frac{e_2 + e_1}{2} + 1 \right] \right\}$$

The work done is

$$W = S_0 K[c_2 \, f(e_2) - c_1 \, f(e_1)]$$

$$= S_0 Ka[(1 + e_2) \, f(e_2) - (1 + e_1) \, f(e_1)]$$

Thus

$$U - W = S_0 Ka \left\{ \int_{e_1}^{e_2} f de - \tfrac{1}{2}[f(e_2) + f(e_1)] \, (e_2 - e_1) \right\}$$

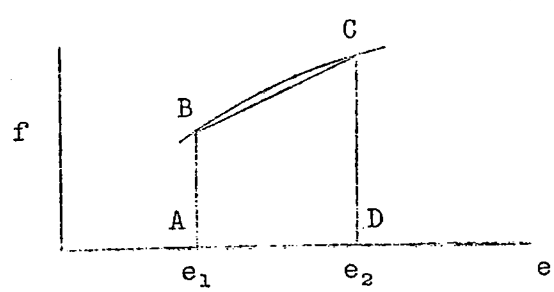


FIG. 2.2

The second term inside the bracket is the area of the trapezium A B C D (Fig. 2.2). Thus if the slope of the force curve decreases as the strain increases, $U - W$ has the same sign as $e_2 - e_1$, while if it increases, $U - W$ has the same sign as $e_1 - e_2$. In the first case 'compression' shocks, in which the tension decreases, are possible, with dissipation of energy, but 'expansion' shocks are not possible. In the second the reverse is true.

This behaviour is related to the accomulation of successive disturbances.

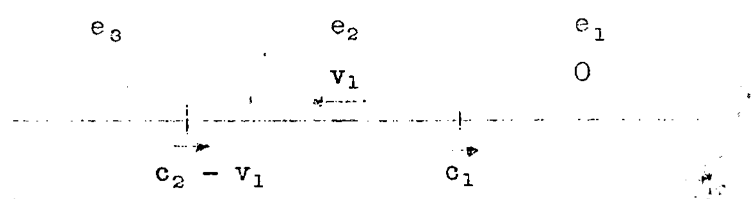


FIG. 2.3

Consider a string, stationary on the right, (Fig. 2.3) along which two small disturbances are travelling. Suppose that they are expansion waves and that (4a), (5a) and (6a) hold.
Then

$$c_1 = (1 + e_1) \sqrt{\frac{K}{\rho_o} \frac{f(e_2) - f(e_1)}{e_2 - e_1}}$$

and

$$c_2 - v_1 = (1 + e_2) \sqrt{\frac{K}{\rho_o} \frac{f(e_3) - f(e_2)}{e_3 - e_2}}$$

$$- (e_2 - e_1) \sqrt{\frac{K}{\rho_o} \frac{f(e_2) - f(e_1)}{e_2 - e_1}}$$

so the difference between the speeds of the two waves is

$$c_1 - c_2 + v_1 = (1 + e_2) \sqrt{\frac{K}{\rho_o}} \times$$

$$\left\{ \sqrt{\frac{f(e_3) - f(e_2)}{e_3 - e_2}} - \sqrt{\frac{f(e_2) - f(e_1)}{e_2 - e_1}} \right\}$$

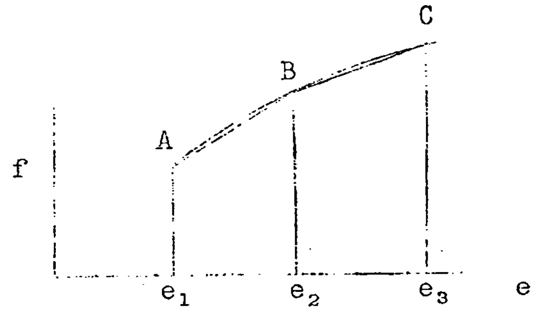


FIG. 2.4

The terms inside the bracket are the square roots of the slopes of AB and CD (Fig. 2.4). If the slope of the force curve decreases as the strain increases, the bracket is positive. The distance between successive expansion waves would then increase as they travelled along the string, and a finite expansion would disintegrate. Successive compression waves, on the other hand, would accumulate, and tend to form a compression shock. When the force is directly proportional to the extension, successive disturbances neither close up nor fall apart: compression and expansion shocks are possible, but there will be no tendency for small disturbances to accumulate. The sections to come will be restricted to this case.

3. KINKS

There is the possibility also that discontinuities of direction may be propogated. Suppose that such a discontinuity or 'kink' is travelling along a string in which the stress is constant. If the speed of the kink is a, one can choose axes attached to the kink and think of the string as running into the kink at a speed a.

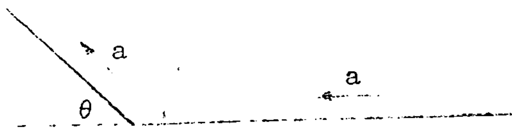


FIG. 3.1

Since the stress is constant there is no change in strain as the string passes the kink, so that it also leaves the kink at a speed a after turning through an angle θ (Fig. 3.1). The total external force equals the rate of change of momentum, therefore

$$Sa\rho.a \sin\theta \;=\; S\sigma \sin\theta$$

$$Sa\rho.a(1 - \cos\theta) \;=\; S\sigma(1 - \cos\theta)$$

and

$$a^2 \;=\; \frac{\sigma}{\rho} \qquad (1)$$

Also there is no energy generated, and the work done by the force at one end of the string balances the work done by the force at **the other end.** The propagation of a kink at a speed $\sqrt{\sigma/\rho}$ relative to the string on either side of the kink should therefore be possible. The vertical and horizontal components of the velocity of the string will change by $\sqrt{\sigma/\rho}\,\sin\theta$ and $\sqrt{\sigma/\rho}\,(1 - \cos\theta)$ across a kink of angle θ.

It is remarkable that a tensioned string, running along itself at a speed $\sqrt{\sigma/\rho}$, should be able to maintain an arbitrary fixed shape in space in the absence of disturbing forces.

Suppose that the force in the string is directly proportional to its extension. Then if S_0 is the area and ρ_0 the density of the unstressed string,

$$\sigma \frac{S}{S_0} = Ke$$

$$\rho S(1 + e) = \rho_0 S_0$$

so that

$$a^2 = \frac{K}{\rho_0} e(1 + e) = c_0{}^2 e(1 + e)$$

The speed of a kink in such a string is therefore mc_0 where

$$m^2 = e + e^2 \qquad (2)$$

and the velocity components change across the kink by $mc_0 \sin\theta$ and $mc_0(1 - \cos\theta)$.

Consider now the more general case in which the stress changes across the kink.

FIG. 3.2

Then the string enters at a speed a_1 and leaves at a speed a_2, (Fig. 3.2) where since the string does not accumulate at the kink

$$\frac{a_2}{a_1} = \frac{1 + e_2}{1 + e_2} \qquad (3)$$

Also

$$\sigma_2 \sin\theta = \rho_1 a_1 a_2 \sin\theta \qquad (4a)$$

$$\sigma_1 - \sigma_2 \cos\theta = \rho_1 a_1(a_1 - a_2 \cos\theta) \qquad (4b)$$

or, if $\theta \neq 0$, substituting from (4a) in (4b)

$$\sigma_1 - \sigma_2 \cos\theta = \rho_1 a_1^2 - \sigma_2 \cos\theta$$

whence

$$a_1^2 = \frac{\sigma_1}{\rho_1} = \frac{Ke_1}{\rho_1} = \frac{K}{\rho_0} e_1(1 + e_1) \qquad (5)$$

But equally if one resolved perpendicular to a_2,

$$a_2^2 = \frac{K}{\rho_0} e_2(1 + e_2)$$

therefore

$$\left(\frac{a_2}{a_1}\right)^2 = \frac{e_2(1 + e_2)}{e_1(1 + e_1)}$$

and by (3)

$$\frac{e_2}{e_1} = \frac{1 + e_2}{1 + e_1}$$

or

$$e_2 = e_1$$

Thus when $\dot{\theta} \neq 0$ the stress is constant: a combined kink and shock is impossible.

4. REFLECTION OF SHOCKS AND KINKS

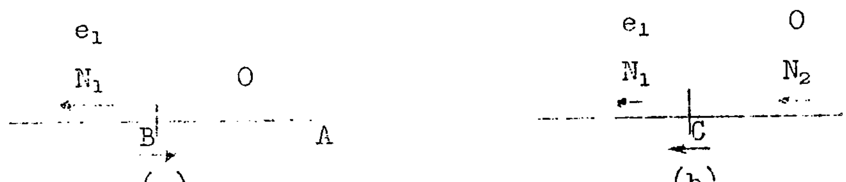


FIG. 4.1

Suppose that a shock B approaches a free end A (Fig. 4.1(a)). The stress is always zero at A, so when the shock arrives, there is a discontinuity in stress at A. Since such a discontinuity cannot remain stationary, it must be propogated back down the string (Fig. 4.1(b)). The shock is thus reflected.
One has

$$N_1 = e_1 \tag{1}$$

$$N_1 + e_1 = N_2 \tag{2}$$

giving

$$N_2 = 2N_1 \tag{3}$$

The speed of the string is doubled.

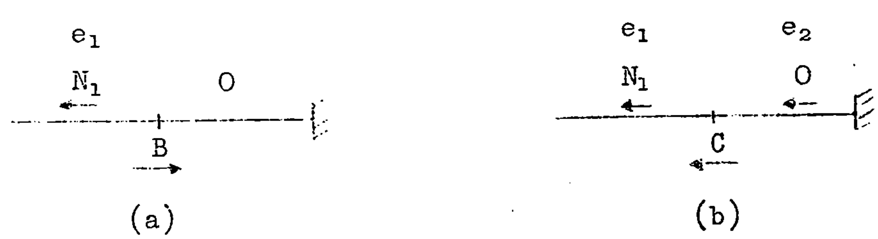


FIG. 4.2

A shock is similarly reflected at a fixed end (Fig. 4.2). In this case

$$N_1 = e_1 \qquad (4)$$

$$N_1 + e_1 = e_2 \qquad (5)$$

giving

$$e_2 = 2e_1 \qquad (6)$$

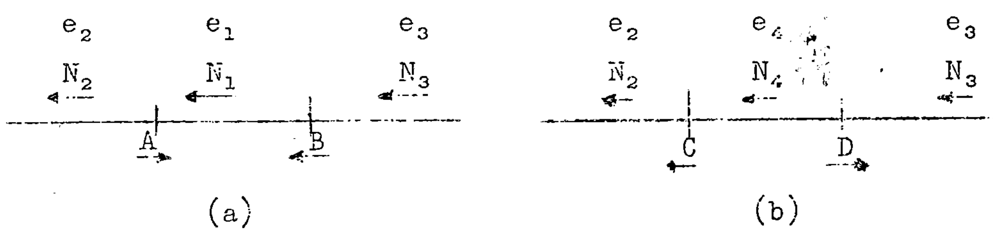


FIG. 4.3

Shocks can also be reflected off each other (Fig. 4.3). If shocks A and B collide, modified shocks C and D are transmitted from the point of collision. For A, B, C and D

$$N_2 - N_1 = e_2 - e_1 \qquad (7)$$

$$N_1 - N_3 = e_3 - e_1 \qquad (8)$$

$$N_2 - N_4 = e_4 - e_2 \qquad (9)$$

$$N_4 - N_3 = e_4 - e_3 \qquad (10)$$

Adding (7) and (8)

$$N_2 - N_3 = e_2 + e_3 - 2e_1$$

and adding (9) and (10)

$$N_2 - N_3 = 2e_4 - (e_2 + e_3)$$

so the strain between the reflected shocks is

$$e_4 = e_2 + e_3 - e_1 \qquad (11)$$

Also subtracting (8) from (7)

$$N_2 + N_3 - 2N_1 = e_2 - e_3$$

and subtracting (9) from (10)

