OPTIMUM TRANSONIC WING DESIGN USING CONTROL THEORY

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1. Introduction

While aerodynamic prediction methods based CFD are now well established, and quite accurate and robust, the ultimate need in the design process is to find the optimum shape which maximizes the aerodynamic performance. One way to approach this objective is to view it as a control problem, in which the wing is treated as a device which controls the flow to produce lift with minimum drag, while meeting other requirements such as low structure weight, sufficient fuel volume, and stability and control constrains. Here we apply the theory of optimal control of systems governed by partial differential equations with boundary control, in this case through changing the shape of the boundary. Using this theory, we can find the Frechet derivative (infinitely dimensional gradient) of the cost function with respect to the shape by solving an adjoint problem, and then we can make an improvement by making a modification in a descent direction. For example, the cost function might be the drag coefficient at a fixed lift, or the lift to drag ratio. During the last decade, this method has been intensively developed, and has proved to be very effective for improving wing section shapes for fixed wing planform [3, 4, 8–11, 13, 14].

In the present work a continuous adjoint formulation has been used to derive the adjoint system of equations, in which the adjoint equations are derived directly from the governing equations and then discretized. This approach has the advantage over the discrete adjoint formulation in that the resulting adjoint equations are independent of the form of discretized flow equations. The adjoint system of equations has a similar form to the governing equations of the flow, and hence the numerical methods developed for the flow equations [1, 2, 5] can be reused for the adjoint equations. Moreover, the gradient can be derived directly from the adjoint solution and the surface motion, independent of the mesh modification.

In order to accelerate the convergence of the descent process the gradient is then smoothed implicitly via a second order differential equation. This is equivalent to redefining the gradient in a Sobolve space. The resulting procedure is very efficient, often yielding the optimum in 10-20 design cycles.

2. The general formulation of the Adjoint Approach to Optimal Design

The aerodynamic properties which define the cost function are functions of the flow-field variables, w, and the physical location of the boundary, which may be represented by the function, $\mathcal{F}$, say. Then

$$I = I(w, \mathcal{F}),$$

and a change in $\mathcal{F}$ results in a change

$$\delta I = \frac{\partial I^T}{\partial w} \delta w + \frac{\partial I^T}{\partial \mathcal{F}} \delta \mathcal{F}, \tag{1}$$

in the cost function. Using control theory, the governing equations of the flow field are introduced as a constraint in such a way that the final expression for the gradient does not require re-evaluation of the flow-field. In order to achieve this, δw must be eliminated from equation (1). Suppose that the governing equation R which expresses the dependence of w and $\mathcal{F}$ within the flow field domain D can be written as

$$R(w, \mathcal{F}) = 0 \tag{2}$$

Then δw is determined from the equation

$$\delta R = \left[\frac{\partial R}{\partial w}\right] \delta w + \left[\frac{\partial R}{\partial \mathcal{F}}\right] \delta \mathcal{F} = 0 \tag{3}$$

Next, introducing a Lagrange Multiplier ψ, we have

$$\delta I = \frac{\partial I^T}{\partial w} \delta w + \frac{\partial I^T}{\partial \mathcal{F}} \delta \mathcal{F} - \psi^T \left(\left[\frac{\partial R}{\partial w}\right] \delta w + \left[\frac{\partial R}{\partial \mathcal{F}}\right] \delta \mathcal{F} \right)$$

which can be rearranged as

$$\delta I = \left(\frac{\partial I^T}{\partial w} - \psi^T \left[\frac{\partial R}{\partial w}\right] \right) \delta w + \left(\frac{\partial I^T}{\partial \mathcal{F}} - \psi^T \left[\frac{\partial R}{\partial \mathcal{F}}\right] \right) \delta \mathcal{F}$$

Choosing ψ to satisfy the adjoint equation

$$\left[\frac{\partial R}{\partial w}\right]^T \psi = \frac{\partial I}{\partial w} \tag{4}$$

the first term is eliminated and we find that

$$\delta I = \mathcal{G}\delta\mathcal{F} \tag{5}$$

where

$$\mathcal{G} = \frac{\partial I^T}{\partial \mathcal{F}} - \psi^T \left[\frac{\partial R}{\partial \mathcal{F}}\right] \tag{6}$$

In this way the gradient with respect to the shape is obtained at the cost of one flow and one adjoint solution.

