INVERSE PROBLEMS
IN
AERODYNAMICS AND CONTROL THEORY

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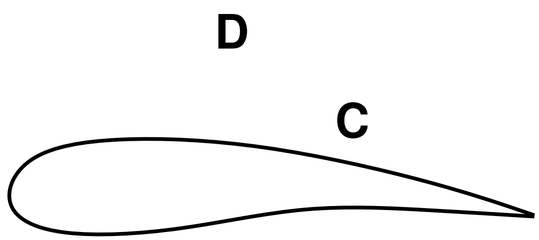
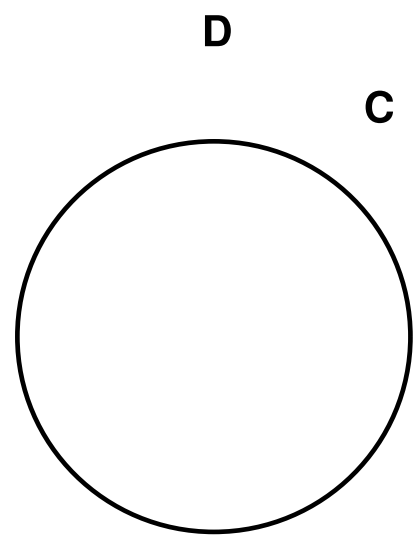
International Conference on Control, PDEs and Scientific Computing
Dedicated to J. L. Lions
Beijing, China
September 10-13, 2004

SOLUTION OF THE INVERSE PROBLEM FOR AIRFOIL DESIGN IN IDEAL FLOW

Lighthill-James-Jameson Method



1a: z-Plane.



1b: σ-Plane.

Transform the profile to a unit circle by a conformal mapping.

On the profile the velocity is

$$q = \frac{1}{h} \left| \boldsymbol{\nabla} \phi \right|,$$

where ϕ is the potential for circulatory flow past a circle, and h is the modulus of the transformation,

$$h = \left| \frac{dz}{d\sigma} \right|. \tag{1}$$

Now set $q = q_t$, where q_t is the target velocity, and solve for h,

$$h = \frac{\boldsymbol{\nabla} \phi}{q_t} = \frac{q_c}{q_t} \tag{2}$$

where q_c is the velocity over the circle.
Since the function defining the mapping is analytic, the knowledge of h on the boundary is sufficient to define the mapping completely

Because the mapping should be one-to-one at infinity,

$$log\frac{dz}{d\sigma} \rightarrow 0 \quad \text{at} \ \infty \tag{3}$$

Thus it can be defined by a Laurent series with inverse powers

$$\log\left(\frac{dz}{d\sigma}\right) = \sum_{n=o}^{\infty} \frac{c_n}{\sigma^n}, \tag{4}$$

On C this can be expanded as

$$\log\frac{ds}{d\theta} + i\left(\alpha - \theta - \frac{\pi}{2}\right) = \sum\left(a_n\cos(n\theta) + b_n\sin(n\theta) + i\sum\left(b_n\cos(n\theta) - a_n\sin(n\theta)\right)\right. \tag{5}$$

where α is the surface tangent angle and s is the arc length.
Also

$$\phi = \left(r + \frac{1}{r}\right)\cos\theta + \frac{\Gamma}{2\pi}\theta \qquad \text{is known} \tag{6}$$

where Γ is the circulation. Now a_n and b_n can be determined as the Fourier coefficients of

$$log\ h = log\frac{q_c}{q_t}$$

Constraints on the target velocity

To preserve q_∞,

$$\frac{dz}{d\sigma} \rightarrow 1 \quad \text{at} \;\; \infty.$$

Hence,

$$c_0 = 0.$$

Also, integration around a circuit gives

$$\Delta z = \oint \frac{dz}{d\sigma} d\sigma = 2\pi i c_1$$

Closure $\rightarrow c_1 = 0$

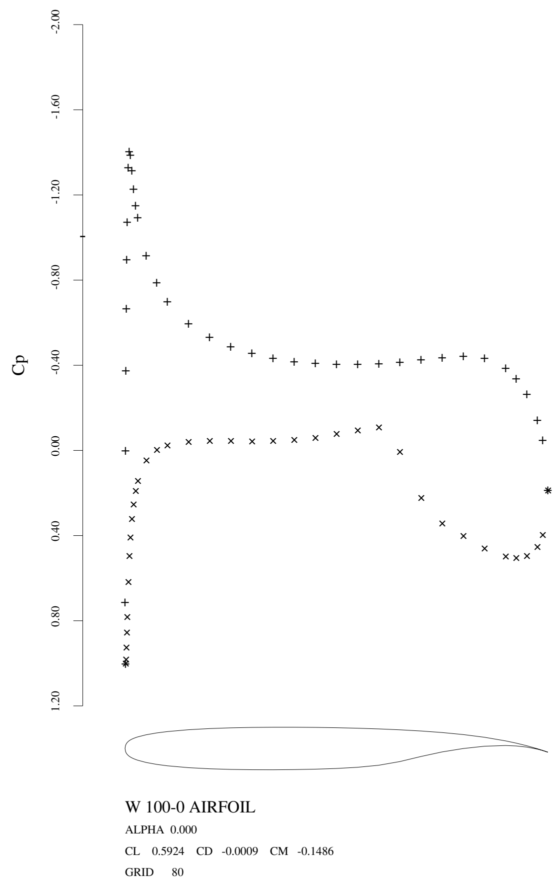
Thus, we require

$$\begin{aligned} \int \log(q_t) d\theta &= 0 \\ \int \log(q_t) \cos(\theta) d\theta &= 0 \\ \int \log(q_t) \sin(\theta) d\theta &= 0 \end{aligned} \tag{7}$$

Here we need $q_t(\theta)$ as a function of the angle around the circle.
However, we can determine ϕ as

$$\phi = \int q_t ds$$

where the integral is over the arc length of the desired profile. This implicitly defines q_t as a function of ϕ. Since ϕ is known, one can find q_t as a function of θ by a Newton iteration. This step, missing from the original Lighthill method, was provided by James and Jameson.



Inverse calculation, recovering Whitcomb airfoil

AIRFOIL DESIGN FOR TRANSONIC POTENTIAL FLOW VIA CONTROL THEORY

Formulation of the Inverse Problem as an Optimization Problem

Because a shape does not necessarily exist for an arbitrary pressure distribution the inverse problem may be ill posed if one tried directly to enforce a specified pressure as a boundary condition.

This difficulty is avoided by formulating the inverse problem as an optimization problem in which one seeks a shape which minimized a cost function such as

$$I = \frac{1}{2}\int (p - p_t)^2 ds$$

where p and p_t are the actual and target pressures.