$$2N_4 - (N_2 + N_3) = e_2 - e_3$$

so the speed of the string between the reflected shocks is $N_4 c_o$ where

$$N_4 = N_1 + e_2 - e_3 \qquad (12)$$

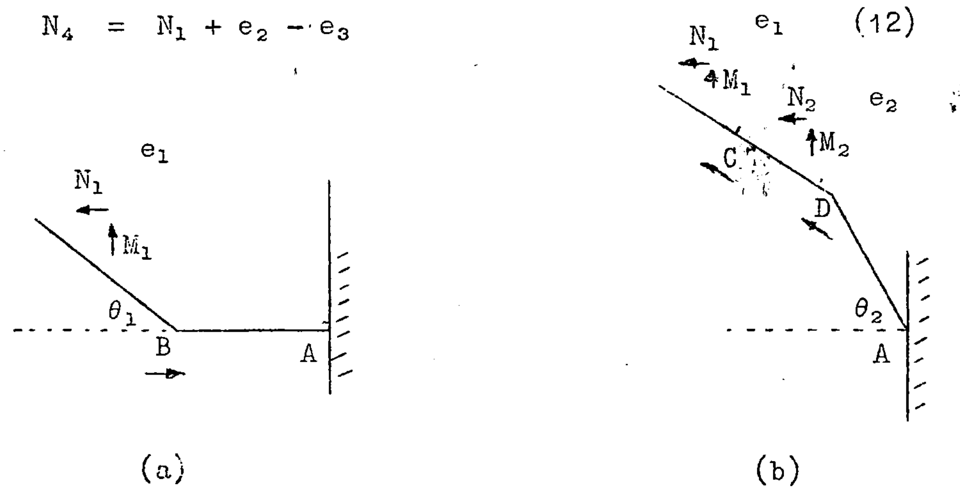


FIG. 4.4

When a kink B travelling along a string in which the stress is constant approaches a fixed end A, (Fig. 4.4(a)) the behaviour is more complicated. No velocity can be imparted to A. A section AD therefore develops behind a reflected kink D, (Fig. 4.4(b)) in which the velocity parallel to AD is zero because A is stationary, and the velocity normal to AD is also zero because the kink angle is constant. If the stress in the string remained unchanged, it would be impossible for the kink to produce the required change in the velocity. It is therefore preceded by a shock C. The stress and speed of the string change across C in such a way that behind D the string is stationary.

Then the speed of the kink B is $m_1 c_o$ where

$$m_1 = e_1 + e_1^2 \qquad (13)$$

and behind it the velocity components are $M_1 c_o$ and $N_1 c_o$ where

$$M_1 = m_1 \sin\theta_1 \qquad (14a)$$

$$N_1 = -m_1(1 - \cos\theta_1) \qquad (14b)$$

Across the shock C the velocity components change to M_2c_0 and N_2c_0 where

$$M_2 = M_1 - (e_2 - e_1)\sin\theta_1 \qquad (15a)$$

$$N_2 = N_1 - (e_2 - e_1)\cos\theta_1 \qquad (15b)$$

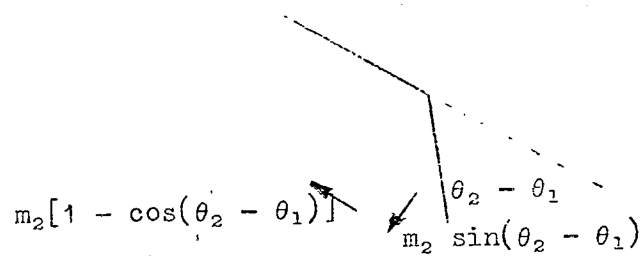


FIG. 4.5

For the kink D (Fig. 4.5)

$$m_2 = e_2 + e_2^2 \qquad (16)$$

$$\begin{aligned} 0 &= M_2 + m_2\left[\sin\theta_1 - \sin\theta_1 \cos(\theta_2 - \theta_1) - \cos\theta_1 \sin(\theta_2 - \theta_1)\right] \\ &= M_2 + m_2(\sin\theta_1 - \sin\theta_2) \qquad (17a) \end{aligned}$$

$$\begin{aligned} 0 &= N_2 + m_2\left[\cos\theta_1 - \cos\theta_1 \cos(\theta_2 - \theta_1) + \sin\theta_1 \sin(\theta_2 - \theta_1)\right] \\ &= N_2 + m_2(\cos\theta_1 - \cos\theta_2) \qquad (17b) \end{aligned}$$

From (14), (15) and (17)

$$(m_1 + e_1 + m_2 - e_2)\sin\theta_1 = m_2 \sin\theta_2 \qquad (18a)$$

$$(m_1 + e_1 + m_2 - e_2)\cos\theta_1 - m_1 = m_2 \cos\theta_2 \qquad (18b)$$

Dividing (18b) by (18a)

$$\cot\theta_2 = \cot\theta_1 - \frac{m_1}{(m_1 + e_1 + m_2 - e_2)\sin\theta_1}$$

where by (16) $m_2 > e_2$, so

$$\cot\theta_2 < \cot\theta_1$$

and

$$\theta_2 > \theta_1$$

(18a) and (18b), when squared and added, give

$$(m_1 + e_1 + m_2 - e_2)^2 - 2m_1 \cos\theta_1(m_1 + e_1 + m_2 - e_2) + m_1^2 - m_2^2 = 0$$

One can obtain a quartic equation either by cancelling the terms in m_2^2, separating the terms in m_2, squaring, and eliminating m_2^2 with the aid of (16), or by making a substitution consistent with (16) such as

$$e_2 = \frac{b^2}{2b + 1}, \quad m_2 = \frac{b(b + 1)}{2b + 1}, \quad m_2 - e_2 = \frac{b}{2b + 1}$$

and e_2, m_2 and θ_2 can then be determined.

When $\theta_1 \to 0$ (18a) shows that θ_2 also $\to 0$, provided that m_2 remains finite. Then

$$\cos\theta_1 \to 1 - \frac{\theta_1^2}{2}, \quad \cos\theta_1 \to 1 - \frac{\theta_2^2}{2}$$

and according to (18b) e_2 differs from e_1 by terms of order θ^2. It can be seen from (14) and (16) that m_2 then differs from m_1 by terms of a similar order, and finally from (18a) that as $\theta_1 \to 0$

$$\theta_2 \to 2\theta_1.$$

5. IMPACT ON A STRING

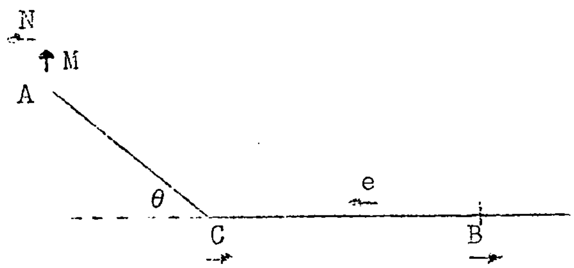


FIG. 5.1

If the end A of a string is suddenly given velocity components Mc_0 and Nc_0 by an impact, (Fig. 5.1) it will be impossible for the deflection to be transmitted along the untensioned string. First, therefore, a shock B travels along the string, establishing a tension. It is followed by a kink C which travels at a speed mc_0 relative to the string, where

$$m^2 = e + e^2 \tag{1}$$

Across the kink the vertical and horizontal components of velocity change by $mc_0 \sin\theta$ and $mc_0(1 - \cos\theta)$. Behind the shock the string has a speed ec_0 to the left, therefore

$$M = m \sin\theta \tag{2a}$$

$$e - N = m(1 - \cos\theta) \tag{2b}$$

Also the speed of the kink in space is

$$(m - e)c_0 = \frac{ec_0}{m + e}$$

Since $m > e$ when $e > 0$, $0 \leqslant m - e \leqslant \frac{1}{2}$.

Squaring and adding (2a) and (2b)

$$M^2 = m^2 - (m - e + N)^2 = (e - N)(2m - e + N) \tag{3}$$

One can obtain a quartic equation either by separating the term in m, squaring, and eliminating m^2 with the aid of (1), or by making a substitution consistent with (1) such as

$$e = \frac{a^2}{2a + 1}, \quad m = \frac{a(a + 1)}{2a + 1}, \quad m - e = \frac{a}{2a + 1}$$

For a fixed angle of impact the ratio of M to N is fixed. It is then easier to solve the problem in reverse for M, given e.

If the impact is normal, setting N = 0 (3) gives

$$M^2 = e(2m - e) \tag{4}$$

Then setting $M = ne$,

$$(n^2 + 1)e = 2m$$

and

$$(n^2 + 1)^2 e^2 = 4m^2 = 4e + 4e^2$$

whence

$$e = \frac{4}{(n^2 - 1)(n^2 + 3)}, \quad m = \frac{2(n^2 + 1)}{(n^2 - 1)(n^2 + 3)}, \quad m - e = \frac{2}{n^2 + 3},$$

$$\cos\theta = \frac{m - e}{m} = \frac{n^2 - 1}{n^2 + 1}, \quad \sin\theta = \frac{M}{m} = \frac{2n}{n^2 + 1}$$

A family of solutions for which M and e are both perfect fractions can be generated by substituting integral values of n: they are associated with Pythagorean number triples such as (3, 4, 5). Some solutions of this type obtained by setting n = 2, 3, 4 and 5 are listed in Table 5.1, and Fig. 5.2 shows the way in which the strain e, the kink angle θ, and the speed $(m - e)c_0$ of the kink depend on the impact speed Mc_0.

As $e \to 0$, $m^2 = e + e^2 \to e$.

Then (4) shows that for a normal impact

$$M^2 \to 2me \to 2e^{\frac{3}{2}}$$

so that

$$e \rightarrow .630M^{\frac{4}{3}} \tag{5a}$$

$$\sin\theta \;=\; \frac{M}{m} \rightarrow \sqrt{2}\; e^{\frac{1}{4}}, \tag{5b}$$

$$\cos\theta \rightarrow 1 - \frac{\theta^2}{2} \rightarrow 1 - m \tag{5c}$$

It can be seen from Fig. 5.2 that e departs rather slowly from its asymptotic value. The approximation

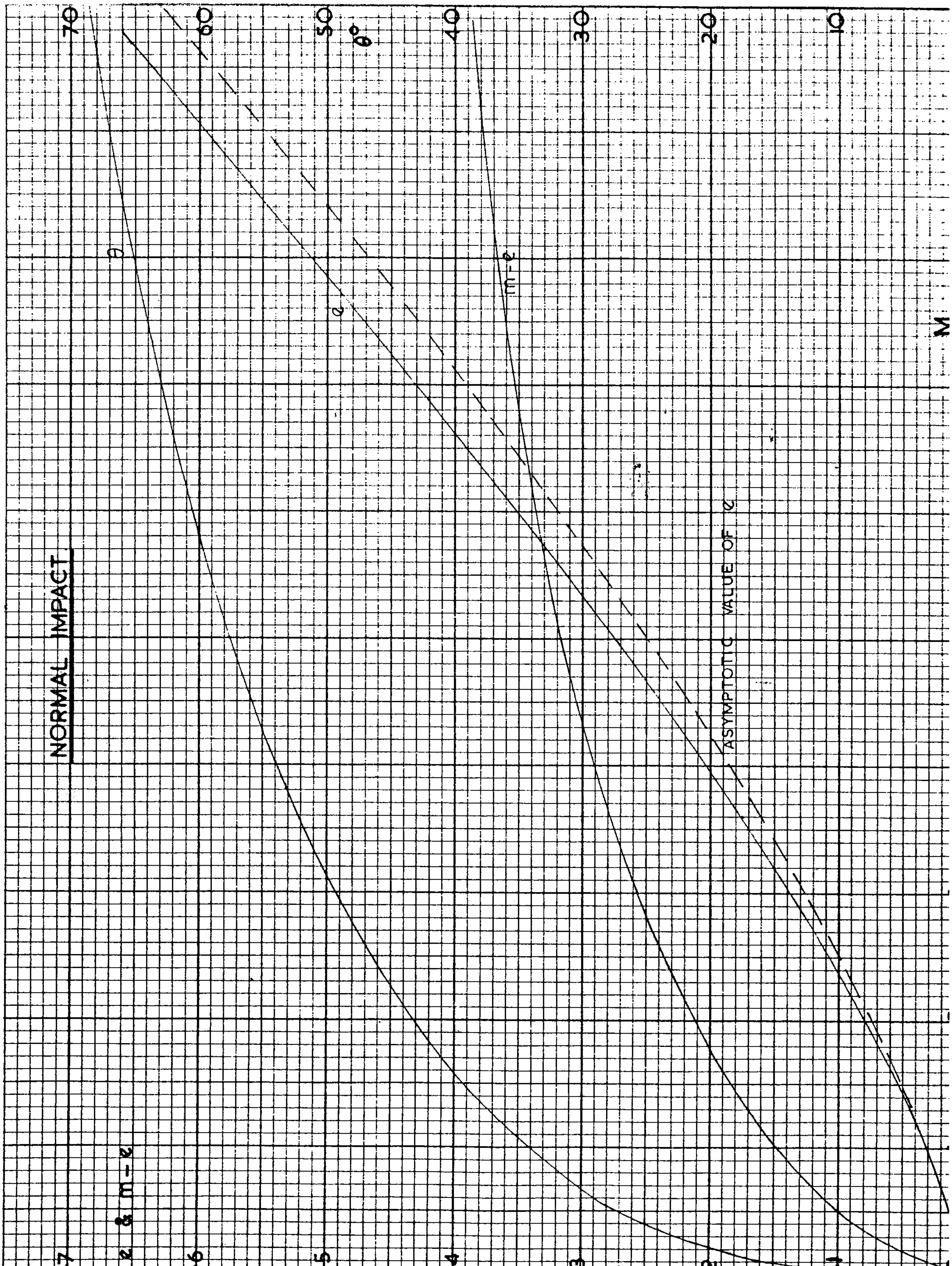
$$e \;=\; \tfrac{2}{3}\, M^{\frac{4}{3}}$$

has an error which ranges from +4% to −4% as M increases from 0 to .5.