After taking a step in the negative gradient direction, the gradient is recalculated and the process is repeated to follow the path of steepest descent until a minimum is reached. In order to avoid violating constraints, such as the minimum acceptable wing thickness, the gradient can be projected into an allowable subspace within which the constraints are satisfied. In this way one can devise procedures which must necessarily converge at least to a local minimum and which can be accelerated by the use of more sophisticated descent methods such as conjugate gradient or quasi-Newton algorithms. There is a possibility of more than one local minimum, but in any case this method will lead to an improvement over the original design.

3. Adjoint and Gradient formulations for the equations of transonic flow

In applying the adjoint method one may apply the above procedure directly to the partial differential equations to derive a continuous adjoint equation, which must then be discretized to obtain a numerical solution. Alternatively one may derive a discrete adjoint equation directly after first discretizing the flow equations. In this work the first procedure has been adopted because it allows more flexibility in the formulation of the gradient.

The procedure is illustrated here for the Euler equations. These are represented in transformed coordinates ξ_i on a fixed computational domain.
Let

$$S = JK^{-1}$$

where

$$K_{ij} = \frac{\partial x_i}{\partial \xi_j}, J = det(K)$$

Then the transformed equations are

$$\frac{\partial F_i}{\partial \xi_i} = \frac{\partial (S_{ij} f_j)}{\partial \xi_i} = 0$$

Consider the case of an inverse problem where one wishes to find the shape which brings the pressure as close as possible to the specified target pressure, p_t. Hence we try to minimize the cost function

$$\mathcal{I} = \frac{1}{2}\int_{\mathcal{B}} (p - p_t)^2 dS$$

over the design surface $\mathcal{B}$, which for convenience is assumed to be the surface $\psi_2 = 0$. Now a shape modification induces a change δp in the pressure and consequently

$$\delta\mathcal{I} = \int_{\mathcal{B}} (p - p_t)\delta p dS + \frac{1}{2}\int_{\mathcal{B}} (p - p_t)^2 d\delta S$$

Also the change in the solution is given by

$$\frac{\partial}{\partial \psi_i}\left(\delta F_i(w)\right) = 0$$

Here the flux changes are

$$\delta F_i = \delta S_{ij} f_j + C_i \delta w$$

where

$$C_i = S_{ij}\frac{\partial f_j}{\partial w}$$

Consequently one can augment the cost variation by

$$\int_{\mathcal{D}} \psi^T \frac{\partial \delta F_i}{\partial \xi_i} d\xi = \int_{\mathcal{B}} n_i \psi^T \delta F_i d\xi_{\mathcal{B}} - \int_{\mathcal{D}} \frac{\partial \psi^T}{\partial \xi}\delta F_i d\xi$$

Now choose ψ to satisfy the adjoint equation

$$C_i^T \frac{\partial \psi}{\partial \xi_i} = 0$$

with the boundary condition

$$\psi_2 \eta_x + \psi_3 \eta_y + \psi_4 \eta_z = p - p_t$$

where η_x, η_y, η_z are the components of the surface normal. Then the boundary integrals involving δp and the field integral involving δw are eliminated and the gradient is reduced to

$$\frac{1}{2}\int_{\mathcal{B}} (p - p_t)^2 d\delta S - \int\int_{\mathcal{B}} \left(\delta S_{21}\psi_2 + \delta S_{22}\psi_3 + \delta S_{23}\psi_4\right) p d\xi_1 d\xi_3 - \int_{\mathcal{D}} \frac{\partial \psi^T}{\partial \xi}(\delta S_{ij} f_j) d\xi$$

where typically the first term is negligible and can be dropped.