Shape Design Based on Control Theory

- Regard the wing as a device to generate lift (with minimum drag) by controlling the flow
- Apply theory of optimal control of systems governed by PDEs (Lions) with boundary control (the wing shape)
- Merge control theory and CFD

Automatic Shape Design via Control Theory

- Apply the theory of control of partial differential equations (of the flow) by boundary control (the shape)
- Find the Frechet derivative (infinite dimensional gradient) of a cost function (performance measure) with respect to the shape by solving the adjoint equation in addition to the flow equation
- Modify the shape in the sense defined by the smoothed gradient
- Repeat until the performance value approaches an optimum

Aerodynamic Shape Optimization: Gradient Calculation

For the class of aerodynamic optimization problems under consideration, the design space is essentially infinitely dimensional. Suppose that the performance of a system design can be measured by a cost function I which depends on a function $\mathcal{F}(x)$ that describes the shape,where under a variation of the design $\delta\mathcal{F}(x)$, the variation of the cost is δI. Now suppose that δI can be expressed to first order as

$$\delta I = \int \mathcal{G}(x)\delta\mathcal{F}(x)dx$$

where $\mathcal{G}(x)$ is the gradient. Then by setting

$$\delta\mathcal{F}(x) = -\lambda\mathcal{G}(x)$$

one obtains an improvement

$$\delta I = -\lambda \int \mathcal{G}^2(x)dx$$

unless $\mathcal{G}(x) = 0$. Thus the vanishing of the gradient is a necessary condition for a local minimum.

Symbolic Development of the Adjoint Method

Let I be the **cost** (or **objective**) function

$$I = I(w, \mathcal{F})$$

where

$$\begin{aligned} w &= \text{flow field variables} \\ \mathcal{F} &= \text{grid variables} \end{aligned}$$

The **first variation** of the cost function is

$$\delta I = \frac{\partial I}{\partial w}^T \delta w + \frac{\partial I}{\partial \mathcal{F}}^T \delta \mathcal{F}$$

The **flow field equation** and its **first variation** are

$$R(w, \mathcal{F}) = 0$$

$$\delta R = 0 = \left[\frac{\partial R}{\partial w}\right] \delta w + \left[\frac{\partial R}{\partial \mathcal{F}}\right] \delta \mathcal{F}$$

Symbolic Development of the Adjoint Method (cont.)

Introducing a **Lagrange Multiplier**, ψ, and using the **flow field equation** as a **constraint**

$$\delta I = \frac{\partial I}{\partial w}^T \delta w + \frac{\partial I}{\partial \mathcal{F}}^T \delta \mathcal{F} - \psi^T \left\{ \left[\frac{\partial R}{\partial w}\right] \delta w + \left[\frac{\partial R}{\partial \mathcal{F}}\right] \delta \mathcal{F} \right\}$$

$$= \left\{ \frac{\partial I}{\partial w}^T - \psi^T \left[\frac{\partial R}{\partial w}\right] \right\} \delta w + \left\{ \frac{\partial I}{\partial \mathcal{F}}^T - \psi^T \left[\frac{\partial R}{\partial \mathcal{F}}\right] \right\} \delta \mathcal{F}$$

By choosing ψ such that it satisfies the **adjoint equation**

$$\left[\frac{\partial R}{\partial w}\right]^T \psi = \frac{\partial I}{\partial w},$$

we have

$$\delta I = \left\{ \frac{\partial I}{\partial \mathcal{F}}^T - \psi^T \left[\frac{\partial R}{\partial \mathcal{F}}\right] \right\} \delta \mathcal{F}$$

This reduces the **gradient** calculation for an arbitrarily large number of design variables at a **single design point** to

One Flow Solution + One Adjoint Solution

The Adjoint Equation for Transonic Potential Flow

Consider the case of two-dimensional compressible inviscid flow. In the absence of shock waves, an initially irrotational flow will remain irrotational, and we can assume that the velocity vector $\mathbf{q}$ is the gradient of a potential ϕ. In the presence of weak shock waves this remains a fairly good approximation.

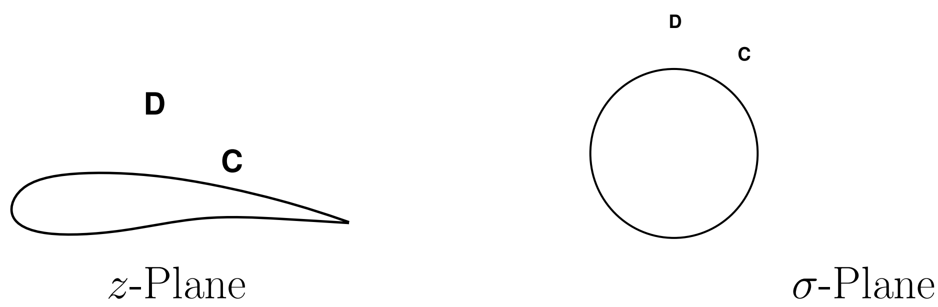


Figure 1: Conformal Mapping.

Let p, ρ, c, and M be the pressure, density, speed-of-sound, and Mach number q/c. Then the potential flow equation is

$$\boldsymbol{\nabla}\cdot\left(\rho\boldsymbol{\nabla}\phi\right)=0, \tag{8}$$

where the density is given by

$$\rho=\left\{1+\frac{\gamma-1}{2}M_\infty^2\left(1-q^2\right)\right\}^{\frac{1}{(\gamma-1)}}, \tag{9}$$

while

$$p=\frac{\rho^\gamma}{\gamma M_\infty^2},\quad c^2=\frac{\gamma p}{\rho}. \tag{10}$$

Here M_∞ is the Mach number in the free stream, and the units have been chosen so that p and q have a value of unity in the far field.

Suppose that the domain D exterior to the profile C in the z-plane is conformally mapped on to the domain exterior to a unit circle in the σ-plane as sketched in Figure 1. Let R and θ be polar coordinates in the σ-plane, and let r be the inverted radial coordinate $\frac{1}{R}$. Also let h be the modulus of the derivative of the mapping function

$$h = \left| \frac{dz}{d\sigma} \right|. \tag{11}$$

Now the potential flow equation becomes

$$\frac{\partial}{\partial \theta}\left(\rho \phi_\theta\right) + r\frac{\partial}{\partial r}\left(r \rho \phi_r\right) = 0 \quad \text{in } D, \tag{12}$$

where the density is given by equation (9), and the circumferential and radial velocity components are

$$u = \frac{r\phi_\theta}{h}, \quad v = \frac{r^2 \phi_r}{h}, \tag{13}$$

while

$$q^2 = u^2 + v^2. \tag{14}$$

The condition of flow tangency leads to the Neumann boundary condition

$$v = \frac{1}{h}\frac{\partial \phi}{\partial r} = 0 \quad \text{on } C. \tag{15}$$

In the far field, the potential is given by an asymptotic estimate, leading to a Dirichlet boundary condition at $r = 0$.