TABLE 5.1

NORMAL IMPACT

M	e	m	m − e	$\cos\theta$	θ
$\frac{5}{168}$	$\frac{1}{168}$	$\frac{13}{168}$	$\frac{1}{14}$	$\frac{12}{13}$	22.6°
$\frac{16}{285}$	$\frac{4}{285}$	$\frac{34}{285}$	$\frac{2}{19}$	$\frac{15}{17}$	28.1°
$\frac{1}{8}$	$\frac{1}{24}$	$\frac{5}{24}$	$\frac{1}{6}$	$\frac{4}{5}$	36.8°
$\frac{8}{21}$	$\frac{4}{21}$	$\frac{10}{21}$	$\frac{2}{7}$	$\frac{3}{5}$	53.2°



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6. REMOVAL OF THE LOAD AFTER A NORMAL IMPACT

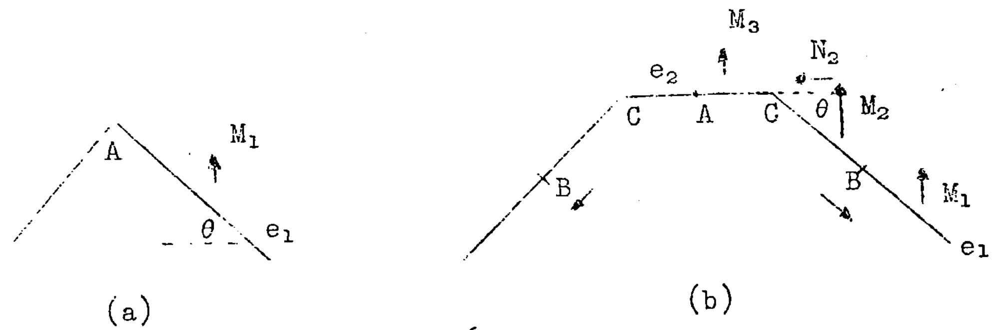

FIG. 6.1

If a string receives a normal impact at A (Fig. 6.1(a)) and the load is suddenly removed, (Fig. 6.1(b)) the kink at A cannot remain stationary, so kinks C must move out each way from A leaving between them a straight section, which will be horizontal because of the symmetry. If the kinks were not preceded by shocks, the parts of the string ahead of the kinks would have a purely vertical velocity, and the kinks would impart opposite horizontal components of velocity to the string either side of A. The kinks must therefore be preceded by shocks B.

The equations for the original impact give

$$m_1 \sin\theta = M_1 \qquad (1a)$$

$$m_1 \cos\theta = m_1 - e_1 \qquad (1b)$$

$$m_1^2 = e_1 + e_1^2 \qquad (2)$$

For the shocks B

$$M_2 = M_1 + (e_2 - e_1)\sin\theta \qquad (3a)$$

$$N_2 = (e_2 - e_1)\cos\theta \qquad (3b)$$

For the kinks C

$$M_3 = M_2 - m_2 \sin\theta \qquad (4a)$$

$$0 = N_2 + m_2 (1 - \cos\theta) \qquad (4b)$$

$$m_2^2 = e_2 + e_2^2 \qquad (5)$$

Eliminating M_2 and N_2 from (3) and (4),

$$M_3 = M_1 - (e_1 - e_2 + m_2)\sin\theta$$

$$(e_1 - e_2)\cos\theta = m_2\,(1 - \cos\theta)$$

It can be seen from the second of these that $e_2 < e_1$.
Using (1) and (2)

$$M_3 = [(m_1 - e_1) - (m_2 - e_2)]\sin\theta = \frac{M_1}{m_1}[(m_1 - e_1) - (m_2 - e_2)]$$

$$e_1 - e_2 = \frac{m_2 e_1}{m_1 - e_1} = m_2(m_1 + e_1) \qquad (6)$$

It can be seen from the first of these that $0 < M_3 < M_1$.
Equation (6) must be solved subject to (2) and (5).
The sideways speed of the kinks C is $m_2 c_0$, and the ratio of this speed to the speed of the original kinks is

$$\frac{m_2}{m_1 - e_1} = 1 - \frac{e_2}{e_1}$$

Setting

$$e_1 = \frac{a^2}{2a + 1}, \quad m_1 = \frac{a(a + 1)}{2a + 1}, \quad m_1 + e_1 = a,$$

$$e_2 = \frac{b^2}{2b + 1}$$

(6) becomes

$$\frac{a^2}{2a + 1} - \frac{b^2}{2b + 1} = \frac{ab(b + 1)}{2b + 1}$$

or

$$b^2(a + 1)(2a + 1) + ab - a^2 = 0$$

whence

$$b = \frac{-a + a\sqrt{8a^2 + 12a + 5}}{2(a + 1)(2a + 1)}$$

The limiting case $e_1 \to 0$ is obtained when $a \to 0$. Then

$$\frac{b}{a} \to \frac{\sqrt{5} - 1}{2},$$

and

$$\frac{e_2}{e_1} \to \frac{3 - \sqrt{5}}{2} = .382$$

Some solutions obtained by setting $a = 2, 1, \frac{1}{2}, \frac{1}{3}$ and $\frac{1}{4}$ are listed in Table 6.1, and Fig. 6.2 shows the way in which the speed $M_3 c_0$ of the central section, and the ratio of the strain e_2 after the removal of the load to the strain e_1 after the original impact depend on the speed $M_1 c_0$ of the impact.

TABLE 6.1

REMOVAL OF THE LOAD AFTER A NORMAL IMPACT

M_1	e_1	M_3	e_2	$\frac{e_2}{e_1}$
1.132	.8	.152	.1077	.135
.576	.333	.116	.0667	.2
.279	.125	.0735	.0329	.263
.172	.0667	.0518	.0196	.294
.125	.0417	.0389	.0130	.312

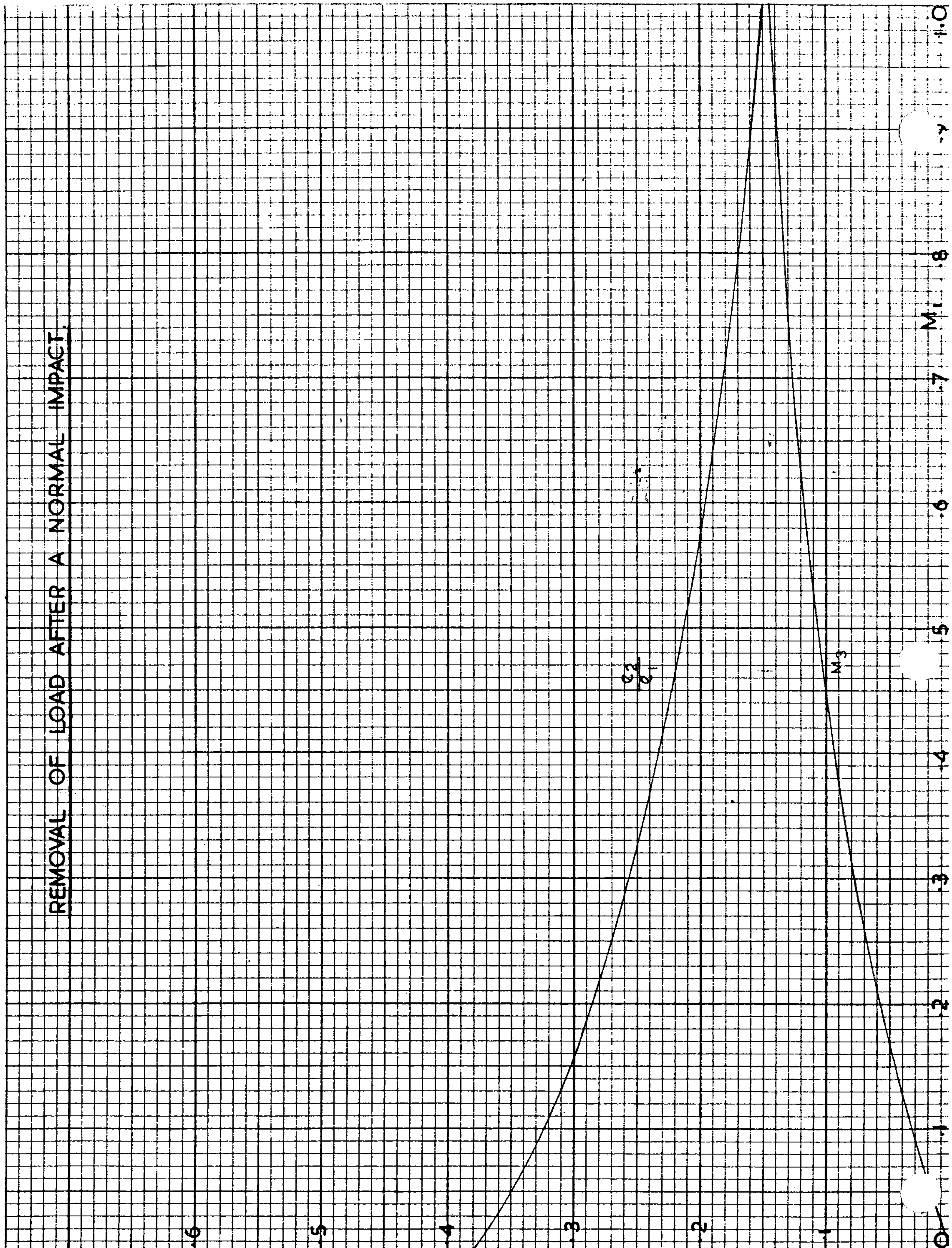

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FIG. No. 6·2

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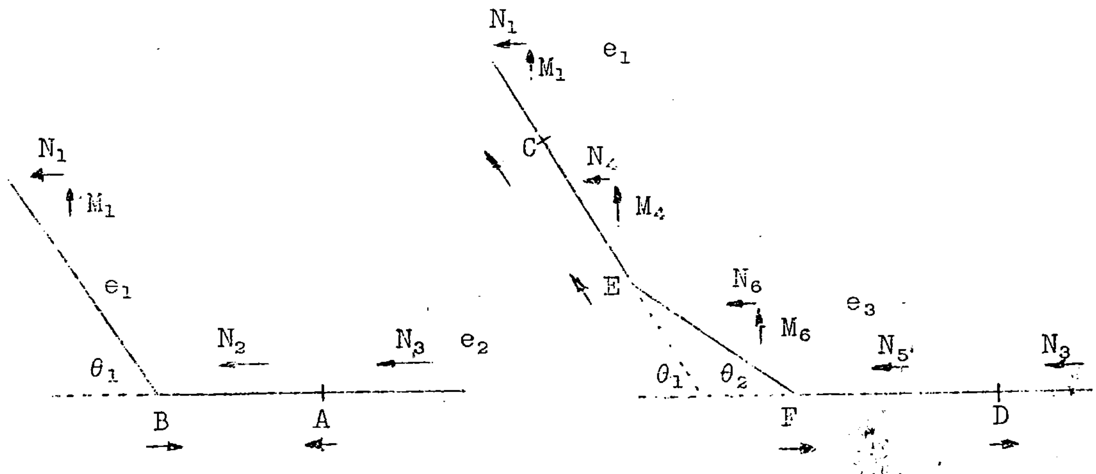
7. COLLISION OF A SHOCK AND KINK



(a) FIG. 7.1 (b)

If a string of finite length suffers an impact, the resulting shock will be reflected when it reaches the end of the string, and will travel back until it hits the following kink. When a shock A collides with a kink B, (Fig. 7.1(a)) the stress behind A is in general not compatible with the velocity changes across B. Accordingly shocks C and D and kinks E and F are transmitted from the point of collision (Fig. 7.1(b)). Since static kinks and shocks are impossible, no discontinuity can remain at the point of collision. The section EF is therefore straight, and the stress is constant from C to D.

