An Implicit-Explicit Hybrid Scheme for Calculating Complex Unsteady Flows
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Motivation

- The current calculations of comlex unsteady flows are prohibitively expensive for use in real engineering applications to turbomachinery design.
- Example:
  In the Stanford ASCI project, we are calculating the unsteady flow through a complete turbine with 9 blade rows; using a mesh with 94 million cells. Using an implicit scheme, the number of time steps required to reach a stationary periodic state is ~ 2500, and the total estimated computer time is 2.0 million CPU hours. Using 512 processors, the calculation requires approximately 8 months.

Outline

- Fully Implicit Backward Difference Formula (BDF)
- Linearized Scheme
- ADI-BDF Scheme
- Hybrid Scheme
- Results
  - 2D Picthing Airfoil Testcase
- Conclusions

Fully Implicit Backward Difference Formula (BDF)

Discretize

$$\frac{\partial w}{\partial t} + \frac{\partial}{\partial x} f(w) + \frac{\partial}{\partial y} g(w) = 0$$

as

$$\frac{3V}{2\Delta t} w^{n+1} - \frac{2V}{\Delta t} w^{n} + \frac{V}{2\Delta t} w^{n-1} + R(w^{n+1}) = 0 \qquad \text{(BDF)}$$

where $R(w^{n+1})$ is the discrete residual

$$R(w) = D_x f(w) + D_y g(w)$$

evaluated at the end of each time step.

Fully Implicit Backward Difference Formula (BDF) Contd.

- **Advantages:**
  - the scheme is second order accurate in time
  - it is A–stable. (i.e., unconditionally stable for any Δt if the physical equations are stable.)
- **Disadvantages:**
  - coupled nonlinear equations have to be solved at each time step by some approximate method.

Fully Implicit Backward Difference Formula (BDF) Contd.

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  - the scheme is second order accurate in time
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- **Disadvantages:**
  - coupled nonlinear equations

Linearized Scheme

Approximate the flux vectors as

$$f(w^{n+1}) = f(w^n) + A\Delta w^n + \mathcal{O}\,||\Delta w||^2$$
$$g(w^{n+1}) = g(w^n) + B\Delta w^n + \mathcal{O}\,||\Delta w||^2$$

where

$$A = \frac{\partial f(w)}{\partial w}, B = \frac{\partial g(w)}{\partial w}, \Delta w^n = w^{n+1} - w^n$$

also

$$\frac{3V}{2\Delta t}w^{n+1} - \frac{2V}{\Delta t}w^n + \frac{V}{2\Delta t}w^{n-1} = \frac{3V}{2\Delta t}\left(\Delta w^n - \frac{1}{3}\Delta w^{n-1}\right)$$

Hence we obtain the linearized scheme

$$\left\{I + \frac{2\Delta t}{3V}(D_x A + D_y B)\right\}\Delta w^n = \frac{1}{3}\Delta w^{n-1} - \frac{2\Delta t}{3V}R(w^n) \qquad \text{(L)}$$

Linearized Scheme Contd.

- **Advantage:** Since $||\Delta w|| = \mathcal{O}(\Delta t)$ the scheme is still second order accurate.
- **Disadvantage:** The cost of inversion is still too great.

Fully Implicit Dual Time Stepping Scheme

- Solve the full nonlinear BDF by inner iterations which advance in pseudo time τ

$$\frac{\partial w}{\partial \tau} + \left[\frac{3wV - 4w^n V + w^{n-1} V}{2\Delta t} + D_x f(w) + D_y g(w)\right] = 0 \tag{DTS}$$

  On convergence to steady state, $\frac{\partial w}{\partial \tau} = 0$, solution of the BDF is recovered.

- We solve (DTS) using
  - explicit multistage scheme with variable local $\Delta\tau$
  - implicit residual averaging
  - multigrid

Fully Implicit Dual Time Stepping Scheme Contd.

- **Advantage:**
  - if the inner iterations converge fast enough, we solve the fully nonlinear BDF, giving an efficient A-stable scheme which allows very large Δt.
- **Disadvantage:**
  - no way of assessing accuracy unless the inner iterations are fully converged.
  - if a large number of inner iterations are required, the scheme becomes expensive.

Alternating Direction Implicit (ADI) Scheme with the Backward Difference Formula (BDF)

- Replace the left hand side of the linearized BDF by an approximate factorization, giving the modified ADI scheme

$$\left(I + \frac{2\Delta t}{3V} D_x A\right)\left(I + \frac{2\Delta t}{3V} D_y B\right)\Delta w^n = \frac{1}{3}\Delta w^{n-1} - \frac{2\Delta t}{3V} R(w^n) \qquad \text{(ADI)}$$

where $R(w^n) = D_x f(w^n) + D_y g(w^n)$.

Alternating Direction Implicit Scheme with the Backward Difference Formula (ADI-BDF)

- Replace the left hand side of the linearized BDF by an approximate factorization, giving the modified ADI scheme

$$\left(I + \frac{2\Delta t}{3V} D_x A\right)\left(I + \frac{2\Delta t}{3V} D_y B\right)\Delta w^n = \frac{1}{3}\Delta w^{n-1} - \frac{2\Delta t}{3V} R(w^n) \tag{ADI}$$

where $R(w^n) = D_x f(w^n) + D_y g(w^n)$.