Suppose that it is desired to achieve a specified velocity distribution q_d on C. Introduce the cost function

$$I = \frac{1}{2}\int_C (q - q_d)^2 \, d\theta,$$

The design problem is now treated as a control problem where the control function is the mapping modulus h, which is to be chosen to minimize I subject to the constraints defined by the flow equations (8–15).

A modification δh to the mapping modulus will result in variations $\delta\phi$, δu, δv, and $\delta\rho$ to the potential, velocity components, and density. The resulting variation in the cost will be

$$\delta I = \int_C (q - q_d) \; \delta q \, d\theta, \tag{16}$$

where, on C, $q = u$. Also,

$$\delta u = r\frac{\delta\phi_\theta}{h} - u\frac{\delta h}{h}, \quad \delta v = r^2\frac{\delta\phi_r}{h} - v\frac{\delta h}{h},$$

while according to equation (9)

$$\frac{\partial\rho}{\partial u} = -\frac{\rho u}{c^2}, \quad \frac{\partial\rho}{\partial v} = -\frac{\rho v}{c^2}.$$

It follows that $\delta\phi$ satisfies

$$L\delta\phi = -\frac{\partial}{\partial\theta}\left(\rho M^2\phi_\theta\frac{\delta h}{h}\right) - r\frac{\partial}{\partial r}\left(\rho M^2 r\phi_r\frac{\delta h}{h}\right)$$

where

$$L \equiv \frac{\partial}{\partial\theta}\left\{\rho\left(1 - \frac{u^2}{c^2}\right)\frac{\partial}{\partial\theta} - \frac{\rho uv}{c^2}r\frac{\partial}{\partial r}\right\} + r\frac{\partial}{\partial r}\left\{\rho\left(1 - \frac{v^2}{c^2}\right)r\frac{\partial}{\partial r} - \frac{\rho uv}{c^2}\frac{\partial}{\partial\theta}\right\}. \tag{17}$$

Then, if ψ is any periodic differentiable function which vanishes in the far field,

$$\int_D \frac{\psi}{r^2} L\,\delta\phi\,dS = \int_D \rho M^2 \nabla\phi \cdot \nabla\psi \frac{\delta h}{h}\,dS, \tag{18}$$

where dS is the area element $r\,dr\,d\theta$, and the right hand side has been integrated by parts. Now we can augment equation (16) by subtracting the constraint (18). The auxiliary function ψ then plays the role of a Lagrange multiplier. Thus,

$$\delta I = \int_C (q - q_d)\, q \frac{\delta h}{h}\,d\theta - \int_C \delta\phi \frac{\partial}{\partial\theta}\left(\frac{q - q_d}{h}\right) d\theta - \int_D \frac{\psi}{r^2} L\delta\phi\,dS + \int_D \rho M^2 \nabla\phi \cdot \nabla\psi \frac{\delta h}{h}\,dS.$$

Now suppose that ψ satisfies the adjoint equation

$$L\psi = 0 \ \text{ in } D \tag{19}$$

with the boundary condition

$$\frac{\partial\psi}{\partial r} = \frac{1}{\rho}\frac{\partial}{\partial\theta}\left(\frac{q - q_d}{h}\right) \quad \text{on } C. \tag{20}$$

Then, integrating by parts,

$$\int_D \frac{\psi}{r^2} L\delta\phi\,dS = -\int_C \rho\psi_r \delta\phi\,d\theta,$$

and

$$\delta I = -\int_C (q - q_d)\, q \frac{\delta h}{h}\,d\theta + \int_D \rho M^2 \nabla\phi \cdot \nabla\psi \frac{\delta h}{h}\,dS. \tag{21}$$

Here the first term represents the direct effect of the change in the metric, while the area integral represents a correction for the effect of compressibility. When the second term is deleted the method reduces to a variation of Lighthill's method.

Equation (21) can be further simplified to represent δI purely as a boundary integral because the mapping function is fully determined by the value of its modulus on the boundary. Set

$$\log \frac{dz}{d\sigma} = \mathcal{F} + i\beta,$$

where

$$\mathcal{F} = \log \left| \frac{dz}{d\sigma} \right| = \log h,$$

and

$$\delta \mathcal{F} = \frac{\delta h}{h}.$$

Then $\mathcal{F}$ satisfies Laplace's equation

$$\Delta \mathcal{F} = 0 \ \text{ in } D,$$

and if there is no stretching in the far field, $\mathcal{F} \rightarrow 0$. Also $\delta \mathcal{F}$ satisfies the same conditions.

Introduce another auxiliary function P which satisfies

$$\Delta P = \rho M^2 \boldsymbol{\nabla}\psi \boldsymbol{\cdot} \boldsymbol{\nabla}\psi \quad \text{in } D, \tag{22}$$

and

$$P = 0 \quad \text{on } C.$$

Then, the area integral in equation (21) is

$$\int_D \Delta P \, \delta\mathcal{F} \, dS = \int_C \delta\mathcal{F} \frac{\partial P}{\partial r} \, d\theta - \int_D P \Delta \, \delta\mathcal{F} \, dS,$$

and finally

$$\delta I = \int_C \mathcal{G} \, \delta\mathcal{F} \, d\theta,$$

where $\mathcal{F}_c$ is the boundary value of $\mathcal{F}$, and

$$\mathcal{G} = \frac{\partial P}{\partial r} - (q - q_d) \, q. \tag{23}$$

This suggests setting

$$\delta\mathcal{F}_c = -\lambda\mathcal{G}$$

so that if λ is a sufficiently small positive quantity

$$\delta I = -\int_C \lambda\mathcal{G}^2 \, d\theta < 0.$$

Arbitrary variations in $\mathcal{F}$ cannot, however, be admitted. The condition that $\mathcal{F} \to 0$ in the far field, and also the requirement that the profile should be closed, imply constraints which must be satisfied by $\mathcal{F}$ on the boundary C. Suppose that $\log\left(\frac{dz}{d\sigma}\right)$ is expanded as a power series

$$\log\left(\frac{dz}{d\sigma}\right) = \sum_{n=o}^{\infty} \frac{c_n}{\sigma^n}, \tag{24}$$

where only negative powers are retained, because otherwise $\left(\frac{dz}{d\sigma}\right)$ would become unbounded for large σ. The condition that $\mathcal{F} \to 0$ as $\sigma \to \infty$ implies

$$c_o = 0.$$

Also, the change in z on integration around a circuit is

$$\Delta z = \int \frac{dz}{d\sigma}\, d\sigma = 2\pi i\, c_1,$$

so the profile will be closed only if

$$c_1 = 0.$$

In order to satisfy these constraints, we can project $\mathcal{G}$ onto the admissible subspace for $\mathcal{F}_c$ by setting

$$c_o = c_1 = 0. \tag{25}$$

Then the projected gradient $\tilde{\mathcal{G}}$ is orthogonal to $\mathcal{G} - \tilde{\mathcal{G}}$, and if we take

$$\delta\mathcal{F}_c = -\lambda\tilde{\mathcal{G}},$$

it follows that to first order

$$\begin{aligned} \delta I &= -\int_C \lambda \mathcal{G}\tilde{\mathcal{G}}\, d\theta = -\int_C \lambda\left(\tilde{\mathcal{G}} + \mathcal{G} - \tilde{\mathcal{G}}\right)\mathcal{G}\, d\theta \\ &= -\int_C \lambda \tilde{\mathcal{G}}^2\, d\theta < 0. \end{aligned}$$