For the kink B

$$m_1^2 = e_1 + e_1^2 \tag{1}$$

$$M_1 = m_1 \sin\theta_1 \tag{2a}$$

$$N_2 - N_1 = m_1 (1 - \cos\theta_1) \tag{2b}$$

For the shock A

$$N_3 - N_2 = e_1 - e_2 \tag{3}$$

For the shocks C and D

$$M_4 = M_1 - (e_3 - e_1)\sin\theta_1 \tag{4a}$$

$$N_4 = N_1 - (e_3 - e_1)\cos\theta_1 \tag{4b}$$

$$M_5 = 0 \tag{5a}$$

$$N_5 = N_3 + e_3 - e_2 \tag{5b}$$

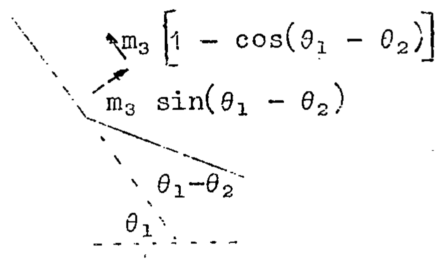


FIG. 7.2

For the kink E (Fig. 7.2)

$$m_3^2 = e_3 + e_3^2 \tag{6}$$

$$\begin{aligned} M_6 &= M_4 + m_3 \left[\sin\theta_1 - \sin\theta_1 \cos(\theta_1 - \theta_2) + \cos\theta_1 \sin(\theta_1 - \theta_2)\right] \\ &= M_4 + m_3(\sin\theta_1 - \sin\theta_2) \end{aligned} \tag{7a}$$

$$\begin{aligned} N_6 &= N_4 + m_3 \left[\cos\theta_1 - \cos\theta_1 \cos(\theta_1 - \theta_2) - \sin\theta_1 \sin(\theta_1 - \theta_2)\right] \\ &= N_4 + m_3(\cos\theta_1 - \cos\theta_2) \end{aligned} \tag{7b}$$

For the kink F

$$M_6 = m_3 \sin_2 \tag{8a}$$

$$N_6 = N_5 - m_3(1 - \cos\theta_2) \tag{8b}$$

(7) and (8) give

$$2m_3 \sin\theta_2 = M_4 + m_3 \sin\theta_1 \tag{9a}$$

$$2m_3 \cos\theta_2 = N_4 - N_5 + m_3(1 + \cos\theta_1) \tag{9b}$$

But (2) and (3) give

$$N_3 - N_1 = e_1 - e_2 + m_1(1 - \cos\theta_1)$$

and (4) and (5) give

$$M_4 = (m_1 + e_1 - e_3)\sin\theta_1$$

$$N_4 - N_5 = N_1 - N_3 + e_1 \cos\theta_1 + e_2 - e_3(1 + \cos\theta_1)$$

so (9a) and (9b) become

$$2m_3 \sin\theta_2 = (m_3 - e_3)\sin\theta_1 + (m_1 + e_1)\sin\theta_1 \qquad (10a)$$

$$2m_3 \cos\theta_2 = (m_3 - e_3)(1 + \cos\theta_1) - (m_1 + e_1)(1 - \cos\theta_1) + 2e_2 \qquad (10b)$$

If $e_2 = e_1$ these equations are evidently satisfied by $e_3 = e_1$, $m_3 = m_1$, $\theta_2 = \theta_1$.

Since

$$\sin^2\theta + (1 \pm \cos\theta)^2 = 2(1 \pm \cos\theta)$$

and

$$(1 + \cos\theta)(1 - \cos\theta) = \sin^2\theta$$

(10a) and (10b), when squared and added, give

$$2e_2^2 - 2e_2\left[(m_1 + e_1)(1 - \cos\theta_1) - (m_3 - e_3)(1 + \cos\theta_1)\right] + (m_1 + e_1)^2(1 - \cos\theta_1) + (m_3 - e_3)^2(1 + \cos\theta_1) - 2m_3^2 = 0 \qquad (11)$$

It is easiest to solve the inverse problem of finding e_2, given e_1 and e_3. According to (11)

$$e_2 = A \pm B$$

where

$$A = \tfrac{1}{2}(m_1 + e_1)(1 - \cos\theta_1) - \tfrac{1}{2}(m_3 - e_3)(1 + \cos\theta_1)$$

and

$$B^2 = m_3^2 - \tfrac{1}{4}(m_1 + e_1 + m_3 - e_3)^2 \sin^2\theta_1$$

Only values of e_3 for which B is positive are possible. When B is positive there are two values of e_2 which will result in the given value of e_3, but one may be negative. According to (10a) $\sin\theta_2$ is the same for these two values, so if $\theta_2 = \alpha$ for one of them, $\theta_2 = \pi - \alpha$ for the other: one is a solution with a reverse kink. (10a) can be rearranged as

$$m_3(2\sin\theta_2 - \sin\theta_1) + e_3\sin\theta_1 = (m_1 + e_1)\sin\theta_1$$

Since m_1, e_1, $\sin\theta_1$, m_3 and e_3 are positive, and m_3 increases with e_3, e_3 is least for a given value of e_1 when $\theta = \pi/2$. Then $B = 0$, and the two solutions for e_2 coincide.

If the kink originated from a normal impact, then according to section 5, $N_2 = e_1$ and

$$m_1\cos\theta_1 = m_1 - e_1 \tag{12}$$

For this case some solutions of (11) for e_2, given e_1 and e_3, are listed in Table 7.1, and Fig. 7.3 shows the way in which e_3 varies with e_2 for different values of e_1.

Apart from the trivial solution $e_1 = e_2 = e_3$, there are solutions for which $e_2 \neq e_1$, and $e_3 = e_1$, so that there is no shock C. When $e_3 = e_1$, the solution of (11) is

$$e_2 = e_1 \text{ or } e_1 - 2m_1\cos\theta_1$$

If the kink originated from a normal impact, substituting from (12) for $m_1\cos\theta_1$, the second solution is

$$e_2 = 3e_1 - 2m_1$$

With the aid of (1) it can be seen that this is positive when $e_1 > \frac{4}{5}$, though such large values of e_1 are probably outside the range which could be encountered in practice.

To determine the asymptotic behaviour as $e_1 \to 0$, suppose that $e_2 = re_1$. Then (11) may be written

$$[m_1 + e_1(1 - r)]\ (1 - \cos\theta_1)$$

$$- (e_3 - re_1)[2m_3 - (e_3 - re_1)]\ (1 + \cos\theta_1) - m_3^2(1 - \cos\theta_1) = 0$$

If e_1 and $m_1 \to 0$ the left hand side would remain finite and negative unless $e_3 \to 0$, so e_3 must $\to 0$ with e_1. Then

$$m_1 \to \sqrt{e_1}\,, \quad \frac{e_1}{m_1} \to 0$$

$$m_3 \to \sqrt{e_3}\,, \quad \frac{e_3}{m_3} \to 0$$

and according to equation 5.5c, if the kink originated from a normal impact,

$$\cos\theta_1 \to 1 - m_1$$

so in the limit, provided that $\frac{e_1}{e_3}$ remains finite,

$$m_3^2\, m_1 + 4m_3(e_3 - re_1) - m_1^3 = 0$$

Putting

$$\frac{m_1}{m_3} = k, \quad \frac{e_1}{e_3} \doteq k^2,$$

this becomes

$$k^3 + 4rk^2 - k - 4 = 0$$

Denoting the left hand side by F

$$\frac{dF}{dk} = 3k^2 + 8rk - 1$$

$$\frac{dF}{dk} = 0 \text{ when } k = \frac{-4r}{3} \pm \sqrt{\left(\frac{4r}{3}\right)^2 + 1}$$

and $F = -4$ when $k = 0$

so this is a cubic with a single positive root.
If $e_2 = 0$, then $F = 0$ when $k = 1.80$, and

$$\lim_{e_1 \to 0} \frac{e_3}{e_1} = .309$$

If $e_2 = 2e_1$ then $F = 0$ when $k = .736$ and

$$\lim_{e_1 \to 0} \frac{e_3}{e_1} = 1.848$$

When $e_2 = 0$, as for a shock reflected off a free end, it is possible to devise a direct method of generating solutions.

(11) now reduces to

$$(m_1 + e_1)^2(1 - \cos\theta_1) + (m_3 - e_3)^2(1 + \cos\theta_1) - 2m_3^2 = 0$$

where, according to (12), if the kink originated from a normal impact,

$$\cos\theta_1 = 1 - \frac{e_1}{m_1}$$

Then setting

$$e_1 = \frac{a^2}{2a + 1}, \quad m_1 = \frac{a(a + 1)}{2a + 1}, \quad m_1 + e_1 = a, \quad \cos\theta_1 = \frac{1}{a + 1}$$

$$e_3 = \frac{b^2}{2b + 1}, \quad m_3 = \frac{b(b + 1)}{2b + 1}, \quad m_3 + e_3 = b, \quad m_3 - e_3 = \frac{b}{2b + 1}$$

one obtains

$$\frac{a^3}{a + 1} + \frac{b^2}{(2b + 1)^2}\,\frac{a + 2}{a + 1} - \frac{2b^2(b + 1)^2}{(2b + 1)^2} = 0$$

or

$$a^3(2b + 1)^2 - ab^2(2b^2 + 4b + 1) - 2b^3(b + 2) = 0$$

whence, if $a = kb$,

$$2b^2(2k^3 - k) + 2b(2k^3 - 2k - 1) + k^3 - k - 4 = 0$$

If $k^3 > k + 4$, all three coefficients are positive, and there is no positive solution. When $k^3 = k + 4$, or $k = 1.80$, there is a solution $a = b = 0$: this is the solution approached in the limit as $e_1 \to 0$. Then

$$k = \frac{a}{b} = \frac{m_1 + e_1}{m_3 + e_3} \to \sqrt{\frac{e_1}{e_3}}$$

and k has the same significance as in the earlier treatment of the asymptotic behaviour.

If $k^2 < \frac{1}{2}$ all three coefficients are negative, and again there is no positive solution. When $k^2 = \frac{1}{2}$ there is a solution $\frac{1}{b} = 0$: this is the solution approached in the limit as $e_1 \to \infty$.

Then

$$\frac{m_1}{e_1} \text{ and } \frac{m_3}{e_3} \to 1$$

so that

$$\frac{e_3}{e_1} \to \frac{1}{k} = \sqrt{2}$$

For large values of e_1, e_3 is thus greater than e_1.