ADI-BDF Scheme Contd.

- **Advantages:**
  1. Nominally second order accurate in time with 3 sources of error:
     (a) the discretization error of the BDF
     (b) the linearization error
     (c) the factorization error
  2. can be solved at low computational cost in two steps.
- **Disadvantages:**
  1. The factorization error dominates at large CFL numbers
  2. The scheme isn't amenable to parallel processing: it may lose its stability if applied separately in each of a large numbers of blocks.

Hybrid Scheme

- The proposed hybrid scheme will take an initial ADI step in real time Δt:

$$\left(I + \frac{2\Delta t}{3V} D_x A\right)\left(I + \frac{2\Delta t}{3V} D_y B\right)\Delta w^{(1)} + \frac{2\Delta t}{3V} R(w^n) - \frac{1}{3}\Delta w^{n-1} = 0 \tag{ADI}$$

yielding a nominal second order accuracy without iterations.

- then follow it with the iterative multistage time stepping scheme augmented by multigrid to drive the solution in the steady state limit towards the fully nonlinear BDF.

$$\Delta w^{(k)} - \Delta w^{(k-1)} + \beta_k \left[\frac{3V}{2\Delta t}\left(\Delta w^{(k)} - \frac{1}{3}\Delta w^{n-1}\right) + R(w^{(k-1)})\right] = 0 \tag{IT}$$

Accuracy of the Hybrid Scheme

- The initial ADI step is already formally $\mathcal{O}(\Delta t^2)$, and subtracting (ADI) multiplied by β_k from (IT) with $k = 2$ we get

$$\begin{aligned} \Delta w^{(2)} - \Delta w^{(1)} &= \beta_1 \frac{4\Delta t^2}{9V^2} D_x A D_y B \Delta w^{(1)} + \mathcal{O}\left(||\Delta w||^2\right) \\ &= \mathcal{O}(\Delta t^2) \end{aligned}$$

  and subsequently any $\Delta w^{(k)} - \Delta w^{(k-1)}$ is also $O(\Delta t^2)$.

Hybrid Scheme Contd.

- **The advantages of this scheme are that:**
  1. We should retain formal second order accuracy with any number of iterations, and it should not be necessary to iterate to convergence within each implicit time step, in contrast to existing dual-time stepping schemes which are only second order accurate if the inner iterations are fully converged.
  2. The additional iterations with multigrid should provide information exchange between processors which is needed to stabilize the ADI scheme run separately in each processor.

Validation

The case selected is the NACA 64A010 airfoil in a pitching oscillation representative of wing flutter.

- Mach Number: 0.796
- Pitching Amplitude: $\pm 1.02^{\circ}$
- $k = \frac{\omega Chord}{2 q_{\infty}} = 0.212$
- Re = 12.56 million

We show comparisons of

- Dual time stepping with the BDF
- Pure ADI with the BDF
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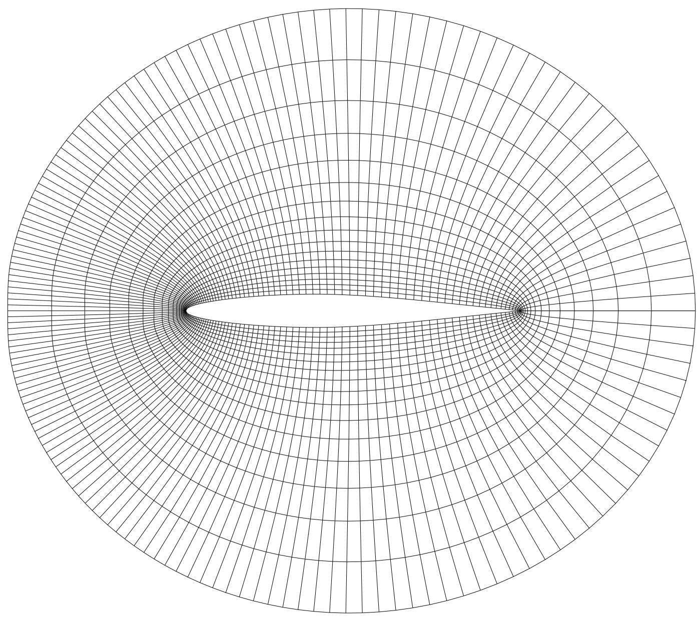
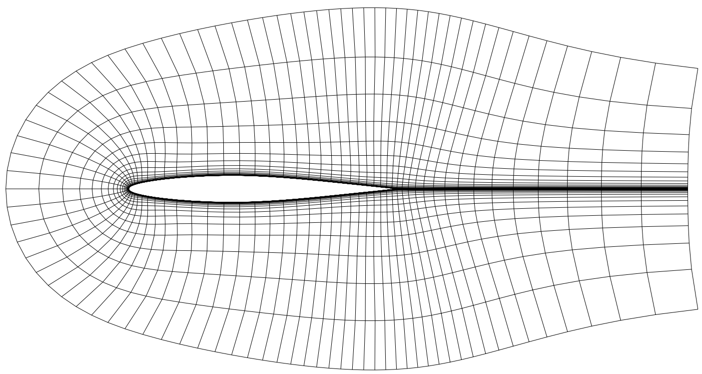
Inviscid 2D Airfoil Testcase



NACA64A010 Airfoil
Mesh Size: 160X32

- Inner part of a grid obtained via conformal mapping. The grid extends to 100 chords.
- Note that the cells at the trailing edge (TE) are very small in comparison with those at mid-chord (MC).