If the flow is subsonic, this procedure should converge toward the desired speed distribution since the solution will remain smooth, and no unbounded derivatives will appear. If, however, the flow is transonic, one must allow for the appearance of shock waves in the trial solutions, even if q_d is smooth. Then $q - q_d$ is not differentiable. This difficulty can be circumvented by a more sophisticated choice of the cost function. Consider the choice

$$I = \frac{1}{2}\int_C \left(\lambda_1 \mathcal{Z}^2 + \lambda_2 \left(\frac{d\mathcal{Z}}{d\theta}\right)^2\right) d\theta, \tag{26}$$

where λ_1 and λ_2 are parameters, and the periodic function $\mathcal{Z}(\theta)$ satisfies the equation

$$\lambda_1 \mathcal{Z} - \frac{d}{d\theta}\lambda_2 \frac{d\mathcal{Z}}{d\theta} = q - q_d. \tag{27}$$

Then

$$\delta I = \int_C \left(\lambda_1 \mathcal{Z}\,\delta\mathcal{Z} + \lambda_2 \frac{d\mathcal{Z}}{d\theta}\frac{d}{d\theta}\,\delta\mathcal{Z}\right) d\theta = \int_C \mathcal{Z}\left(\lambda_1\,\delta\mathcal{Z} - \frac{d}{d\theta}\lambda_2 \frac{d}{d\theta}\,\delta\mathcal{Z}\right) d\theta = \int_C \mathcal{Z}\,\delta q\, d\theta.$$

Thus, $\mathcal{Z}$ replaces $q - q_d$ in the previous formulas, and if one modifies the boundary condition (20) to

$$\frac{\partial \psi}{\partial r} = \frac{1}{\rho}\frac{\partial}{\partial \theta}\left(\frac{\mathcal{Z}}{h}\right) \quad \text{on } C, \tag{28}$$

the formula for the gradient becomes

$$\mathcal{G} = \frac{\partial P}{\partial r} - \mathcal{Z} q \tag{29}$$

instead of equation (23). Smoothing can also be introduced directly in the descent procedure by choosing $\delta \mathcal{F}_c$ to satisfy

$$\delta \mathcal{F}_c - \frac{\partial}{\partial \theta}\beta \frac{\partial}{\partial \theta}\,\delta \mathcal{F}_c = -\lambda \mathcal{G}, \tag{30}$$

where β is a smoothing parameter. Then to first order

$$\int \mathcal{G}\,\delta \mathcal{F} = -\frac{1}{\lambda}\int \left(\delta \mathcal{F}_c^2 - \delta \mathcal{F}_c \frac{\partial}{\partial \theta}\beta \frac{\partial}{\partial \theta}\,\delta \mathcal{F}_c\right) d\theta = -\frac{1}{\lambda}\int \left(\delta \mathcal{F}_c^2 + \beta \left(\frac{\partial}{\partial \theta}\,\delta \mathcal{F}_c\right)^2\right) d\theta < 0.$$

The smoothed correction should now be projected onto the admissible subspace.

The final design procedure is thus as follows. Choose an initial profile and corresponding mapping function $\mathcal{F}$. Then:

1. Solve the flow equations (8–15) for ϕ, u, v, q, ρ.

2. Solve the ordinary differential equation (27) for $\mathcal{Z}$.

3. Solve the adjoint equation (17 and 19) or ψ subject to the boundary condition (28).

4. Solve the auxiliary Poisson equation (22) for P.

5. Evaluate $\mathcal{G}$ by equation (29)

6. Correct the boundary mapping function $\mathcal{F}_c$ by $\delta\mathcal{F}_c$ calculated from equation (30), projected onto the admissible subspace defined by (25).

7. Return to step 1.

THREE DIMENSIONAL TRANSONIC INVERSE DESIGN USING THE EULER EQUATIONS

Design using the Euler Equations

The three-dimensional Euler equations may be written as

$$\frac{\partial w}{\partial t} + \frac{\partial f_i}{\partial x_i} = 0 \quad \text{in } D, \tag{31}$$

where

$$w = \begin{Bmatrix} \rho \\ \rho u_1 \\ \rho u_2 \\ \rho u_3 \\ \rho E \end{Bmatrix}, \quad f_i = \begin{Bmatrix} \rho u_i \\ \rho u_i u_1 + p\delta_{i1} \\ \rho u_i u_2 + p\delta_{i2} \\ \rho u_i u_3 + p\delta_{i3} \\ \rho u_i H \end{Bmatrix} \tag{32}$$

and δ_{ij} is the Kronecker delta function. Also,

$$p = (\gamma - 1)\, \rho \left\{ E - \frac{1}{2} \left(u_i^2 \right) \right\}, \tag{33}$$

and

$$\rho H = \rho E + p \tag{34}$$

where γ is the ratio of the specific heats.

Design using the Euler Equations

In order to simplify the derivation of the adjoint equations, we map the solution to a fixed computational domain with coordinates ξ_1, ξ_2, ξ_3 where

$$K_{ij} = \left[\frac{\partial x_i}{\partial \xi_j} \right], \quad J = \det(K), \quad K_{ij}^{-1} = \left[\frac{\partial \xi_i}{\partial x_j} \right],$$

and

$$S = JK^{-1}.$$

The elements of S are the cofactors of K, and in a finite volume discretization they are just the face areas of the computational cells projected in the x_1, x_2, and x_3 directions. Using the permutation tensor ϵ_{ijk} we can express the elements of S as

$$S_{ij} = \frac{1}{2} \epsilon_{jpq} \epsilon_{irs} \frac{\partial x_p}{\partial \xi_r} \frac{\partial x_q}{\partial \xi_s}. \tag{35}$$