When k lies between these limits

$$b = \frac{1 + 2k - k^3 + \sqrt{1 - 4k + 2k^3 + 12k^2 - 2k^4}}{4k^3 - 2k}$$

Solutions in perfect fractions are obtained when $k = 1$ and $\frac{12}{11}$. These are

$$e_1 = \tfrac{4}{5},\ e_2 = 0,\ e_3 = \tfrac{4}{5}$$

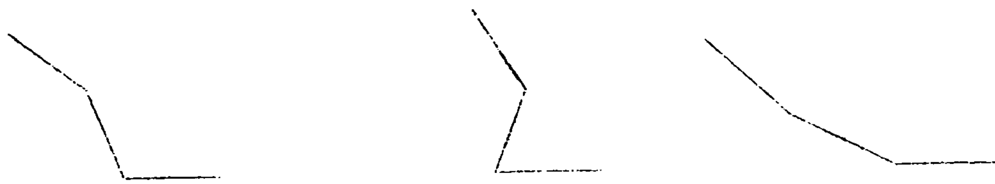
and

$$e_1 = \tfrac{256}{473},\ e_2 = 0,\ e_3 = \tfrac{16}{33}$$

Solutions obtained by setting $k = 1.2, 1.3, 1.4, 1.5$ and 1.6 are listed in Table 7.2, and Fig. 7.4 shows the way in which the final strain e_3 and the angles θ_1 and θ_2 depend on the initial strain e_1.

When the kink collides with a shock reflected off an anchored end, $e_2 = 2e_1$. Some solutions obtained from the curves of e_3 against e_2 when the first impact is normal are listed in Table 7.3, and Fig. 7.5 shows the way in which the final strain e_3 and the angles θ_1 and θ_2 then depend on the initial strain e_1.

It can be seen from Figs. 7.4 and 7.5 that for a collision of a kink with a shock reflected from a free end $\theta_2 > \theta_1$, whereas for a collision with a shock reflected from an anchored end $\theta_2 < \theta_1$. In the first case the angle steepens, the string assuming a shape as in Fig. 7.6(a), or even developing an overhang as in Fig. 7.6(b). In the second case the angle becomes shallower, the string assuming a shape as in Fig. 7.6(c).



(a) (b) (c)

FIG. 7.6

Fig. 7.7 shows the way in which the ratio of the final strain e_3 to the initial strain e_1 varies with e_1 for the two cases. For a collision of a kink with a shock reflected off a free end $e_3 < e_1$ for small values of e_1, but $e_3 > e_1$ when $e_1 > \frac{4}{5}$, while for a collision with a shock reflected off a fixed end $e_1 < e_3 < e_2$.

TABLE 7.1

COLLISION OF A KINK FROM A NORMAL IMPACT AND A SHOCK

e_1	e_3	roots for e_2	
.1	.0667	.036,	-.245
	.1	.1 ,	-.363
	.15	.172,	-.492
	.2	.235,	-.597
	.3	.353,	-.773
	.4	.464,	-.924
	.5	.572,	-1.062
.2	.15	.086,	-.226
	.2	.2 ,	-.380
	.3	.335,	-.589
	.4	.484,	-.757
	.5	.604,	-.905
.3	.222	-.015,	-.015
	.25	.174,	-.200
	.3	.3 ,	-.349
	.4	.471,	-.557
	.5	.612,	-.724

TABLE 7.2

COLLISION OF A SHOCK REFLECTED OFF A FREE END

WITH THE FOLLOWING KINK AFTER A NORMAL IMPACT

e_1	m_1	$\cos\theta_1$	θ_1	e_3	m_3	$\cos\theta_2$	θ_2	$\frac{e_3}{e_1}$
$\frac{4}{5}$	$\frac{6}{5}$	$\frac{1}{3}$	70.5°	$\frac{4}{5}$	$\frac{6}{5}$	$-\frac{1}{3}$	109.5°	1
$\frac{256}{273}$	$\frac{432}{273}$	$\frac{11}{27}$	66.0°	$\frac{16}{33}$	$\frac{28}{33}$	$-\frac{13}{63}$	101.9°	$\frac{43}{48}$
.351	.687	.491	60.6°	.274	.590	-.047	92.7°	.780
.233	.536	.565	55.5°	.160	.431	.104	84.0°	.688
.155	.423	.634	50.7°	.0935	.320	.246	75.7°	.603
.0847	.303	.721	43.8°	.0441	.214	.433	65.3°	.521
.0392	.202	.805	36.3°	.0175	.133	.613	52.2°	.447

TABLE 7.3

COLLISION OF A SHOCK REFLECTED OFF AN ANCHORED END WITH THE FOLLOWING KINK AFTER A NORMAL IMPACT

e_1	m_1	$\cos\theta_1$	θ_1	θ_2	m_3	$\cos\theta_2$	θ_2	$\frac{e_3}{e_1}$
.1	.332	.697	45.7°	.172	.449	.825	34.4°	1.72
.2	.490	.592	53.7°	.334	.668	.788	38.0°	1.67
.3	.624	.820	58.7°	.490	.859	.765	40.1°	1.63

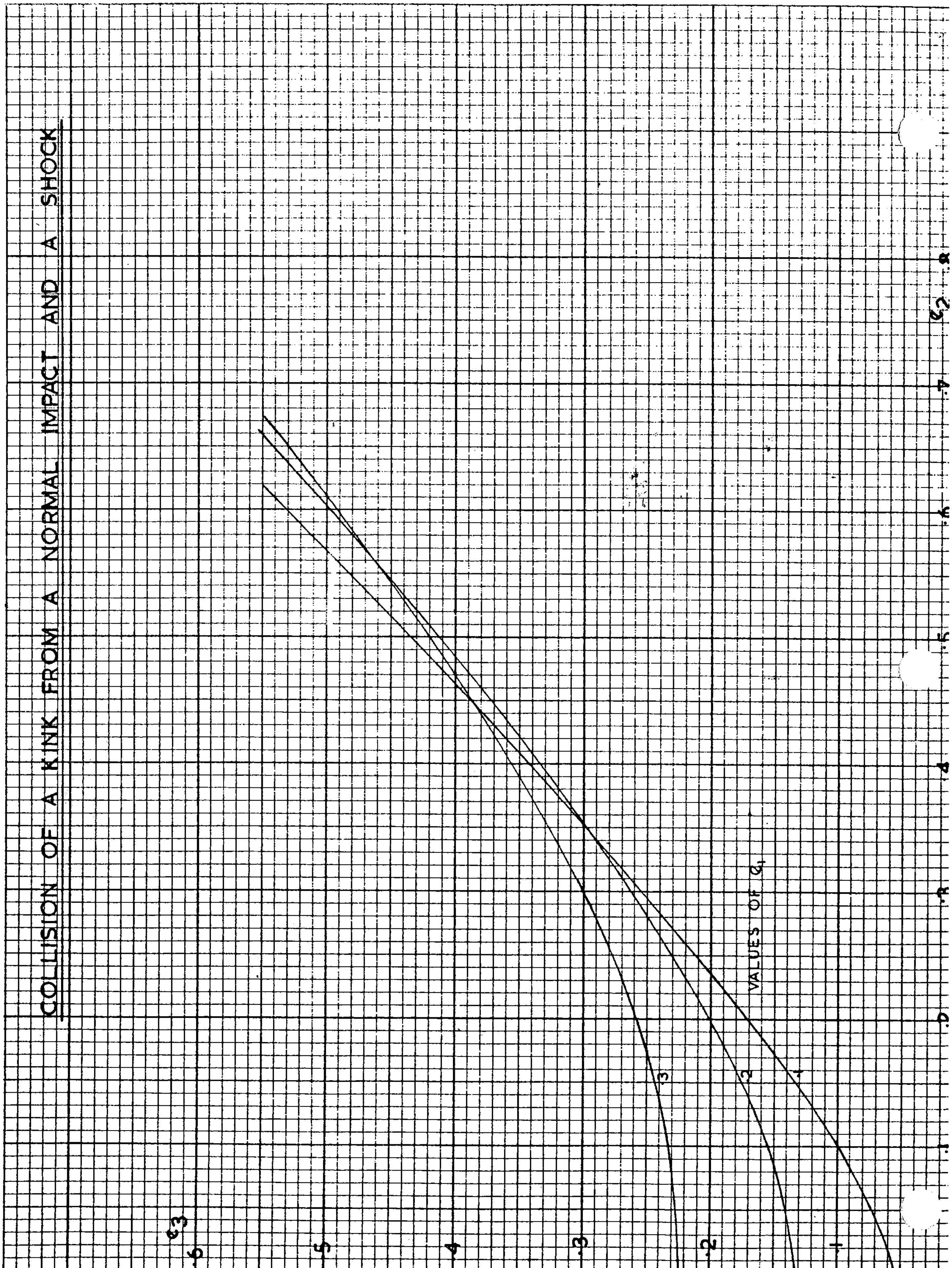


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FIG. No. 7·3

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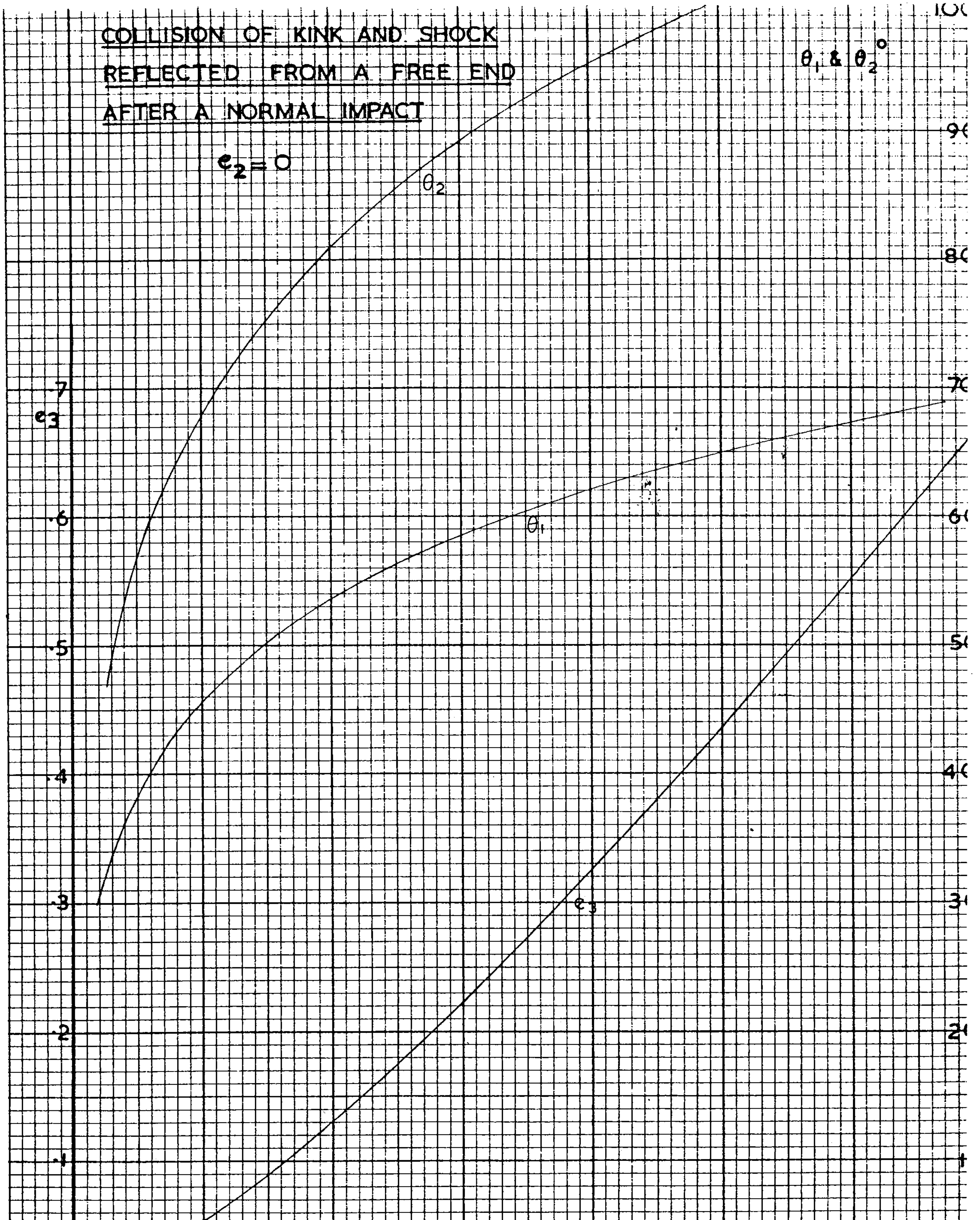

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FIG. No. 7.4.