The evaluation of the field integral requires the evaluation of the metric variations δS_{ij} throughout the domain. However, the true gradient should not depend on the way the mesh is modified. Consider the case of a mesh variation with a fixed boundary. Then

$$\delta I = 0$$

but there is a variation in the transformed flux

$$\delta F_i = \delta S_{ij} f_j + S_{ij} \frac{\partial f_j}{\partial w} \delta w$$

Here the true solution is unchanged, so the variation δw is actually the variation δw^* due to the mesh movement δx at fixed ξ. Therefore

$$\delta w = \delta w^* = \frac{\partial w}{\partial x_j} \delta x_j$$

and since

$$\frac{\partial \delta F_i}{\partial \xi} = 0$$

it follows that

$$\int_{\mathcal{D}} \psi^T \frac{\partial (\delta S_{ij} f_j)}{\partial \xi_i} d\xi = - \int_{\mathcal{D}} \psi^T S_{ij} \frac{\partial f_j}{\partial w} \delta w^* d\xi$$

or

$$\int_{\mathcal{D}} \psi^T \frac{\partial (\delta S_{ij} f_j)}{\partial \xi_i} d\xi = \int_{\mathcal{D}} C_i \frac{\partial w}{\partial x_j} \delta x_j d\xi$$

A similar relationship can be verified in the general case with boundary movement. Now,

$$\begin{aligned} \int_{\mathcal{D}} \psi^T \delta R d\xi &= \int_{\mathcal{D}} \frac{\partial}{\partial \xi_i} C_i (\delta w - \delta w^*) d\xi \\ &= \int_{\mathcal{D}} \frac{\partial \psi^T}{\partial \xi_i} C_i (\delta w - \delta w^*) d\xi \\ &= \int_{\mathcal{B}} \psi^T C_i (\delta w - \delta w^*) d\xi_{\mathcal{B}} \end{aligned} \qquad (7)$$

Hence on the wall boundary

$$C_2 \delta w = \delta F_2 - \delta S_{2j} f_j$$

Thus by choosing ψ to satisfy the adjoint equation and the adjoint boundary condition, we have the following expression for the reduced gradient:

$$\begin{aligned} \delta I = \int \int_{\mathcal{B}} \psi^T (\delta S_{2j} f_j + C_2 \delta w^*) d\xi_1 d\xi_3 - \\ \int \int_{\mathcal{B}} (\delta S_{21} \psi_2 + \delta S_{22} \psi_3 + \delta S_{23} \psi_4) p d\xi_1 d\xi_3 \end{aligned} \qquad (8)$$

It has been confirmed in numerical experiments that these alternate formulations yield computed values of the gradient which are in close agreement, and that the optimization procedure converges to essentially the same result whichever is used. On a structured mesh one can explicitly define mesh deformations which allow the field terms to be evaluated easily. On an unstructured mesh this is not the case and the reduction to a boundary integral yields large savings in the computational cost. The discrete adjoint does not provide for such a transformation.

The need for a Sobolev inner product in the definition of the gradient

Another key issue for successful implementation of the continuous adjoint method is the choice of an appropriate inner product for the definition of the gradient. It turns out that there is an enormous benefit from the use of a modified Sobolev gradient, which enables the generation of a sequence of smooth shapes. This can be illustrated by considering the simplest case of a problem in calculus of variations.