Viscous 2D Airfoil Testcase

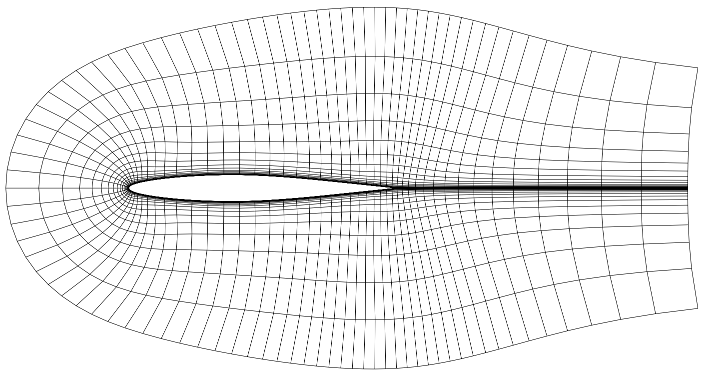


NACA64A010 Airfoil

Mesh Size: 254X64

- Baldwin-Lomax Turbulence Model
- Re = 12.56 Million

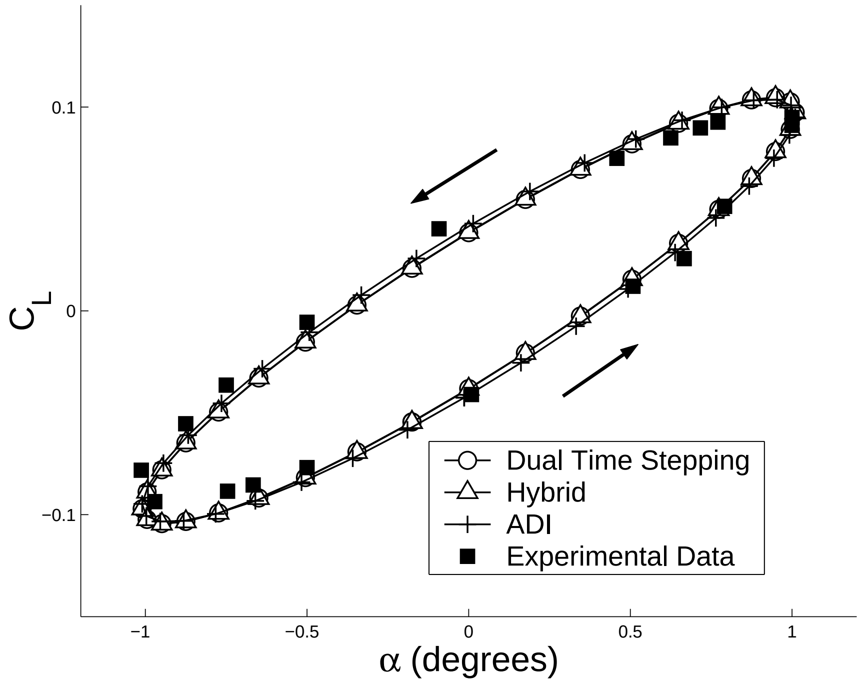
Viscous 2D Airfoil Testcase



NACA64A010 Airfoil
Mesh Size: 254X64

- Baldwin-Lomax Turbulence Model
- Adiabatic Wall BC

Inviscid 2D Airfoil Test Case



Scheme	CFL(TE)	CFL(MC)	Iterations	Time Steps
Dual Time Stepping	1764	74	15	36
Hybrid Scheme	1764	74	15	36
Pure ADI	85	4	1	720

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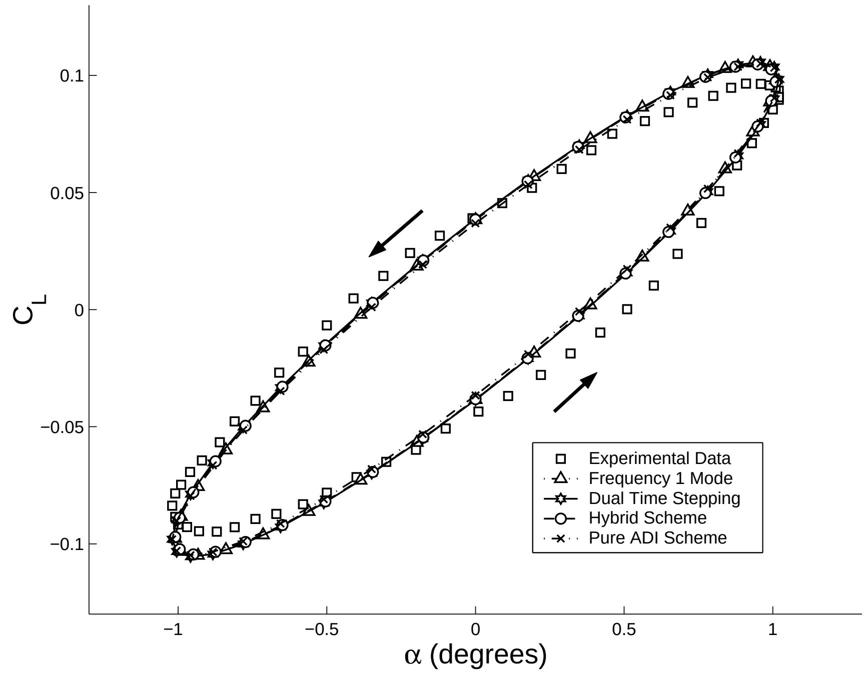