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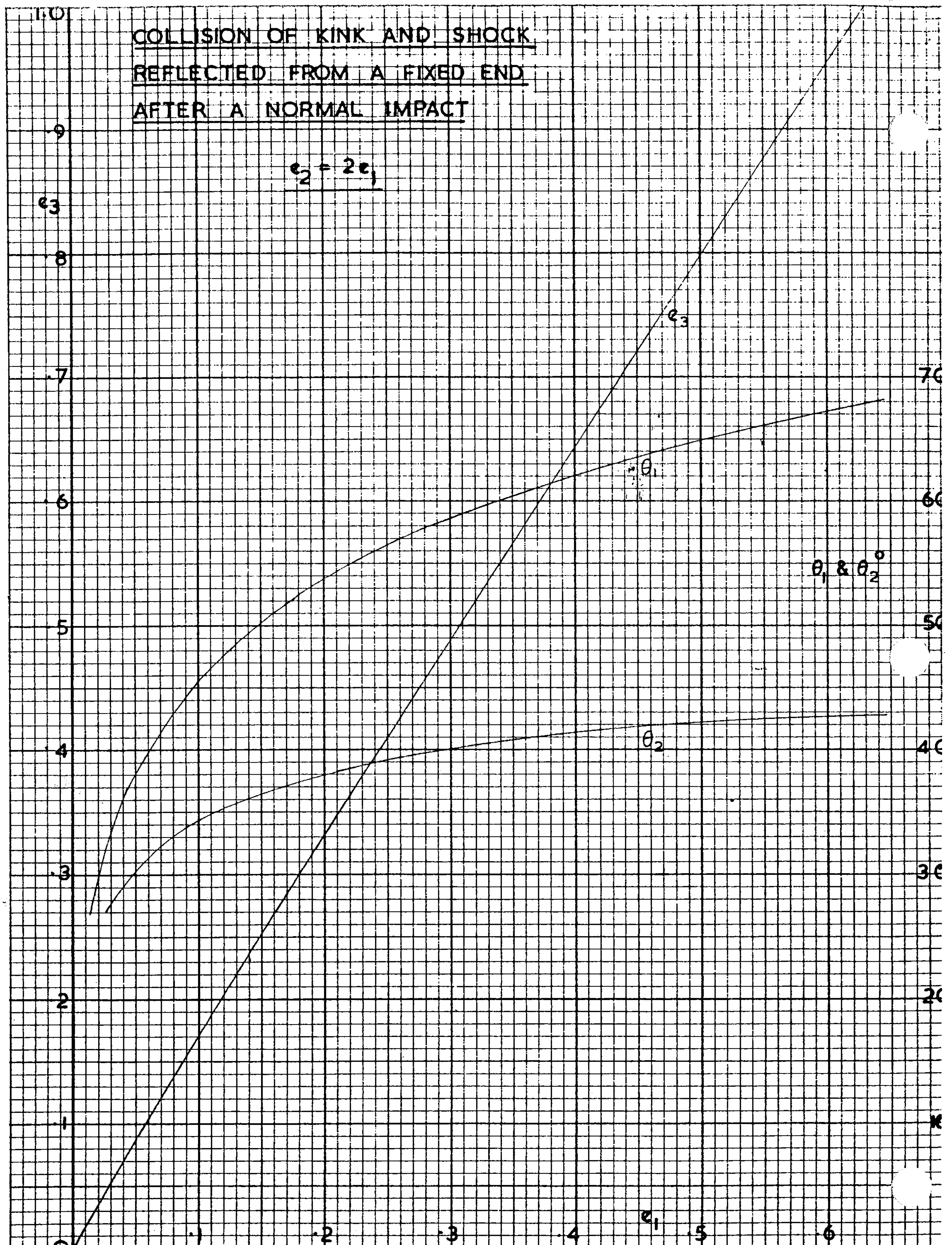

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FIG. No. 7.5.

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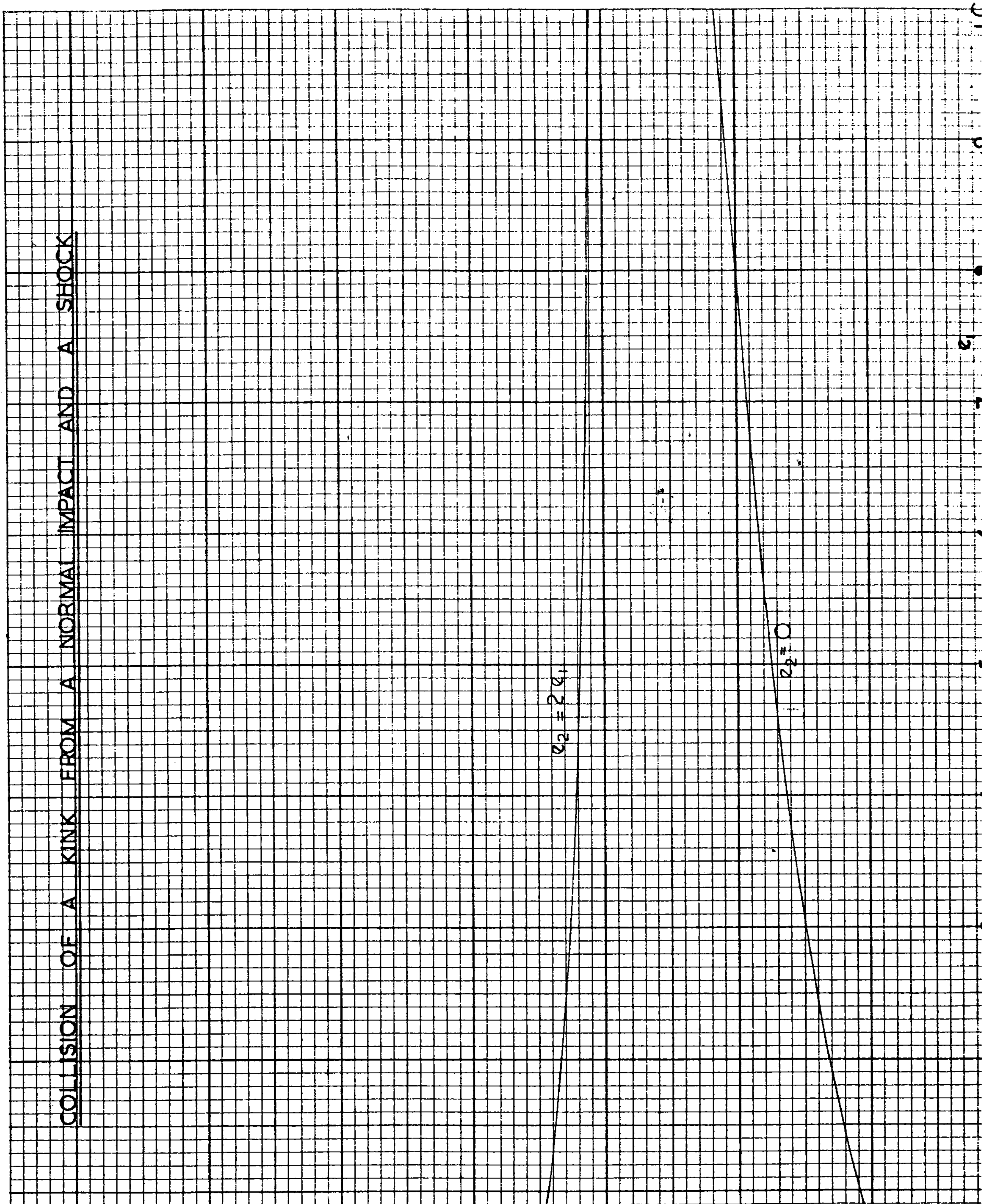

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FIG. No. 7·7

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8. DEVELOPMENT OF THE MOTION OF A STRING AFTER AN IMPACT

From the results of the previous sections it is possible to predict, within the limitations of the theory, the way in which the motion of a string will develop after an impact.

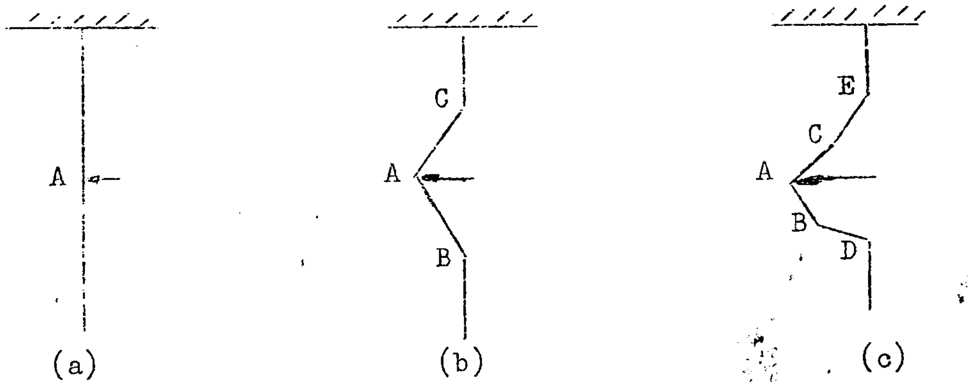


FIG. 8.1

Suppose that a string hanging from its upper end, with its lower end free, is struck at a point A by a bullet travelling horizontally (Fig. 8.1.(a)). Then according to section 5 a notch BAC will spread out from A (Fig. 8.1(b)) . The notch will be preceded by shocks which will be reflected from the ends of the string. According to section 7 the collision of the expanding notch with the shock reflected off the free end will cause a segment BD to spread out at a steeper angle, undercutting the notch (Fig. 8.1(c)). The collision with the shock reflected off the anchored end, on the other hand, will cause a segment CE to spread out at a shallower angle. The outgoing shocks from these collisions will themselves in turn be reflected from the ends. Repeated collisions of the front of the notch with shocks reflected off the tip will cause the string to fold back on itself. If the string is struck near the tip this process may be quite advanced by the time the first reflected shock from the anchored end reaches the notch.

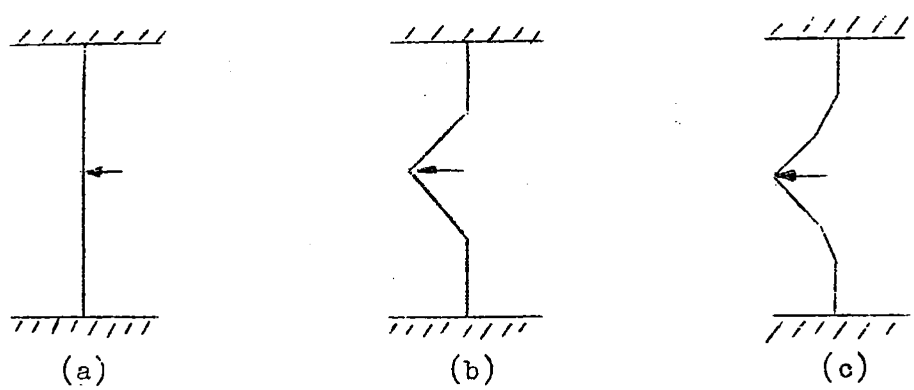


FIG. 8.2

If a string anchored at both ends is struck in the centre by a bullet, the motion will develop symmetrically as in Fig. 8.2. Repeated reflections of shocks off a fixed end will cause the stress to rise, until either the string breaks or the stress ceases to depend linearly on the strain, and the theory ceases to be valid.

FIG. 8.3

If the bullet comes off the string, then according to section 6 a flat should develop as in Fig. 8.3. The expansion waves from the flat will relieve the stress, opposing the rise in stress caused by reflections off a fixed end.