Choose $y(x)$ to minimize

$$I = \int_a^b F(y, y')dx$$

with fixed end points $y(a)$ and $y(b)$. Under a variation $\delta y(x)$,

$$\begin{aligned} \delta I &= \int_a^b \left(\frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}\delta y' \right) dx \\ &= \int_a^b \left(\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} \right) \delta y dx \end{aligned}$$

Thus defining the gradient as

$$g = \frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'}$$

and the inner product as

$$(u, v) = \int_a^b uv dx$$

we find that

$$\delta I = (g, \delta y)$$

Then if we set

$$\delta y = -\lambda g, \qquad \lambda > 0$$

we obtain an improvement

$$\delta I = -\lambda(g,g) \leq 0$$

unless $g = 0$, the necessary condition for a minimum. Note that g is a function of $y, y^{'}, y^{''}$,

$$g = g(y, y^{'}, y^{''})$$

Now each step

$$y^{n+1} = y^n - \lambda^n g^n$$

reduces the smoothness of y by two classes. Thus the computed trajectory becomes less and less smooth, leading to instability.

In order to prevent this we can introduce a modified Sobolev inner product [18]

$$\langle u, v \rangle = \int (uv + \epsilon u^{'} v^{'}) dx$$

where ϵ is a parameter that controls the weight of the derivatives. If we define a gradient $\overline{g}$ such that

$$\delta I = \langle \overline{g}, \delta y \rangle$$

Then we have

$$\begin{aligned} \delta I &= \int (\overline{g}\delta y + \epsilon \overline{g}^{'} \delta y^{'}) dx \\ &= \int (\overline{g} - \frac{\partial}{\partial x} \epsilon \frac{\partial \overline{g}}{\partial x}) \delta y dx \\ &= (g, \delta y) \end{aligned}$$

where

$$\overline{g} - \frac{\partial}{\partial x} \epsilon \frac{\partial \overline{g}}{\partial x} = g$$

and $\overline{g} = 0$ at the end points. Thus $\overline{g}$ is obtained from g by a smoothing equation. Now the step

$$y^{n+1} = y^n - \lambda^n \overline{g}^n$$

gives an improvement

$$\delta I = -\lambda^n \langle \overline{g}^n, \overline{g}^n \rangle$$

but y^{n+1} has the same smoothness as y^n, resulting in a stable process.

In applying control theory for aerodynamic shape optimization, the use of a Sobolev gradient is equally important for the preservation of the smoothness class of the redesigned surface, and it has been employed to obtain all the results in the next section.

4. Redesign of the Boeing 747 wing

Over the last decade the adjoint method has been successfully used to refine a variety of designs for flight at both transonic and supersonic cruising speeds. In the case of transonic flight, it is often possible to produce a shock free flow which eliminates the shock drag by making very small changes, typically no larger than the boundary layer displacement thickness. Consequently viscous effects need to be considered in order to realize the full benefits of the optimization.

Here the optimization of the wing of the Boeing 747-200 is presented to illustrate the kind of benefits that can be obtained. In these calculations the flow was modeled by the Reynolds Averaged Navier-Stokes equations. A Baldwin Lomax turbulence model was considered sufficient, since the optimization is for the cruise condition with attached flow. The calculation were performed to minimize the drag coefficient at a fixed lift coefficient, subject to the additional constraints that the span loading should not be altered and the thickness should not be reduced. It might be possible to reduce the induced drag by modifying the span loading to an elliptic distribution, but this would increase the root bending moment, and consequently require an increase in the skin thickness and structure weight. A reduction in wing thickness would not only reduce the fuel volume, but it would also require an increase in skin thickness to support the bending moment. Thus these constraints assure that there will be no penalty in either structure weight or fuel volume.

Figure 1 displays the result of an optimization at a Mach number of 0.86, which is roughly the maximum cruising Mach number attainable by the existing design before the onset of significant drag rise. The lift coefficient of 0.42 is the contribution of the exposed wing. Allowing for the fuselage to total lift coefficient is about 0.47. It can be seen that the redesigned wing is essentially shock free, and the drag coefficient is reduced from 0.01269 (127 counts) to 0.01136 (114 counts). The total drag coefficient of the aircraft at this lift coefficient is around 270 counts, so this would represent a drag reduction of the order of 5 percent.