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Student Version of MATLAB

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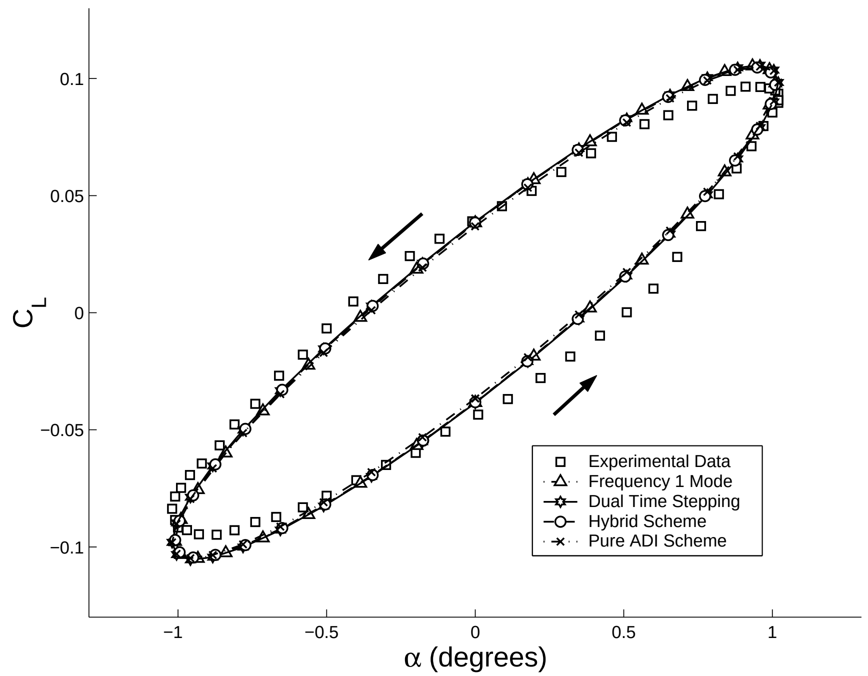
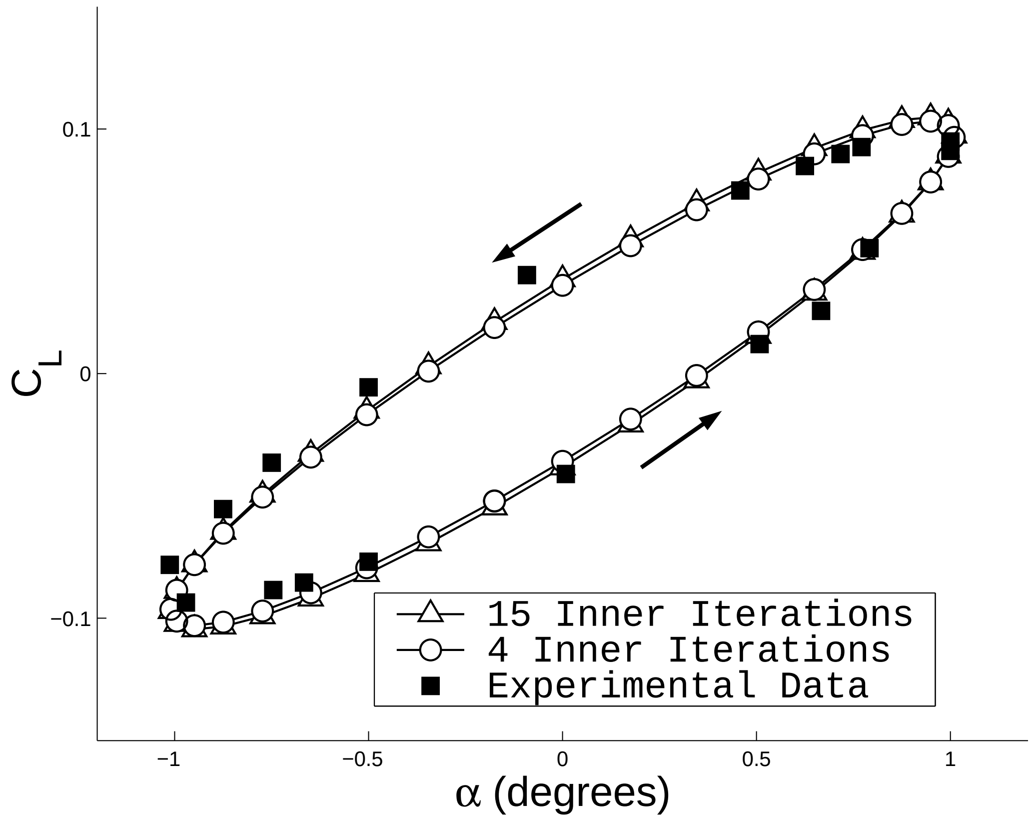