Design using the Euler Equations

Then

$$\begin{aligned} \frac{\partial}{\partial \xi_i} S_{ij} &= \frac{1}{2}\epsilon_{jpq}\epsilon_{irs}\left(\frac{\partial^2 x_p}{\partial \xi_r \partial \xi_i}\frac{\partial x_q}{\partial \xi_s} + \frac{\partial x_p}{\partial \xi_r}\frac{\partial^2 x_q}{\partial \xi_s \partial \xi_i}\right) \\ &= 0. \end{aligned} \tag{36}$$

Also in the subsequent analysis of the effect of a shape variation it is useful to note that

$$\begin{aligned} S_{1j} &= \epsilon_{jpq}\frac{\partial x_p}{\partial \xi_2}\frac{\partial x_q}{\partial \xi_3}, \\ S_{2j} &= \epsilon_{jpq}\frac{\partial x_p}{\partial \xi_3}\frac{\partial x_q}{\partial \xi_1}, \\ S_{3j} &= \epsilon_{jpq}\frac{\partial x_p}{\partial \xi_1}\frac{\partial x_q}{\partial \xi_2}. \end{aligned} \tag{37}$$

Design using the Euler Equations

Now, multiplying equation(31) by J and applying the chain rule,

$$J\frac{\partial w}{\partial t} + R\left(w\right) = 0 \tag{38}$$

where

$$R\left(w\right) = S_{ij}\frac{\partial f_j}{\partial \xi_i} = \frac{\partial}{\partial \xi_i}\left(S_{ij} f_j\right), \tag{39}$$

using (36). We can write the transformed fluxes in terms of the scaled contravariant velocity components

$$U_i = S_{ij} u_j$$

as

$$F_i = S_{ij} f_j = \begin{bmatrix} \rho U_i \\ \rho U_i u_1 + S_{i1} p \\ \rho U_i u_2 + S_{i2} p \\ \rho U_i u_3 + S_{i3} p \\ \rho U_i H \end{bmatrix}.$$

Design using the Euler Equations

For simplicity, it will be assumed that the portion of the boundary that undergoes shape modifications is restricted to the coordinate surface $\xi_2 = 0$. Then equations for the variation of the cost function and the adjoint boundary conditions may be simplified by incorporating the conditions

$$n_1 = n_3 = 0, \quad n_2 = 1, \quad d\mathcal{B}_\xi = d\xi_1 d\xi_3,$$

so that only the variation δF_2 needs to be considered at the wall boundary. The condition that there is no flow through the wall boundary at $\xi_2 = 0$ is equivalent to

$$U_2 = 0, \qquad \text{so that} \qquad \delta U_2 = 0$$

when the boundary shape is modified. Consequently the variation of the inviscid flux at the boundary reduces to

$$\delta F_2 = \delta p \begin{Bmatrix} 0 \\ S_{21} \\ S_{22} \\ S_{23} \\ 0 \end{Bmatrix} + p \begin{Bmatrix} 0 \\ \delta S_{21} \\ \delta S_{22} \\ \delta S_{23} \\ 0 \end{Bmatrix}. \tag{40}$$

Design using the Euler Equations

In order to design a shape which will lead to a desired pressure distribution, a natural choice is to set

$$I = \frac{1}{2}\int_{\mathcal{B}} (p - p_d)^2 \, dS$$

where p_d is the desired surface pressure, and the integral is evaluated over the actual surface area. In the computational domain this is transformed to

$$I = \frac{1}{2}\iint_{\mathcal{B}_w} (p - p_d)^2 \, |S_2| \, d\xi_1 d\xi_3,$$

where the quantity

$$|S_2| = \sqrt{S_{2j}S_{2j}}$$

denotes the face area corresponding to a unit element of face area in the computational domain.

Design using the Euler Equations

In the computational domain the adjoint equation assumes the form

$$C_i^T \frac{\partial \psi}{\partial \xi_i} = 0 \tag{41}$$

where

$$C_i = S_{ij} \frac{\partial f_j}{\partial w}.$$

To cancel the dependence of the boundary integral on δp, the adjoint boundary condition reduces to

$$\psi_j n_j = p - p_d \tag{42}$$

where n_j are the components of the surface normal

$$n_j = \frac{S_{2j}}{|S_2|}.$$

Design using the Euler Equations

This amounts to a transpiration boundary condition on the co-state variables corresponding to the momentum components. Note that it imposes no restriction on the tangential component of ψ at the boundary.
We find finally that

$$\begin{aligned}\delta I \; &= -\int_{\mathcal{D}} \frac{\partial \psi^T}{\partial \xi_i} \delta S_{ij} f_j d\mathcal{D} \\ &- \iint_{B_W} (\delta S_{21}\psi_2 + \delta S_{22}\psi_3 + \delta S_{23}\psi_4)\, p\, d\xi_1 d\xi_3. \end{aligned} \tag{43}$$

Here the expression for the cost variation depends on the **mesh variations throughout the domain which appear in the field integral**. However, the **true gradient** for a shape variation should not depend on the way in which the mesh is deformed, but **only on the true flow solution**. In the next section we show how the field integral can be eliminated to produce a reduced gradient formula which depends only on the boundary movement.

The Reduced Gradient Formulation

Consider the case of a mesh variation with a fixed boundary. Then,

$$\delta I = 0$$

but there is a variation in the transformed flux,

$$\delta F_i = C_i \delta w + \delta S_{ij} f_j.$$

Here the true solution is unchanged. Thus, the variation δw is due to the mesh movement δx at each mesh point. Therefore

$$\delta w = \nabla w \cdot \delta x = \frac{\partial w}{\partial x_j} \delta x_j \; (= \delta w^*)$$

and since

$$\frac{\partial}{\partial \xi_i} \delta F_i = 0,$$

it follows that

$$\frac{\partial}{\partial \xi_i} \left(\delta S_{ij} f_j \right) = - \frac{\partial}{\partial \xi_i} \left(C_i \delta w^* \right). \tag{44}$$

It has been verified by Jameson and Kim* that this relation holds in the general case with boundary movement.

* *Reduction of the Adjoint Gradient Formula in the Continuous Limit*, A.Jameson and S. Kim, 41 st AIAA Aerospace Sciences Meeting & Exhibit, AIAA Paper 2003-0040, Reno, NV, January 6-9, 2003.