Figure 2 displays the result of an optimization at Mach 0.90. In this case the shock waves are not eliminated, but their strength is significantly weakened, while the drag coefficient is reduced from 0.01819 (182 counts) to 0.01293 (129 counts). Thus the redesigned wing has essentially the same drag at Mach 0.9 as the original wing at Mach 0.86. The Boeing 747 wing could apparently be modified to allow such an increase in the cruising Mach number because it has a higher sweep-back than later designs, and a rather thin wing section with a thickness to chord ratio of 8 percent. Figures 3 and 4 verify that the span loading and thickness were not changed by the redesign, while figures 5 and 6 indicate the required section changes at 42 percent and 68 percent span stations.

5. Conclusions

The accumulated experience of the last decade suggests that most existing aircraft which cruise at transonic speeds are amenable to a drag reduction of the order of 3 to 5 percent, or an increase in the drag rise Mach number of at least .02. These improvements can be achieved by very small shape modifications, which are too subtle to allow their determination by trial and error methods. The potential economic benefits are substantial, considering the fuel costs of the entire airline fleet. Moreover, if one were to take full advantage of the increase in the lift to drag ratio during the design process, a smaller aircraft could be designed to perform the same task, with consequent further cost reductions. It seems inevitable that some method of this type will provide a basis for aerodynamic designs of the future.

6. Acknowledgement

This work has benefited greatly from the support of the Air Force Office of Science Research under grant No. AF F49620-98-1-2002.

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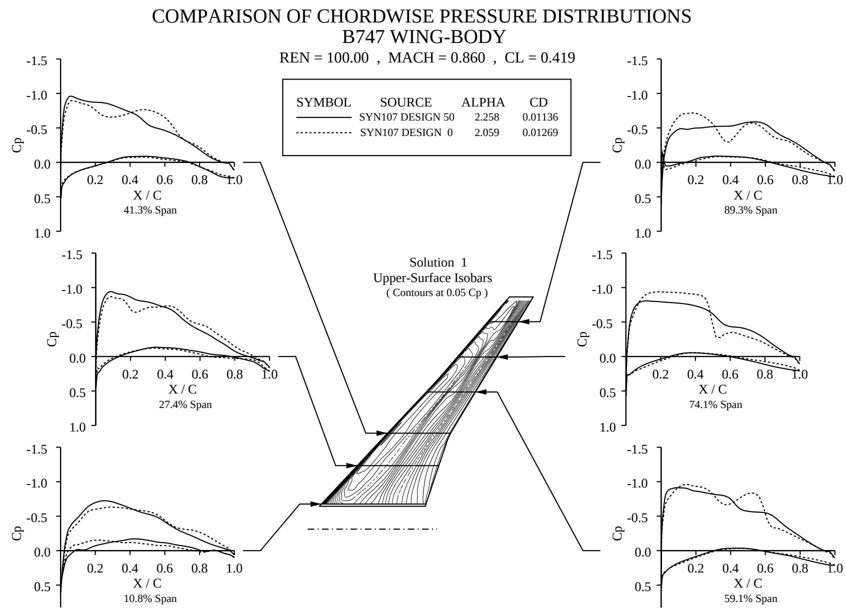