Student Version of MATLAB

Inviscid 2D Airfoil Test Case
Comparison of Hybrid Scheme with Different Number of Iterations



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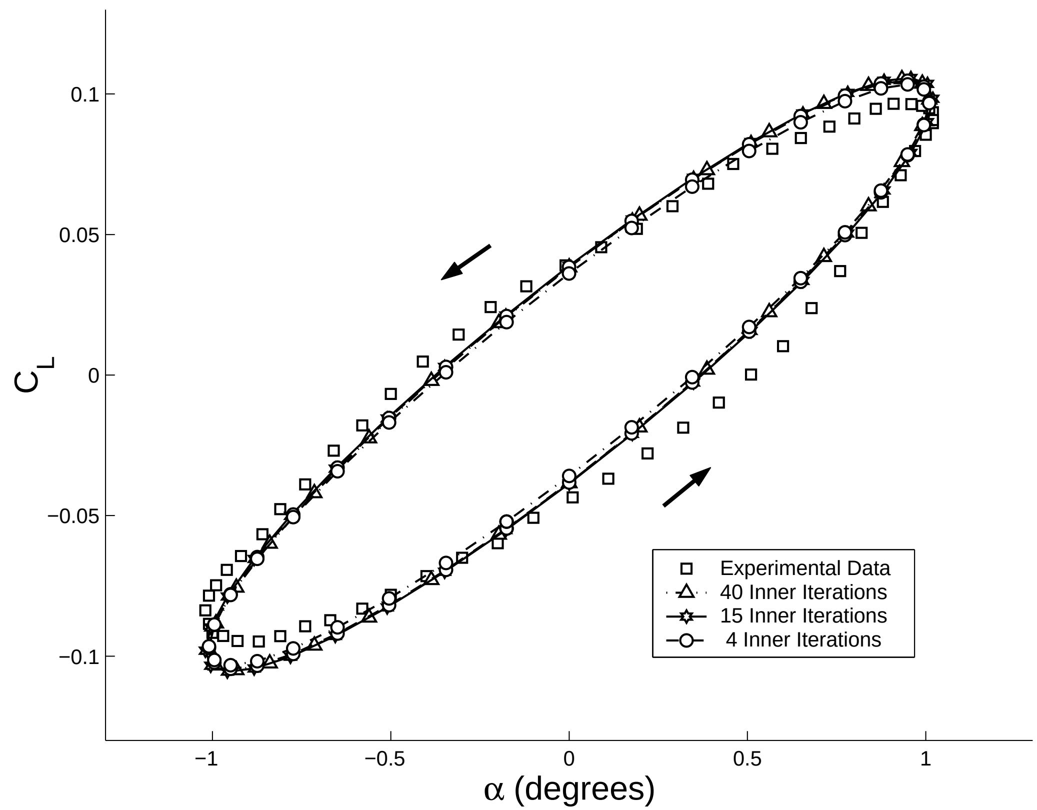

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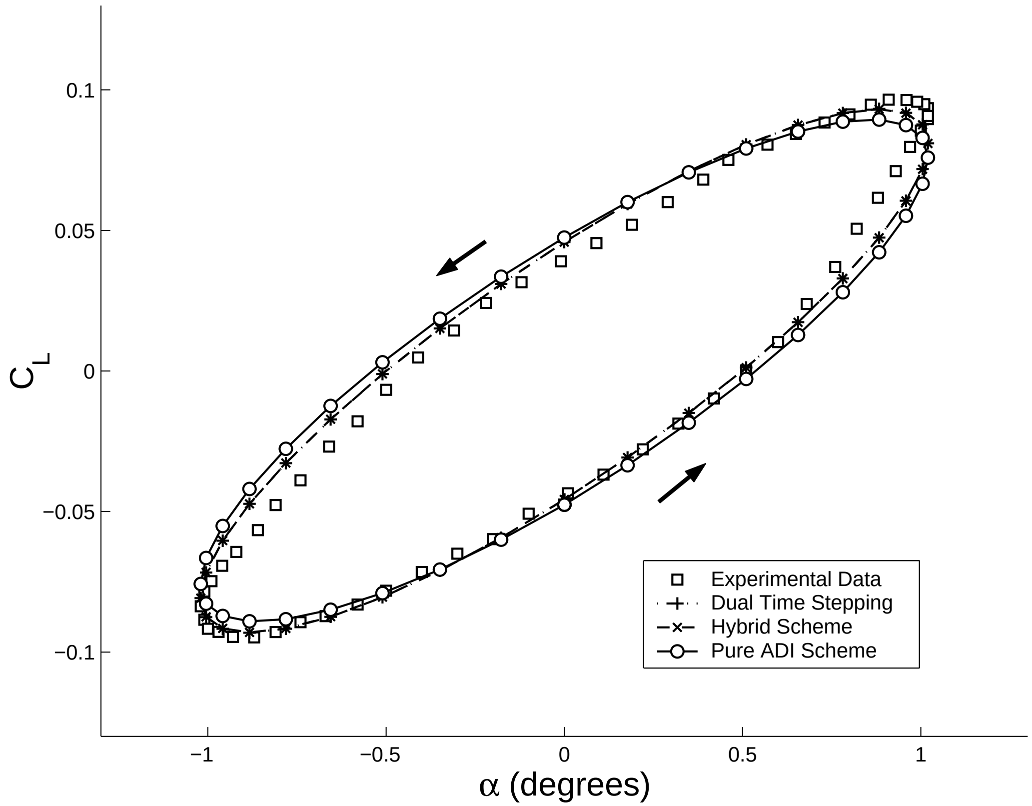