The Reduced Gradient Formulation

Now

$$\int_{\mathcal{D}} \phi^T \delta R d\mathcal{D} = \int_{\mathcal{D}} \phi^T \frac{\partial}{\partial \xi_i} C_i \left(\delta w - \delta w^*\right) d\mathcal{D}$$
$$= \int_{\mathcal{B}} \phi^T C_i \left(\delta w - \delta w^*\right) d\mathcal{B}$$
$$- \int_{\mathcal{D}} \frac{\partial \phi^T}{\partial \xi_i} C_i \left(\delta w - \delta w^*\right) d\mathcal{D}. \quad (45)$$

Here on the wall boundary

$$C_2 \delta w = \delta F_2 - \delta S_{2j} f_j. \quad (46)$$

Thus, by choosing ϕ to satisfy the adjoint equation and the adjoint boundary condition, we reduce the cost variation to a boundary integral which depends only on the surface displacement:

$$\delta I = \int_{\mathcal{B}_{\mathcal{W}}} \psi^T \left(\delta S_{2j} f_j + C_2 \delta w^*\right) d\xi_1 d\xi_3$$
$$- \iint_{B_W} (\delta S_{21} \psi_2 + \delta S_{22} \psi_3 + \delta S_{23} \psi_4)\, p \, d\xi_1 d\xi_3. \quad (47)$$

Sobolev Gradient

"Key issue for successful implementation of the Continuous adjoint method."

Define the gradient with respect to the **Sobolev inner product**

$$\delta I \quad = \quad < \bar{g}, \delta f > \quad = \quad \int \left(\bar{g}\delta f + \epsilon \bar{g'} \delta f' \right) dx$$

Set

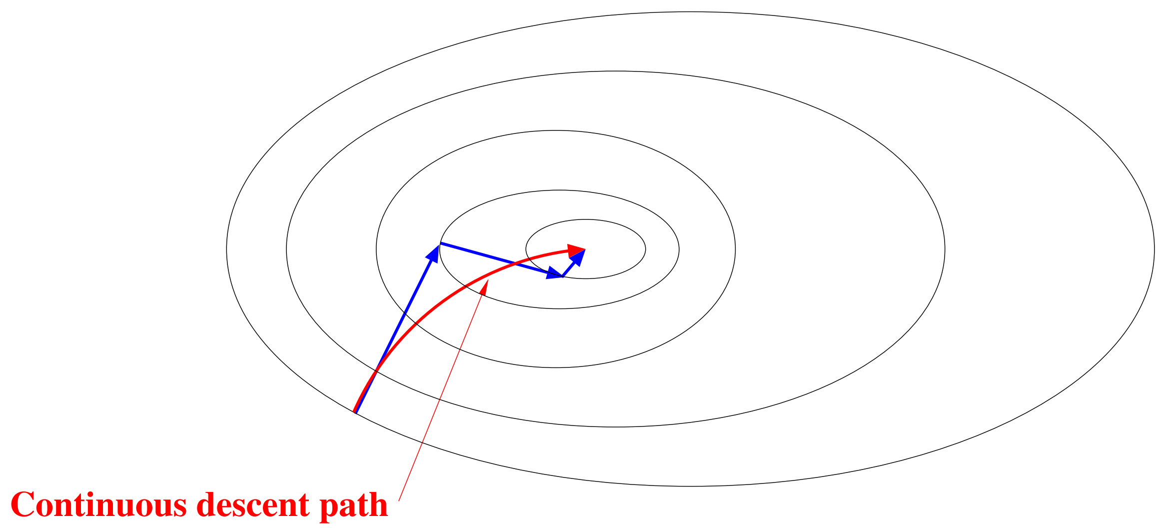
$$\delta f = -\lambda \bar{g}, \quad \delta I = -\lambda < \bar{g}, \bar{g} >$$

This approximates a continuous descent process

$$\frac{df}{dt} = -\bar{g}$$

The **Sobolev** gradient $\bar{g}$ is obtained from the simple gradient g by the smoothing equation

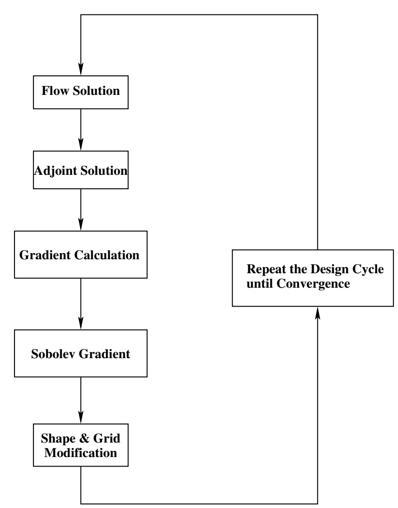
$$\bar{g} - \frac{\partial}{\partial x} \epsilon \frac{\partial \bar{g}}{\partial x} = g$$



Outline of the Design Process

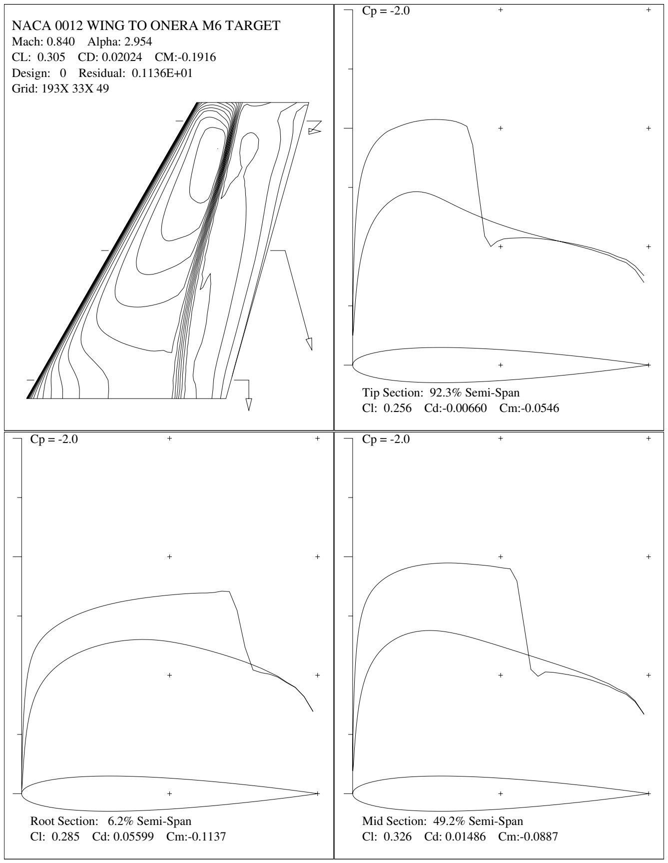
The design procedure can finally be summarized as follows:

1. Solve the flow equations for ρ, u_1, u_2, u_3, p.
2. Solve the adjoint equations for ψ subject to appropriate boundary conditions.
3. Evaluate $\mathcal{G}$ and calculate the corresponding Sobolev gradient $\bar{\mathcal{G}}$.
4. Project $\bar{\mathcal{G}}$ into an allowable subspace that satisfies any geometric constraints.
5. Update the shape based on the direction of steepest descent.
6. Return to 1 until convergence is reached.