Figure 1. Redesigned Boeing 747 wing at Mach 0.86, Cp distributions

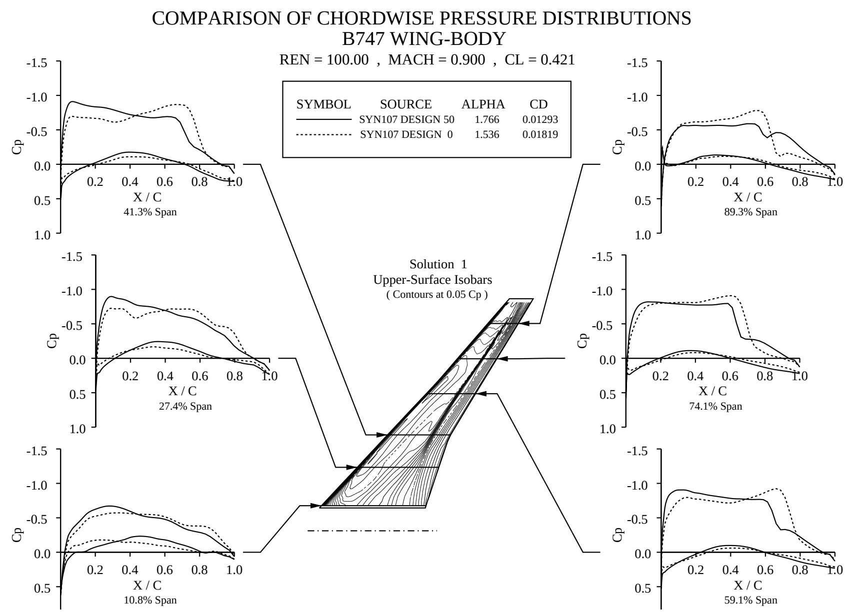

Figure 2. Redesigned Boeing 747 wing at Mach 0.90, Cp distributions

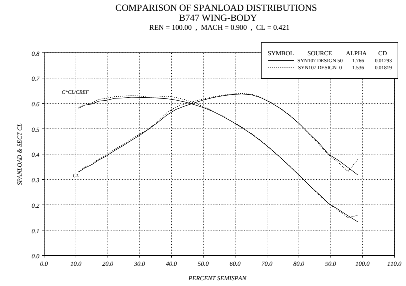

Figure 3. Span loading, Redesigned Boeing 747 wing at Mach 0.90

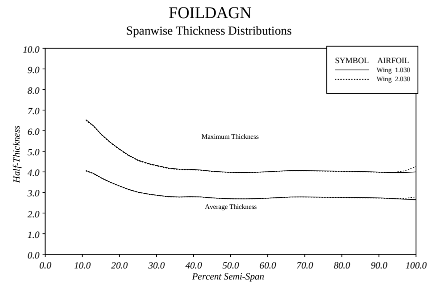

Figure 4. Spanwise thickness distribution, Redesigned Boeing 747 wing at Mach 0.90

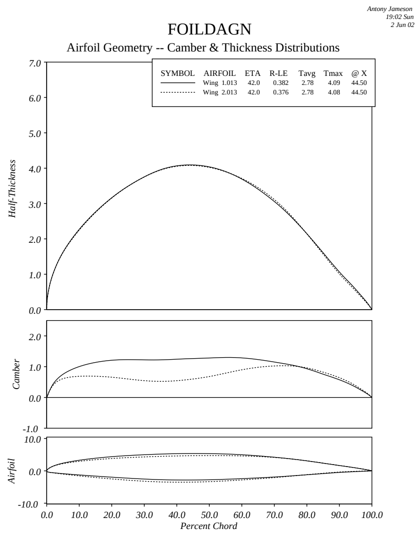

Figure 5. Section geometry at η = 0.42, redesigned Boeing 747 wing at Mach 0.90

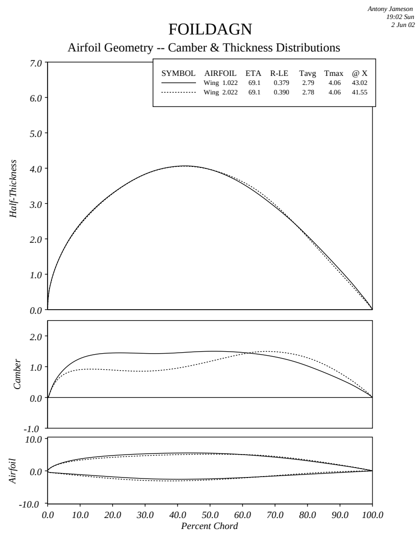

Figure 6. Section geometry at η = 0.68, redesigned Boeing 747 wing at Mach 0.90