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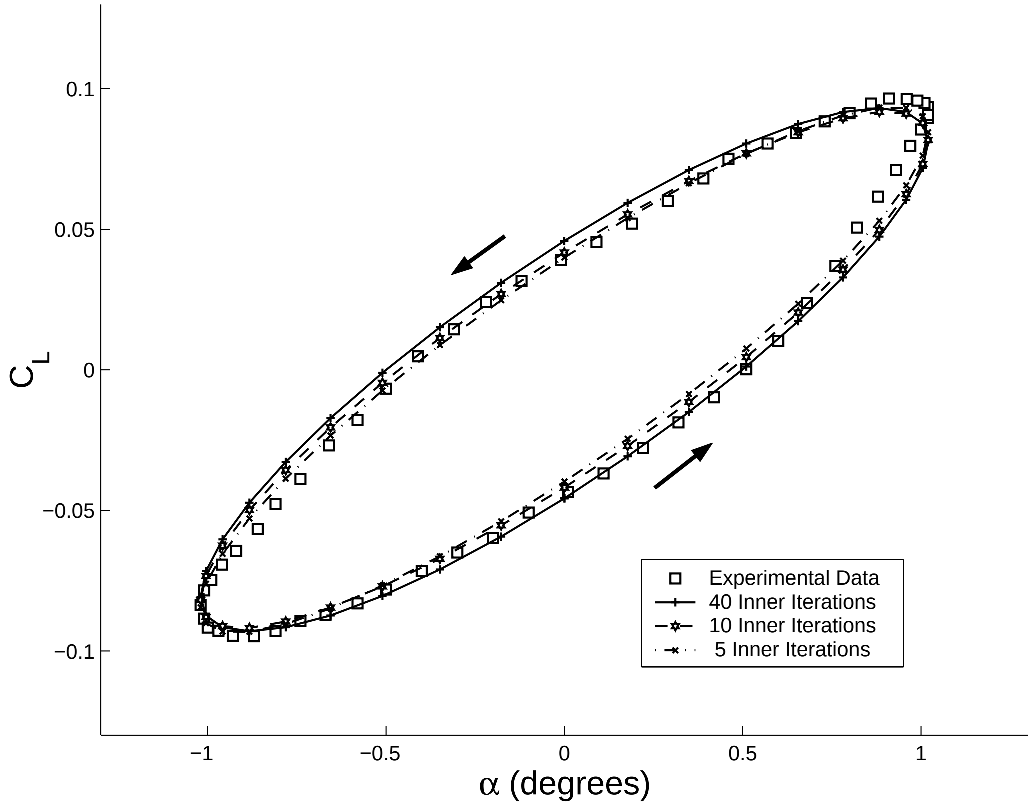
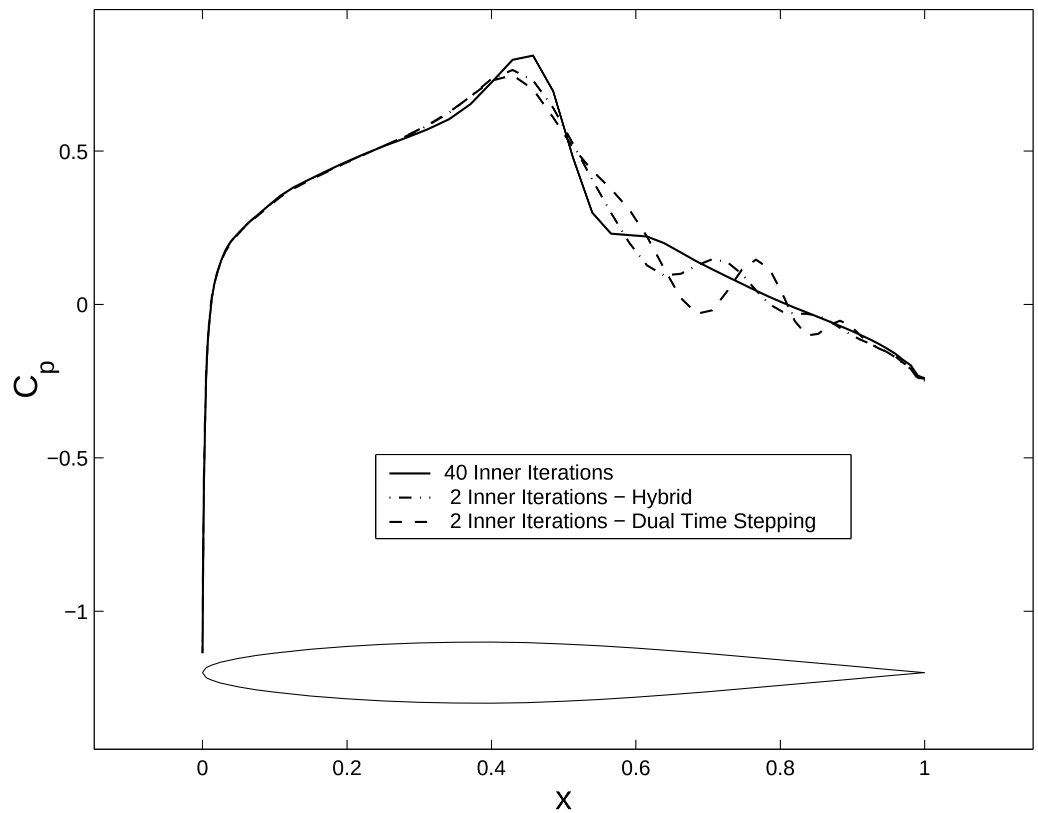


