The Jacobian Matrix for the conservative variables

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August 8, 2006

The Euler equations for the three dimensional flow of an inviscid gas can be written as

$$\frac{d}{dt}\int_{\Sigma} w dV + \int_{d\Sigma} F_i n_i dS = 0$$

where Σ is the domain, $d\Sigma$ its boundary, $\bar{n}$ the normal to the boundary, and dV and dS are the volume and area elements. Let $x_i, u_i, \rho, p, EandH$ denote the Cartesian coordinates velocity, density, pressure, energy and enthalpy. In differential form

$$\frac{\partial w}{\partial t} + \frac{\partial}{\partial x_i} F_i(w) = 0$$

where

$$w = \rho \begin{bmatrix} 1 \\ u_1 \\ u_2 \\ u_3 \\ E \end{bmatrix}, \quad F_i = \rho u_i \begin{bmatrix} 1 \\ u_1 \\ u_2 \\ u_3 \\ H \end{bmatrix} + p \begin{bmatrix} 0 \\ \delta_{i1} \\ \delta_{i2} \\ \delta_{i3} \\ 0 \end{bmatrix} \tag{1}$$

Also,

$$p = (\gamma - 1)\rho(E - \frac{u^2}{2}), \quad H = E + \frac{p}{\rho} = \frac{c^2}{\gamma - 1} + \frac{u^2}{2}$$

where u is the speed and c is the speed of sound

$$u^2 = u_i^2, c^2 = \frac{\gamma p}{\rho}$$

Let m_i and e denote the momentum components and total energy y,

$$m_i = \rho u_i, e = \rho E = \frac{p}{\gamma - 1} + \frac{m_i^2}{2\rho}$$

Then w and F can be expressed as

$$w = \begin{bmatrix} \rho \\ m_1 \\ m_2 \\ m_3 \\ e \end{bmatrix}, \quad F_i = u_i \begin{bmatrix} \rho \\ m_1 \\ m_2 \\ m_3 \\ e \end{bmatrix} + p \begin{bmatrix} 0 \\ \delta_{i1} \\ \delta_{i2} \\ \delta_{i3} \\ u_i \end{bmatrix} \tag{2}$$

In a finite volume scheme the flux needs to be calculated across the interface between each pair of cells. Denoting the face normal and area by n_i and S, the flux is FS where

$$F = n_i F_i$$

The flux can be expressed in terms of the conservative variables w as

$$F = u_n \begin{bmatrix} \rho \\ m_1 \\ m_2 \\ m_3 \\ e \end{bmatrix} + p \begin{bmatrix} 0 \\ n_1 \\ n_2 \\ n_3 \\ u_n \end{bmatrix} \tag{3}$$

where u_n is the normal velocity

$$u_n = n_i u_i = \frac{n_i m_i}{\rho}$$

Also

$$p = (\gamma - 1)(e - \frac{m_i^2}{\rho})$$

All the entries in F_i and F are homogenous of degree 1 in the conservative variables w. It follows that F_i and F satisfy the identities

$$F_i = A_i w, F = Aw$$

where A_i and A are the Jacobian matrices

$$A_i = \frac{\partial F_i}{\partial w}, A = n_i A_i = \frac{\partial F}{\partial w}$$

This is the consequence of the fact that if the function f can be expressed in terms of a vector w as

$$f = \sum w_i^{\alpha_i}, \sum \alpha_i = \alpha,$$

then

$$\begin{aligned} \frac{\partial f}{\partial w_i} w_i &= \sum \alpha_i w_i^{\alpha_i - 1} w_i \\ &= \sum \alpha_i f \\ &= \alpha f \end{aligned} \tag{4}$$

In order to evaluate A note that

$$\frac{\partial u_n}{\partial w} = \frac{1}{\rho} [-u_n, n_1, n_2, n_3, 0]^T$$

and

$$\frac{\partial p}{\partial w} = (\gamma - 1) \left[\frac{u^2}{2}, -u_1, -u_2, -u_3, 1 \right]^T$$

Then

$$\frac{\partial F}{\partial w} = \frac{\partial}{\partial w}(u_n w) + \frac{\partial p}{\partial w} \begin{bmatrix} 0 \\ n_1 \\ n_2 \\ n_3 \\ u_n \end{bmatrix} + p \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \frac{\partial u_n}{\partial w} \end{bmatrix} \tag{5}$$

Then the Jacobian matrix can be assembled as

$$A = u_n I + a_1 b_1^T + a_2 b_2^T$$

where

$$\begin{aligned} a_1^T &= [1, u_1, u_2, u_3, H] \\ a_2^T &= [0, n_1, n_2, n_3, u_n] \\ b_1^T &= \frac{1}{\rho} \left[\frac{\partial u_n}{\partial w} \right]^T = [-u_n, n_1, n_2, n_3, 0] \\ b_2^T &= \left[\frac{\partial p}{\partial w} \right]^T = (\gamma - 1)[\frac{u^2}{2}, -u_1, -u_2, -u_3, 1] \end{aligned} \tag{6}$$

It is easily verified that

$$F = Aw$$

because u_n is homogeneous of degree 0 with the consequence that $\frac{\partial u_n}{\partial w}$ is orthogonal to w.