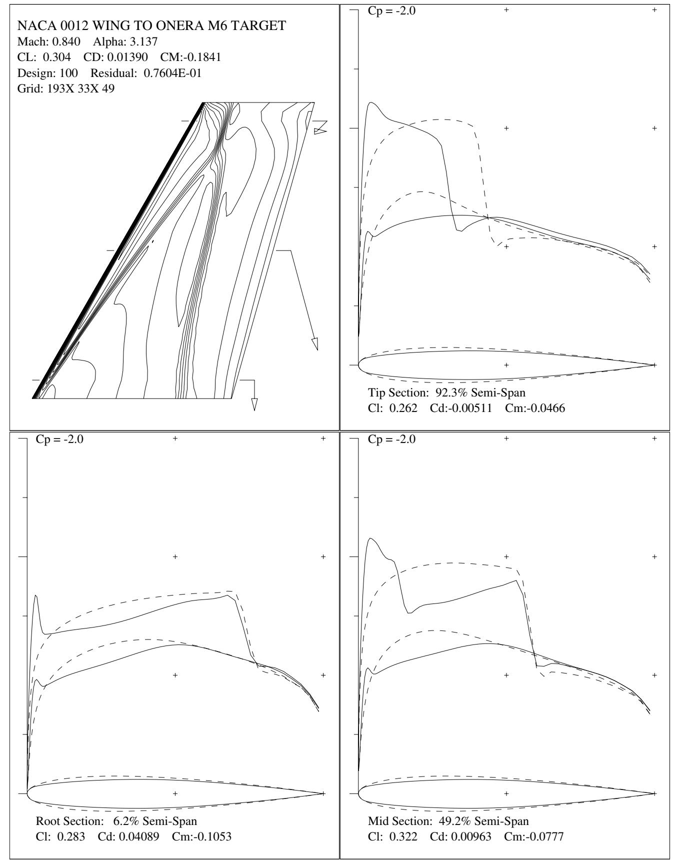


Design cycle

NACA 0012 WING TO ONERA M6 TARGET



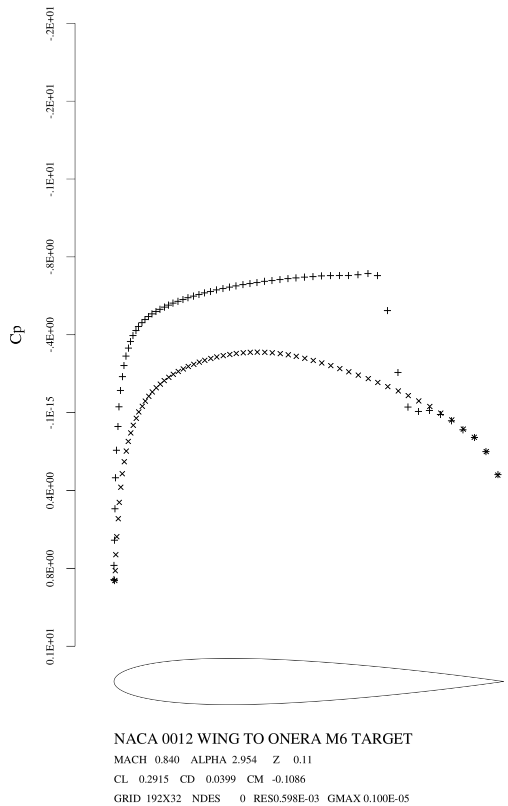
Starting wing: NACA 0012



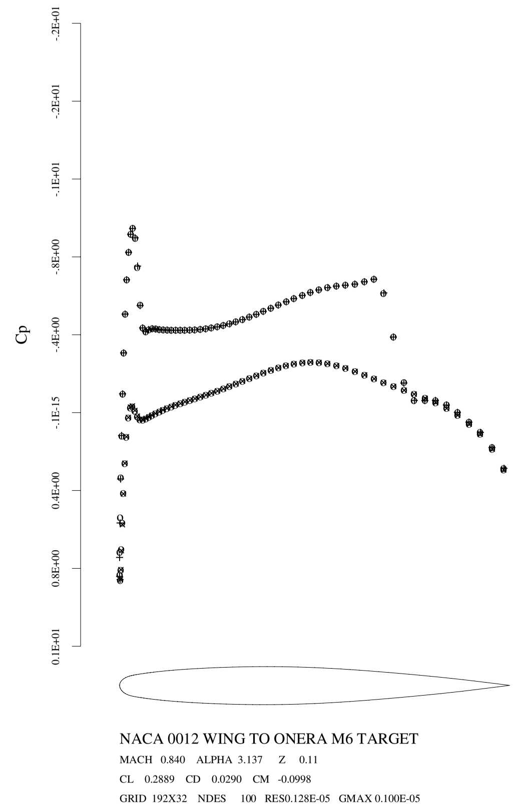
Target wing: ONERA M6

This is a difficult problem because of the presence of the shock wave in the target pressure and because the profile to be recovered is symmetric while the target pressure is not.

Pressure Profiles at 11% span



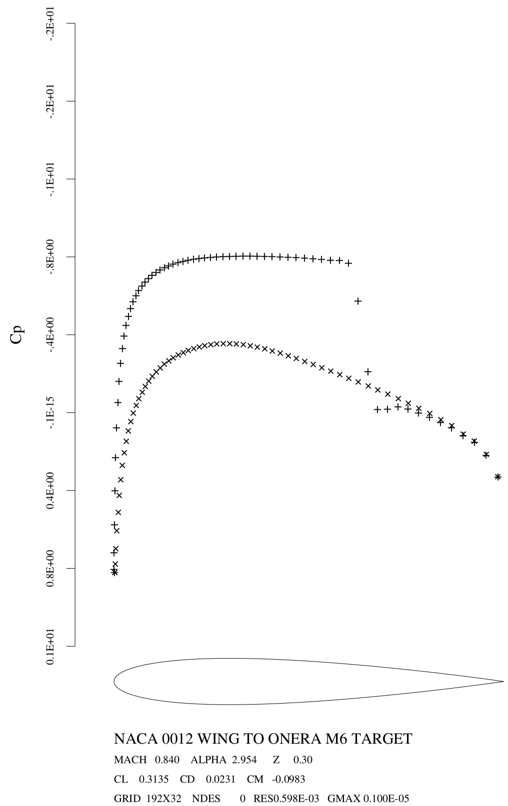
Starting C_p



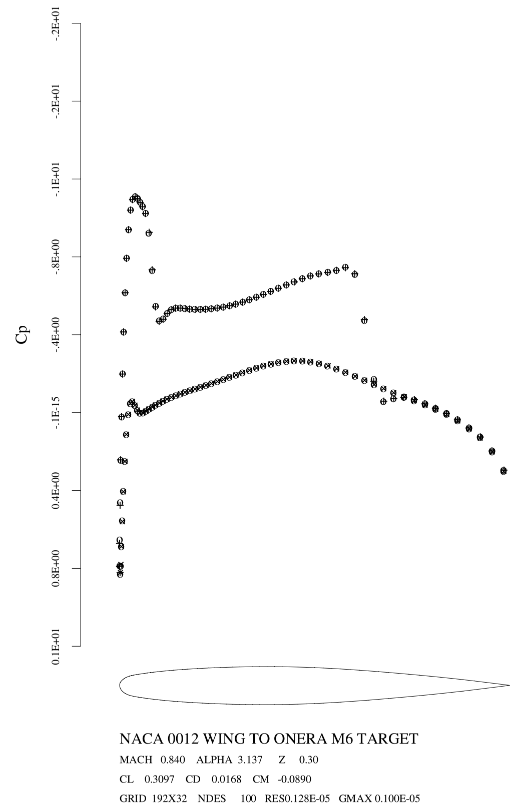
Target C_p

The pressure distribution of the final design match the specified target, even inside the shock.