Comparison of ADI-BDF and Dual-Time-Stepping Algorithms



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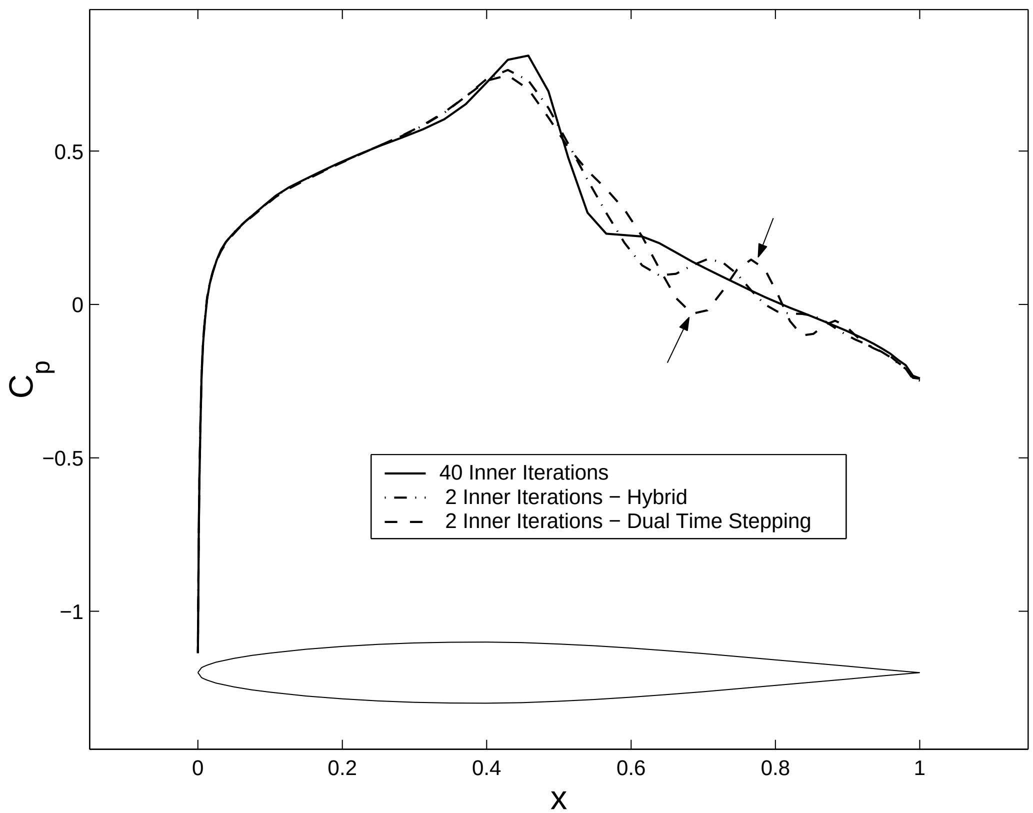
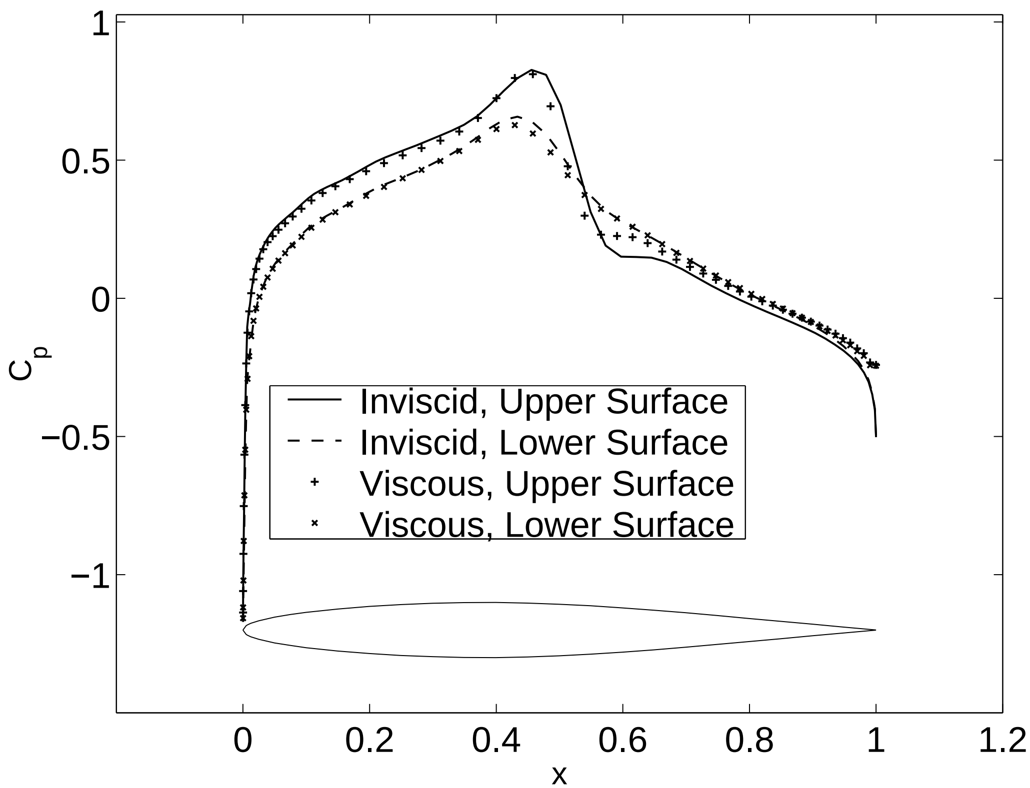