$$\left[\frac{\partial u_n}{\partial w}\right]^T w = 0$$

while p is homogenous of degree 1 so that

$$\left[\frac{\partial p}{\partial w}\right]^T w = p$$

Thus

$$Aw = u_n w + \begin{bmatrix} 1 \\ n_1 \\ n_2 \\ n_3 \\ u_n \end{bmatrix} \left[\frac{\partial p}{\partial w}\right]^T w = F$$

The special structure of the JAcobian matrix enables the direct identification of its eigenvalues and eigenvectors. Any vector in the 3 dimensional subspace orthogonal to the vectors b_1 and b_2 is an eigenvector corresponding to the eigenvalue u_n. Thus u_n is a triple eigenvalue. It is easy to verify that the vectors

$$v_0 = \begin{bmatrix} 1 \\ u_1 \\ u_2 \\ u_3 \\ \frac{u^2}{2} \end{bmatrix}, \quad v_1 = \begin{bmatrix} 0 \\ 0 \\ n_3 \\ -n_2 \\ u_2 n_3 - u_3 n_2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ -n_3 \\ 0 \\ n_1 \\ u_3 n_1 - u_1 n_3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 0 \\ n_2 \\ -n_1 \\ 0 \\ u_1 n_2 - u_2 n_1 \end{bmatrix}, \tag{7}$$

are orthogonal to both b_1 and b_2. However v_1, v_2 and v_3 are not independent since

$$n_k v_k = 0$$

Three independent eigenvectors can be obtained as

$$r_1 = n_1 v_0 + c v_1, r_2 = n_2 v_0 + c v_2, r_3 = n_3 v_0 + c v_3$$

To verify this note that the middle three elements of $v_k, k = 1, 2, 3$ are equal to $\bar{i}_k \times \bar{n}$, where $\bar{i}_k$ is the unit vector in the k^{th} coordinate direction. Also the last element of v_k is equal to $\bar{i}_k \cdot (\bar{n}_k \times \bar{u})$. If the vectors r_k are not independent they must satisfy the relation of the form

$$\alpha_k r_k = 0$$

for some non zero vector $\bar{\alpha}$. The first element is

$$\alpha_k n_k = \bar{\alpha} \cdot \bar{n}$$

And for the next three elements to be zero

$$\alpha_k \left(\bar{i}_k \times \bar{n}\right) = \bar{\alpha} \times \bar{n} = 0$$

which is only possible if $\bar{\alpha}$ is parallel to $\bar{n}$, so that $\bar{\alpha} \cdot \bar{n} \neq 0$.
To identify the remaining eigenvectors note that

$$b_1^T a_1 = 0, b_1^T a_2 = 1,$$

$$b_2^T a_1 = c^2, b_2^T a_2 = 0.$$

Now consider a vector of the form

$$r = a_1 + \alpha a_2$$

Then

$$\begin{aligned} Ar &= u_n(a_1 + \alpha a_2) + \alpha a_1 + c^2 a_2 \\ &= \lambda(a_1 + \alpha a_2) = \lambda r \end{aligned} \tag{8}$$

if

$$u_n + \alpha = \lambda, \alpha u_n + c^2 = \lambda$$

or

$$\alpha^2 = c^2$$

Thus the vectors

$$r_4 = a_1 + c a_2, r_5 = a_1 - c a_2$$

are the eigenvectors corresponding to the eigenvalues $u_n + c, u_n - c$.

The left eiegenvectors are the eigenvectors of

$$A^T = u_n I + b_1 a_1^T + b_2 a_2^T$$

Vectors in the three dimensional subspace orthogonal to a_1 and a_2 are eigenvectors corresponding to the eigenvalue u_n. The vectors

$$\begin{aligned} p_0^T &= \frac{\gamma-1}{c^2}[H-u^2, u_1, u_2, u_3, -1] && (9) \\ p_1^T &= [u_2 n_3 - u_3 n_2, 0, -n_3, n_2, 0] \\ p_2^T &= [u_3 n_1 - u_1 n_3, n_3, 0, -n_1, 0] \\ p_3^T &= [u_1 n_2 - u_2 n_1, -n_2, n_1, 0, 0] \\ & && (10) \end{aligned}$$

are orthogonal to both a_1 and a_2. They also satisfy the relations

$$p_0^T v_0 = 1, p_0^T v_k = 0,$$

$$p_j^T v_0 = 0, p_j^T v_k = n_j n_k - \delta_{jk},$$

for $j = 1$ to 3 and $k = 1$ to 3.
Consequently it can be verified that the vectors

$$l_1 = n_1 p_0 - \frac{1}{c} p_1, \;\; l_2 = n_2 p_0 - \frac{1}{c} p_2, \;\; l_3 = n_3 p_0 - \frac{1}{c} p_3$$

are left eigenvectors which are orthonormal to the right eigenvectors r_1, r_2 and r_3 and also orthogonal to r_4 and r_5 which are linear combinations of a_1 and a_2. Finally the remaining two eigenvectors corresponding to the eigenvalues $u_n + c$ and $u_n - c$ can be expressed as linear combinations of b_1 and b_2. When scaled so that they are orthonormal to r_4 and r_5, they assume the form

$$l_4 = \frac{1}{2c^2}(b_2 + cb_1), \;\; l_5 = \frac{1}{2c^2}(b_2 - cb_1)$$