(a) (b)

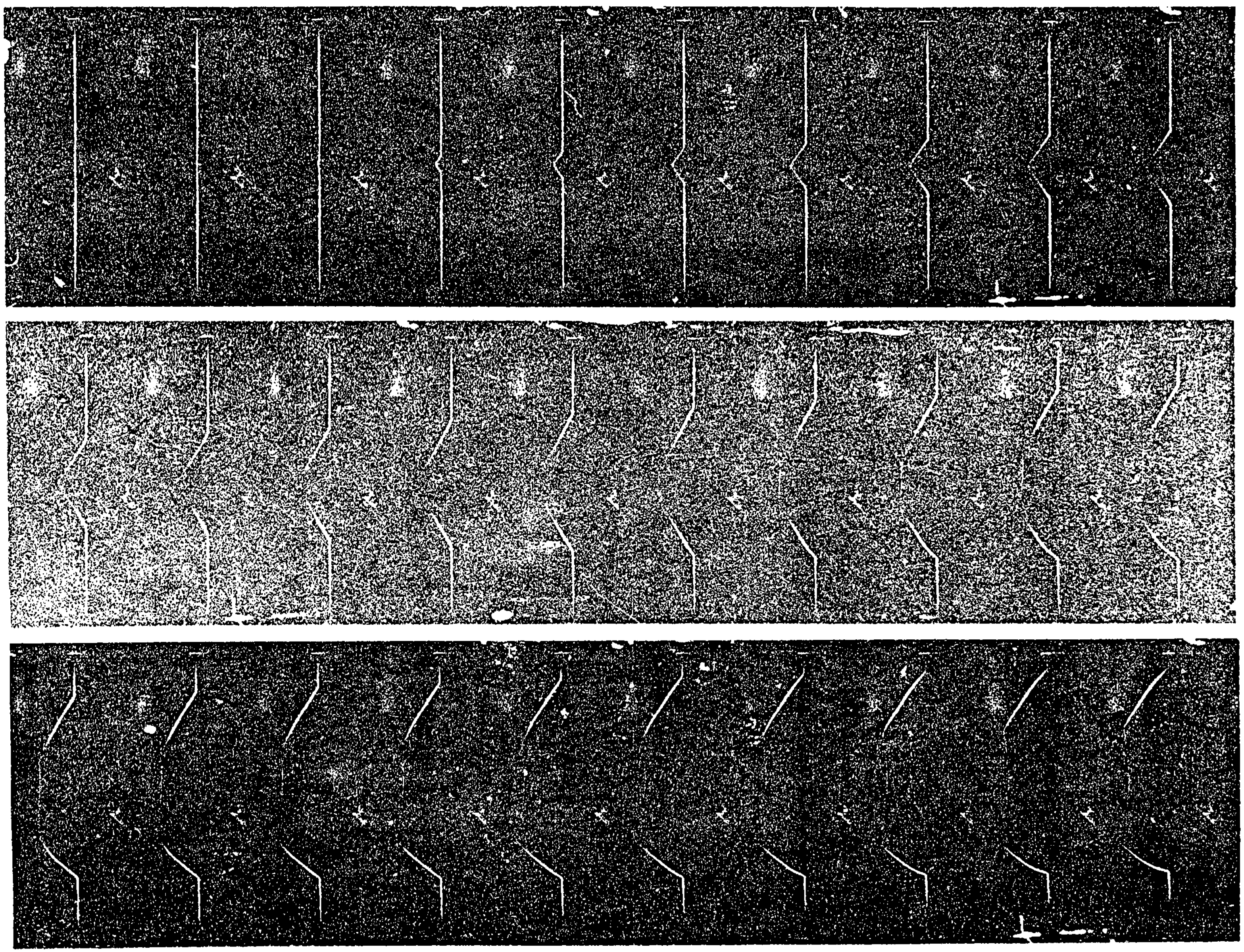
FIG. 8.4

When the kink at the front of the notch reaches a fixed end, it may be expected from section 4 that it will be reflected with a reversal of angle, as in Fig. 8.4, though this will be complicated by the action of the reflected shocks.

Some experiments were carried out under the direction of Mr. K.T. Wardle. An airgun was used to fire flat nosed bullets at strings, and the impacts were photographed with a FASTAX high speed camera by Mr. Derek Lowe. The edge of the film was exposed to the light of a neon bulb supplied with alternating current at a frequency of 50 cycles per sec. From a count of the number of frames between successive light and dark patches it was thus possible to estimate the speed of the film. The accuracy was limited to a few per cent by the lack of sharp edges between the patches and by apparent fluctuations in the speed. Given the speed of the film, the speed of the bullet in each experiment could be estimated from a knowledge of the scale of the photographs. Some of the photographs are reproduced in Figs. 8.5, 8.6 and 8.7. It can be seen that the effects predicted by the theory were in fact observed.

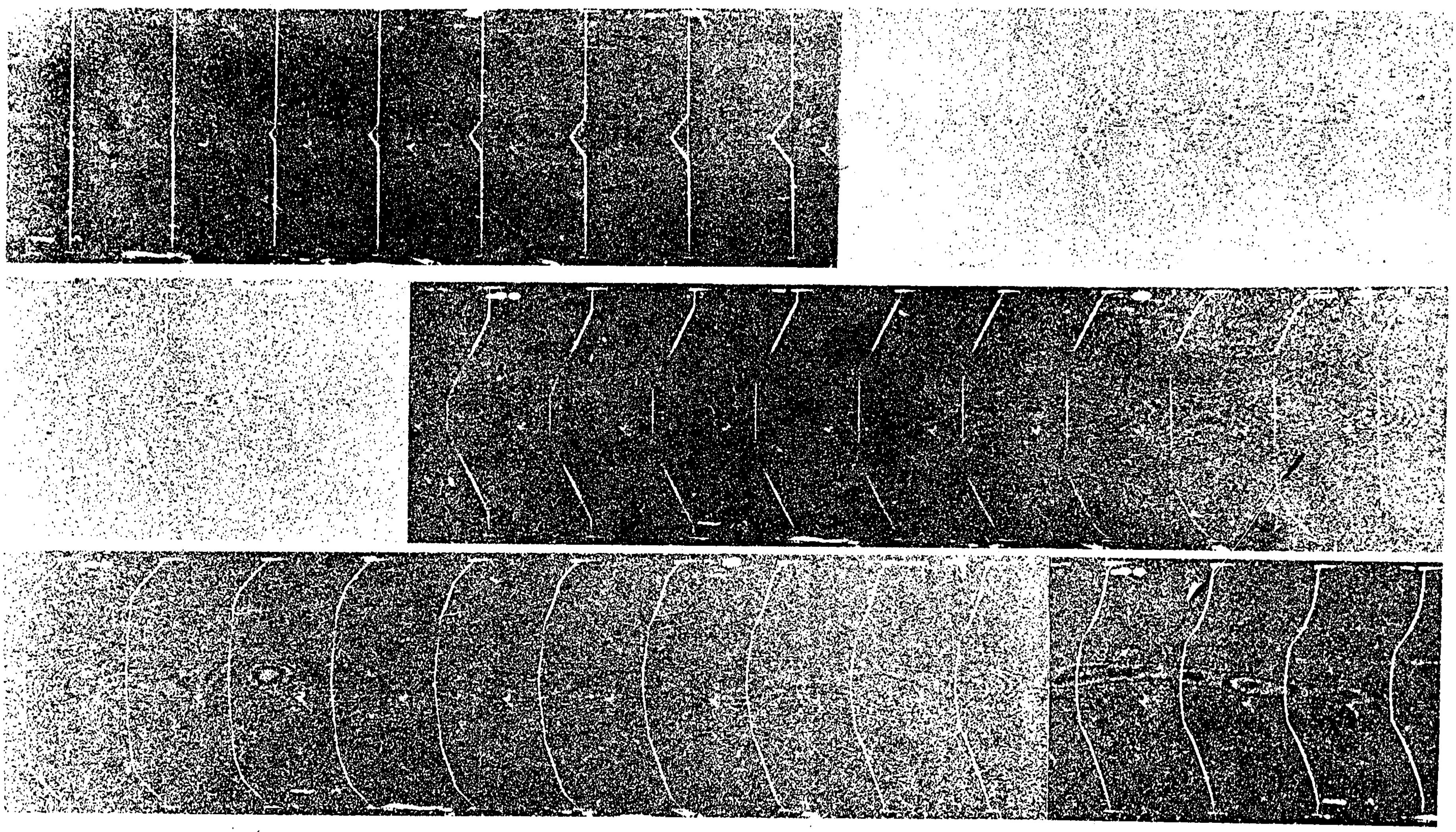
The photographs shown in Figs. 8.5 and 8.6 were taken with direct lighting. The sequence should be read from left to right, but the bullet travelled from right to left. The thread used in these two experiments was a twisted nylon yarn with a weight of .267 oz. per ft. and a breaking strength of 22 - 25 pounds. The results of a test of its extension under load are shown in Fig. 8.8. The slope of the force curve increased as the extension increased. The initial slope was .55 pounds for a 1 per cent extension, and the corresponding speed of a shock wave would be 3250 ft. per sec. In Fig. 8.5 the string was hanging from its upper end. The lower end was embedded in a piece of plasticine to stop the string swinging around, but was otherwise free. In Fig. 8.6 the string was anchored at both ends. The figures both show a notch spreading out from the bullet in the expected manner. In Fig. 8.5 the speed of the film was about 13700 frames per sec., and in Fig. 8.6 it was about 14100 frames per sec. From this information the speed of the bullet was estimated to be about 500 ft. per sec. in each case. In Fig. 8.5 the angle at the front of the notch was about 36° and in Fig. 8.6 it was about 34°. From Fig. 5.2 it can be seen that the corresponding extension of the string would be only a few per cent in each case, and assuming that the wave speed was 3250 ft. per sec., the ratio M of the speed of the bullet to the wave speed was about .154. According to the theory the angle at the front of the notch should then have been 39.6°. When the lower end was free, it can be seen from Fig. 8.5 that an indentation appeared on the lower side of the notch, this marking the return of the shock reflected from the tip. As the indentation increased, the string began to fold on itself. When both ends were anchored, it can be seen from Fig. 8.6 that the angle of the kinks at the front of the notch decreased slightly as the notch expanded. When the kinks reached the ends they were reflected with their angles reversed. In both experiments the bullet came off the string and a flat developed at the centre of the string.

Fig. 8.7 was obtained by a shadow technique, the apparatus being placed between the camera and a perspex screen illuminated from behind. This resulted in a sharper definition, and the bullet is clearly visible in the photographs. In this case the bullet travelled from left to right. The thread was a braided terylene yarn with a weight of .494 oz. per ft. and a breaking strength of 26 - 27 pounds. The results of a test of its extension under load are shown in Fig. 8.9. The thread exhibited a linear dependence of force on extension. The force for a 1% extension was 1.32 pounds, and the corresponding speed of a shock wave would be 3720 ft. per sec. As in Fig. 8.5, the lower end of the string was free to pull out from a lump of plasticine. A notch again developed from the bullet after the impact. The speed of the film was about 13900 frames per sec., and from this it was estimated that the speed of the bullet was about 580 ft. per sec. Assuming that the wave speed was 3720 ft. per sec., M had a value of about .156, and according to Fig. 5.2 the angle at the front of the notch should have been 39.8°. In fact it was about 36°. The subsequent course of events was very similar to the sequence shown in Fig. 8.5. The bullet came off the string and a flat developed. When the shock reached the end of the string, the tip came out of the plasticine and started travelling upwards. When the reflected shock reached the kink, an indentation developed and the string began to fold on itself.