Comparison of Inviscid and Viscous Results



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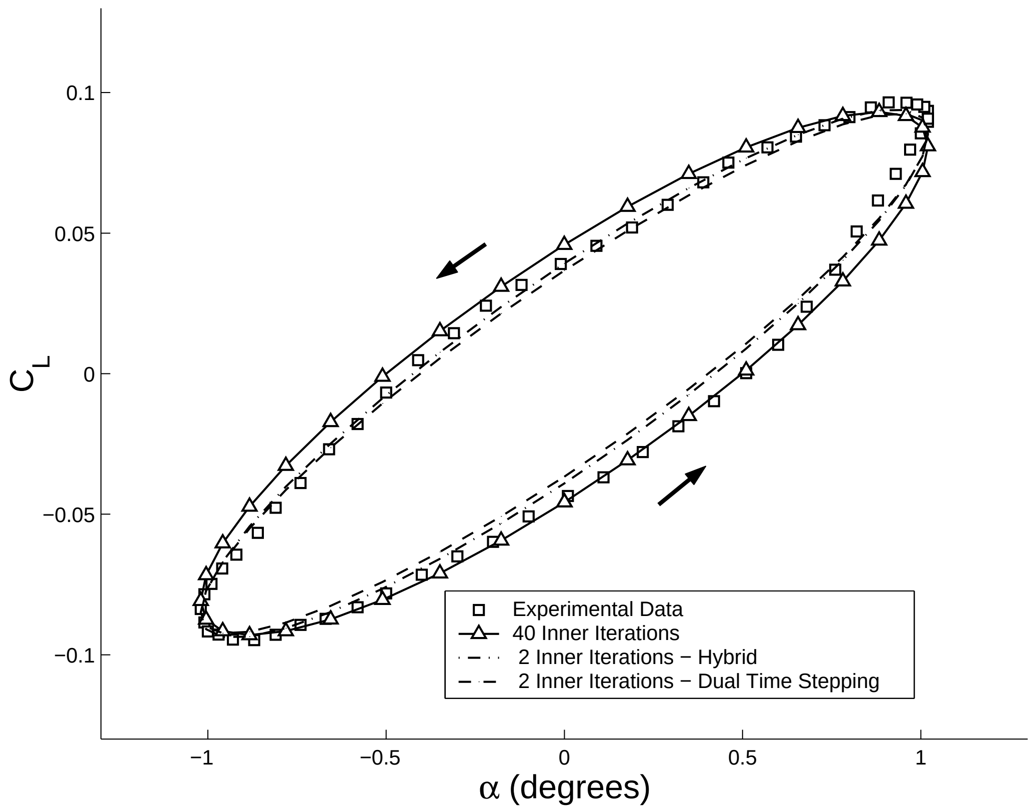

Conclusions

- We can obtain second order accuracy without the need to iterate to convergence, using 1 ADI-BDF step followed by small numbers of dual-time-stepping iterations (of the order of 4 or 5 for inviscid, 10 to 15 for viscous calculations).
- It has been successfully applied to our 2D airfoil testcase
- The scheme may allow a substantial reduction in the cost of unsteady flow simulations in turbomachinery, for which we currently use 30 inner iterations.

Future Work

- Futher refinement of the hybrid scheme
  - diagonally dominant ADI (DDADI)
  - LU-SGS