The Jacobian Matrix for the conservative variables

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The Euler equations for the three dimensional flow of an inviscid gas can be written as

\[
\frac{d}{dt} \int_\Sigma w dV + \int_{d\Sigma} F_i n_i dS = 0
\]

where \(\Sigma\) is the domain, \(d\Sigma\) its boundary, \(n\) the normal to the boundary, and \(dV\) and \(dS\) are the volume and area elements. Let \(x_i, u_i, \rho, p, E\) and \(H\) denote the Cartesian coordinates velocity, density, pressure, energy and enthalpy. In differential form

\[
\frac{\partial w}{\partial t} + \frac{\partial}{\partial x_i} F_i(w) = 0
\]

where

\[
w = \begin{bmatrix} 1 \\ u_1 \\ u_2 \\ u_3 \\ E \end{bmatrix}, \quad F_i = \rho u_i \begin{bmatrix} 1 \\ u_1 \\ u_2 \\ u_3 \\ H \end{bmatrix} + p \begin{bmatrix} 0 \\ \delta_{i1} \\ \delta_{i2} \\ \delta_{i3} \\ 0 \end{bmatrix}
\]

(1)

Also,

\[p = (\gamma - 1)\rho(E - \frac{u^2}{2}), \quad H = E + \frac{p}{\rho} = \frac{c^2}{\gamma - 1} + \frac{u^2}{2}\]

where \(u\) is the speed and \(c\) is the speed of sound

\[u^2 = u_i^2, \quad c^2 = \frac{\gamma p}{\rho}\]

Let \(m_i\) and \(e\) denote the momentum components and total energy \(y\),
\[ m_i = \rho u_i, e = \rho E = \frac{p}{\gamma - 1} + \frac{m_i^2}{2\rho} \]

Then \( w \) and \( F \) can be expressed as

\[
w = \begin{bmatrix} \rho \\ m_1 \\ m_2 \\ m_3 \\ e \end{bmatrix}, \quad F_i = u_i \begin{bmatrix} \rho \\ m_1 \\ m_2 \\ m_3 \\ e \end{bmatrix} + p \begin{bmatrix} 0 \\ \delta_{i1} \\ \delta_{i2} \\ \delta_{i3} \\ u_i \end{bmatrix}
\]

In a finite volume scheme the flux needs to be calculated across the interface between each pair of cells. Denoting the face normal and area by \( n_i \) and \( S \), the flux is \( FS \) where

\[ F = n_i F_i \]

The flux can be expressed in terms of the conservative variables \( w \) as

\[
F = u_n \begin{bmatrix} \rho \\ m_1 \\ m_2 \\ m_3 \\ e \end{bmatrix} + p \begin{bmatrix} 0 \\ n_1 \\ n_2 \\ n_3 \\ u_n \end{bmatrix}
\]

where \( u_n \) is the normal velocity

\[ u_n = n_i u_i = \frac{n_i m_i}{\rho} \]

Also

\[ p = (\gamma - 1)(e - \frac{m_i^2}{\rho}) \]

All the entries in \( F_i \) and \( F \) are homogenous of degree 1 in the conservative variables \( w \). It follows that \( F_i \) and \( F \) satisfy the identities

\[ F_i = A_i w, F = A w \]

where \( A_i \) and \( A \) are the Jacobian matrices

\[ A_i = \frac{\partial F_i}{\partial w}, A = n_i A_i = \frac{\partial F}{\partial w} \]
This is the consequence of the fact that if the function $f$ can be expressed in terms of a vector $w$ as

$$f = \sum w_i^{\alpha_i}, \sum \alpha_i = \alpha,$$

then

$$\frac{\partial f}{\partial w_l} w_l = \sum \alpha_i w_i^{\alpha_i-1} w_l = \sum \alpha_i f = \alpha f$$  \hspace{1cm} (4)

In order to evaluate $A$ note that

$$\frac{\partial u_n}{\partial w} = \frac{1}{\rho} [-u_n, n_1, n_2, n_3, 0]^T$$

and

$$\frac{\partial p}{\partial w} = (\gamma - 1) \left[ \frac{u^2}{2}, -u_1, -u_2, -u_3, 1 \right]^T$$

Then

$$\frac{\partial F}{\partial w} = \frac{\partial}{\partial w} (u_n w) + \frac{\partial p}{\partial w} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \\ u_n \end{bmatrix} + p \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{\partial u_n}{\partial w} \end{bmatrix}$$  \hspace{1cm} (5)

Then the Jacobian matrix can be assembled as

$$A = u_n I + a_1 b_1^T + a_2 b_2^T$$

where

$$a_1^T = [1, u_1, u_2, u_3, H]$$

$$a_2^T = [0, n_1, n_2, n_3, v_n]$$

$$b_1^T = \frac{1}{\rho} \left[ \frac{\partial u_n}{\partial w} \right]^T = [-u_n, n_1, n_2, n_3, 0]$$

$$b_2^T = \left[ \frac{\partial p}{\partial w} \right]^T = (\gamma - 1) \left[ \frac{u^2}{2}, -u_1, -u_2, -u_3, 1 \right]$$  \hspace{1cm} (6)
It is easily verified that

\[ F = Aw \]

because \( u_n \) is homogeneous of degree 0 with the consequence that \( \frac{\partial u_n}{\partial w} \) is orthogonal to \( w \).