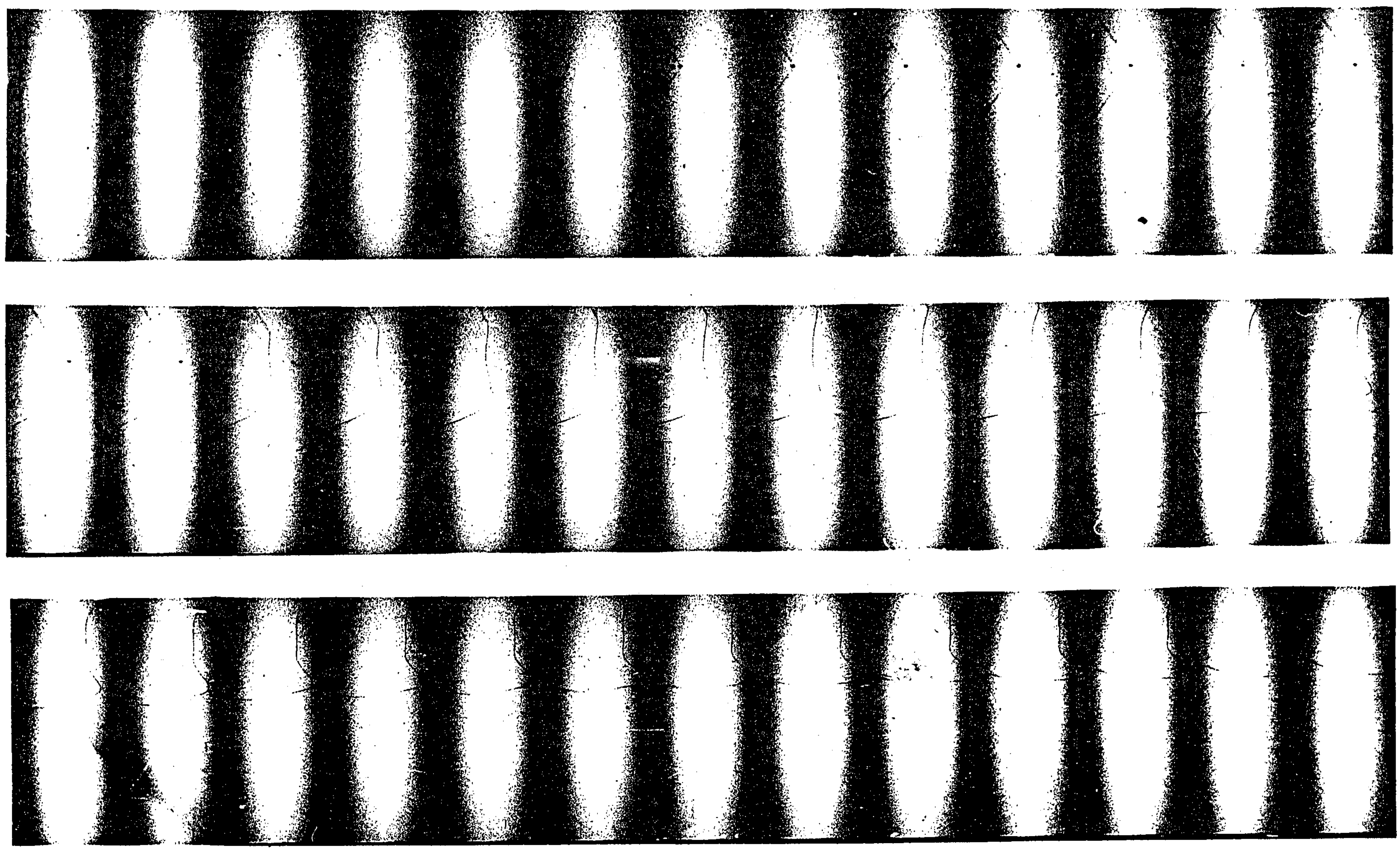
IMPACT OF A BULLET ON A TWISTED NYLON STRING HANGING FROM ITS UPPER END.

13700 FRAMES PER SECOND. IMPACT AT THE CENTRE OF A 36 INCH LENGTH. FIG. No. 8.5



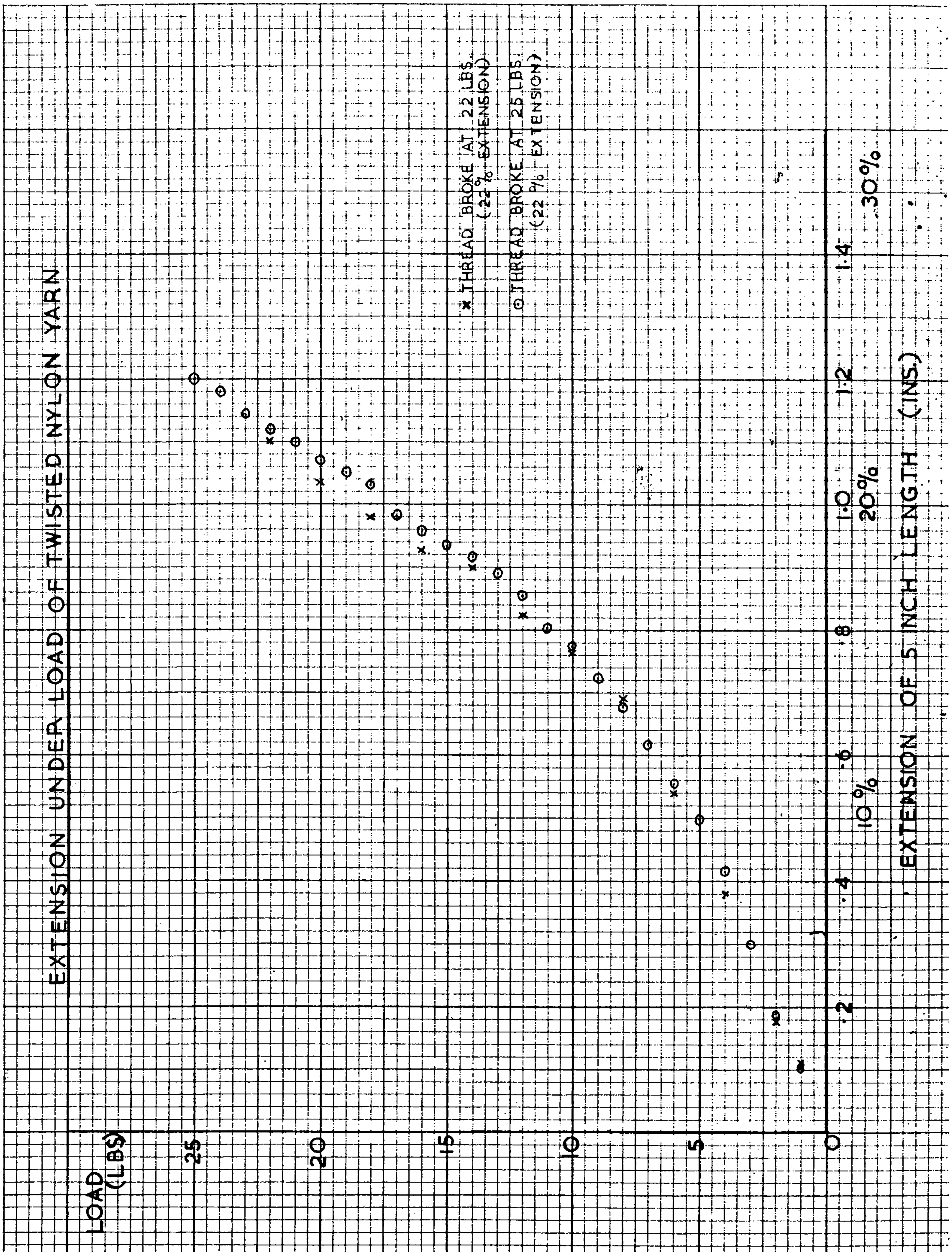
IMPACT OF A BULLET ON A TWISTED NYLON STRING ANCHORED AT BOTH ENDS.

14100 FRAMES PER SECOND. IMPACT AT THE CENTRE OF A 36 INCH LENGTH. FIG. No. 8·6



IMPACT OF A BULLET ON A BRAIDED TERYLENE STRING HANGING FROM ITS UPPER END.

13900 FRAMES PER SECOND. IMPACT AT 17·25 INCHES FROM THE FREE END. FIG. No. 8·7



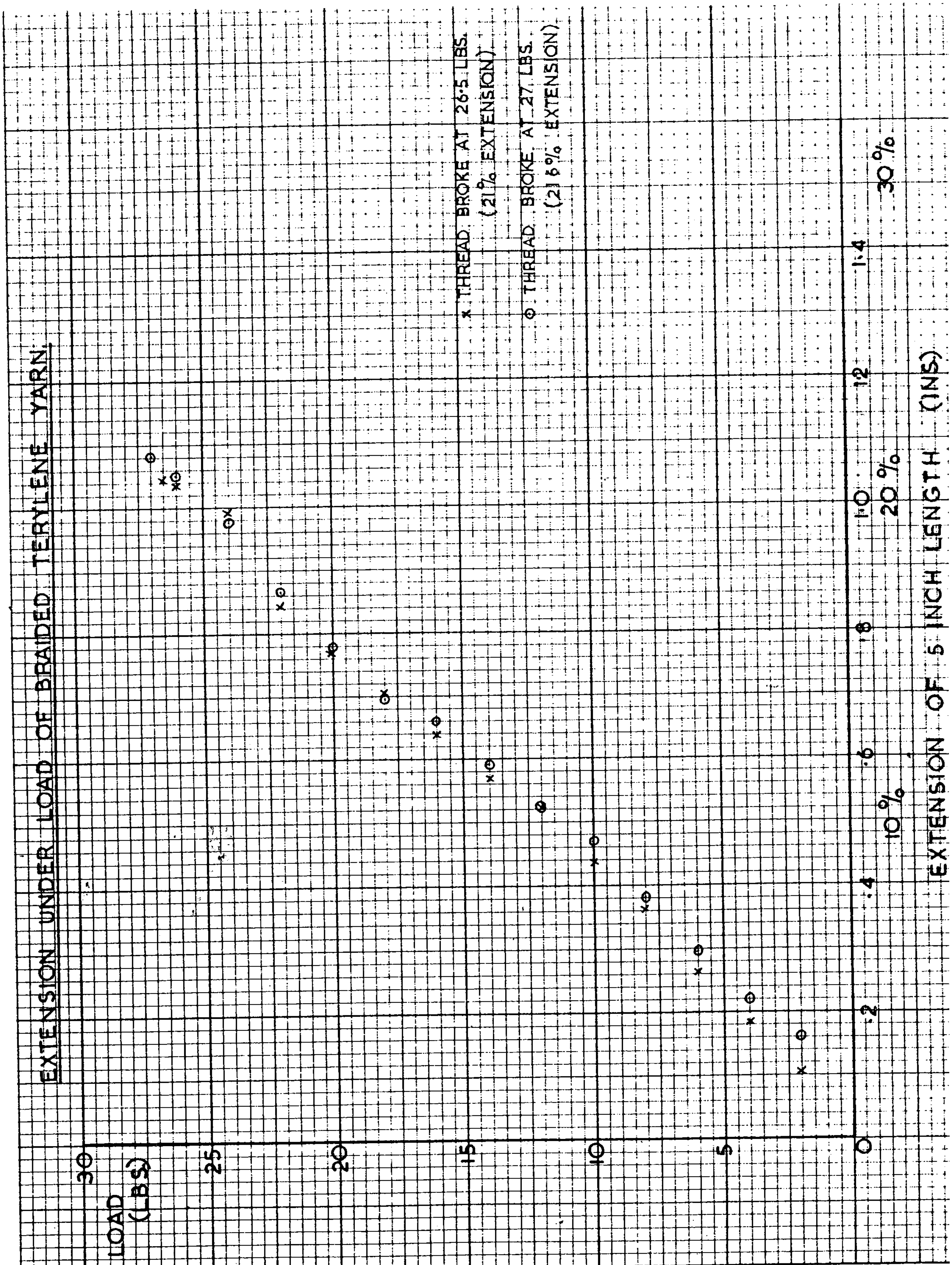
REF. No.

ISSUE No.

DATE :-

FIG. No. 8.8.

AQ. No.



REF. No.

FIG. No. 8.9.

I SUE No.

DATE :-

AQ. No.

9. ACKNOWLEDGEMENTS

The initial impetus for this work came from Mr. D.J. Carey, who, together with Mr. K.T. Wardle, organized the experiments. The photographs were taken by Mr. Derek Lowe. The result of section 5 is in agreement with Ringleb's result in Ref. 1. The typing was by Miss Jacqueline Edwards.

A.J.

LIST OF REFERENCES

REFERENCE No.	AUTHOR	TITLE
1	F.O. Ringleb	Motion and stress and an elastic cable due to impact. J. Appl. Mech. September, 1957.