\[ \left[ \frac{\partial u_n}{\partial w} \right]^T w = 0 \]

while \( p \) is homogenous of degree 1 so that

\[ \left[ \frac{\partial p}{\partial w} \right]^T w = p \]

Thus

\[ Aw = u_n w + \begin{bmatrix} n_1 \\ n_2 \\ n_3 \\ u_n \end{bmatrix} \left[ \frac{\partial p}{\partial w} \right]^T w = F \]

The special structure of the Jacobian matrix enables the direct identification of its eigenvalues and eigenvectors. Any vector in the 3 dimensional subspace orthogonal to the vectors \( b_1 \) and \( b_2 \) is an eigenvector corresponding to the eigenvalue \( u_n \). Thus \( u_n \) is a triple eigenvalue. It is easy to verify that the vectors

\[ v_0 = \begin{bmatrix} 1 \\ u_1 \\ u_2 \\ u^2 \end{bmatrix}, \quad v_1 = \begin{bmatrix} 0 \\ 0 \\ n_3 \\ -n_2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ -n_3 \\ 0 \\ n_1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 0 \\ n_2 \\ -n_1 \\ 0 \end{bmatrix} \]

are orthogonal to both \( b_1 \) and \( b_2 \). However \( v_1, v_2 \) and \( v_3 \) are not independent since

\[ n_k v_k = 0 \]

Three independent eigenvectors can be obtained as

\[ r_1 = n_1 v_0 + cv_1, r_2 = n_2 v_0 + cv_2, r_3 = n_3 v_0 + cv_3 \]
To verify this note that the middle three elements of \( v_k, k = 1, 2, 3 \) are equal to \( \vec{i}_k \times \vec{n} \), where \( \vec{i}_k \) is the unit vector in the \( k^{th} \) coordinate direction. Also the last element of \( v_k \) is equal to \( \vec{i}_k \cdot (\vec{n}_k \times \vec{u}) \). If the vectors \( r_k \) are not independent they must satisfy the relation of the form

\[
\alpha_k r_k = 0
\]

for some non-zero vector \( \vec{\alpha} \). The first element is

\[
\alpha_k n_k = \vec{\alpha} \cdot \vec{n}
\]

And for the next three elements to be zero

\[
\alpha_k (\vec{i}_k \times \vec{n}) = \vec{\alpha} \times \vec{n} = 0
\]

which is only possible if \( \vec{\alpha} \) is parallel to \( \vec{n} \), so that \( \vec{\alpha} \cdot \vec{n} \neq 0 \).

To identify the remaining eigenvectors note that

\[
\begin{align*}
\langle 1 \rangle & = b_1^T a_1 = 0, b_1^T a_2 = 1, \\
\langle 2 \rangle & = b_2^T a_1 = c, b_2^T a_2 = 0.
\end{align*}
\]

Now consider a vector of the form

\[
r = a_1 + \alpha a_2
\]

Then

\[
Ar = u_n (a_1 + \alpha a_2) + \alpha a_1 + c^2 a_2 = \lambda r
\]

if

\[
u_n + \alpha = \lambda, \alpha u_n + c^2 = \lambda
\]

or

\[
\alpha^2 = c^2
\]

Thus the vectors

\[
r_4 = a_1 + \alpha a_2, r_5 = a_1 - \alpha a_2
\]

are the eigenvectors corresponding to the eigenvalues \( u_n + c, u_n - c \).
The left eigenvectors are the eigenvectors of

\[ A^T = u_n I + b_1 a_1^T + b_2 a_2^T \]

Vectors in the three dimensional subspace orthogonal to \( a_1 \) and \( a_2 \) are eigenvectors corresponding to the eigenvalue \( u_n \). The vectors

\[
\begin{align*}
p_0^T &= \frac{\gamma - 1}{c^2} [H - u^2, u_1, u_2, u_3, -1] \\
p_1^T &= [u_2 n_3 - u_3 n_2, 0, -n_3, n_2, 0] \\
p_2^T &= [u_3 n_1 - u_1 n_3, n_3, 0, -n_1, 0] \\
p_3^T &= [u_1 n_2 - u_2 n_1, -n_2, n_1, 0, 0]
\end{align*}
\]  

are orthogonal to both \( a_1 \) and \( a_2 \). They also satisfy the relations

\[ p_0^T v_0 = 1, p_0^T v_k = 0, \]

\[ p_j^T v_0 = 0, p_j^T v_k = n_j n_k - \delta_{jk}, \]

for \( j = 1 \) to \( 3 \) and \( k = 1 \) to \( 3 \).

Consequently it can be verified that the vectors

\[ l_1 = n_1 p_0 - \frac{1}{c} p_1, \quad l_2 = n_2 p_0 - \frac{1}{c} p_2, \quad l_3 = n_3 p_0 - \frac{1}{c} p_3 \]

are left eigenvectors which are orthonormal to the right eigenvectors \( r_1, r_2 \) and \( r_3 \) and also orthogonal to \( r_4 \) and \( r_5 \) which are linear combinations of \( a_1 \) and \( a_2 \). Finally the remaining two eigenvectors corresponding to the eigenvalues \( u_n + c \) and \( u_n - c \) can be expressed as linear combinations of \( b_1 \) and \( b_2 \). When scaled so that they are orthonormal to \( r_4 \) and \( r_5 \), they assume the form

\[ l_4 = \frac{1}{2c^2} (b_2 + cb_1), \quad l_5 = \frac{1}{2c^2} (b_2 - cb_1) \]