Pressure Profiles at 30% span



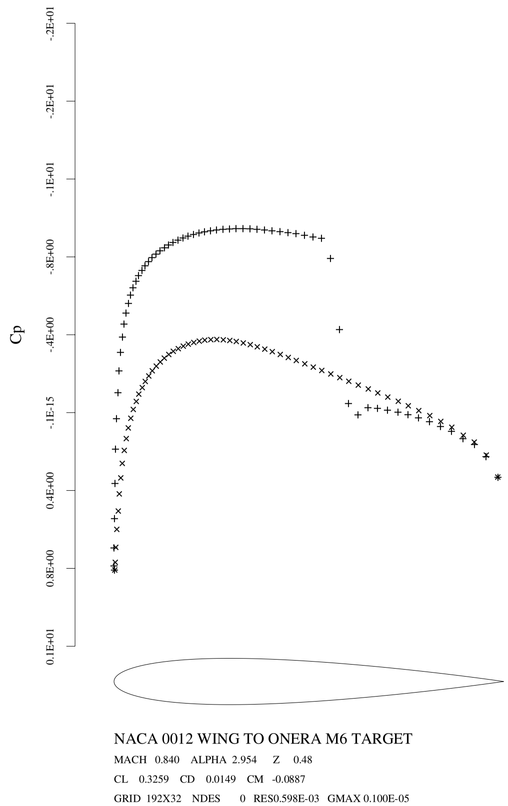
Starting C_p



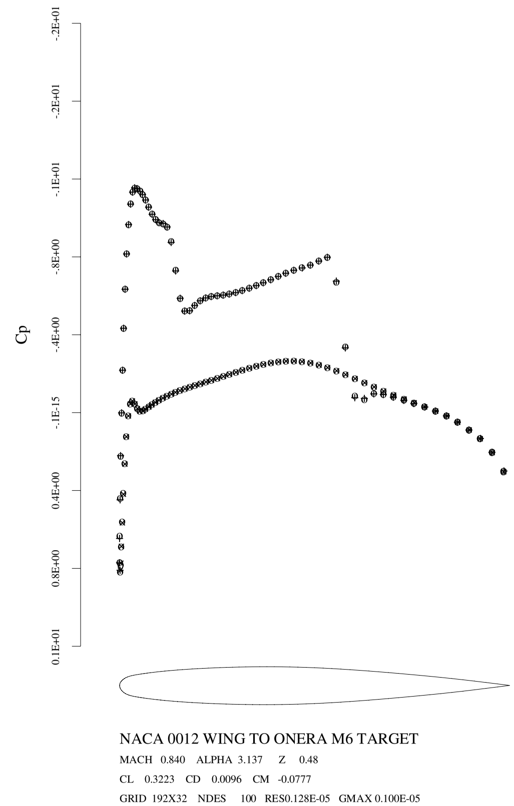
Target C_p

The pressure distribution of the final design match the specified target, even inside the shock.

Pressure Profiles at 48% span



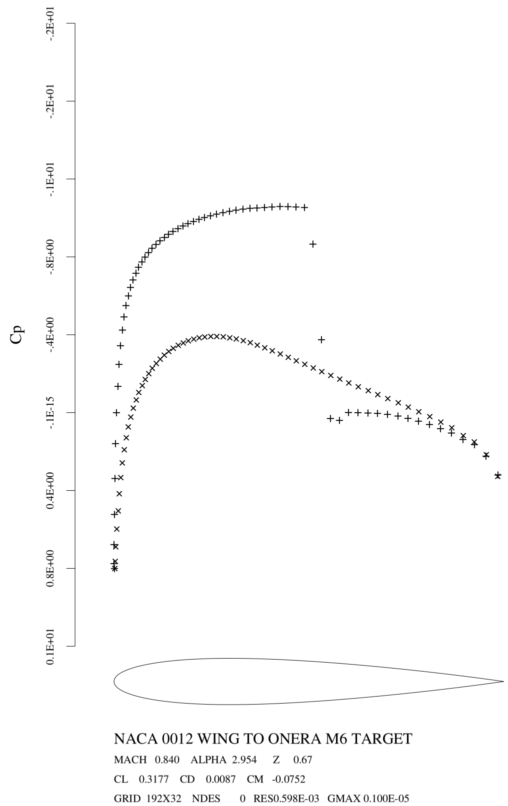
Starting C_p



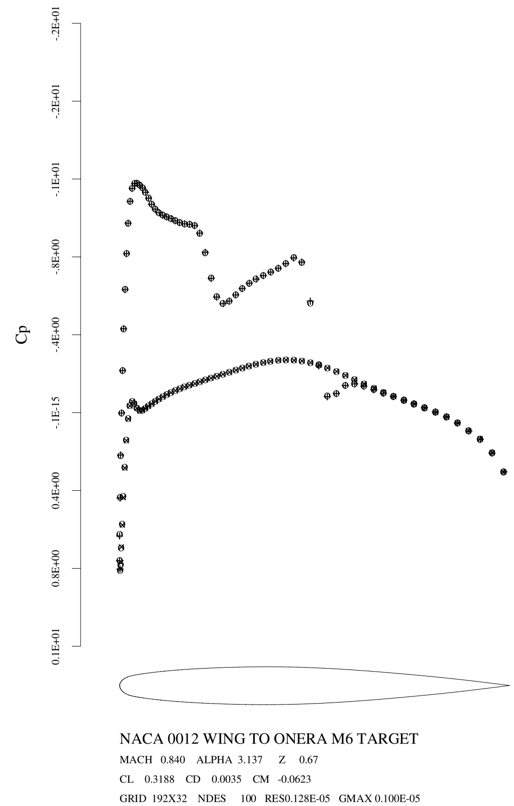
Target C_p

The pressure distribution of the final design match the specified target, even inside the shock.

Pressure Profiles at 67% span



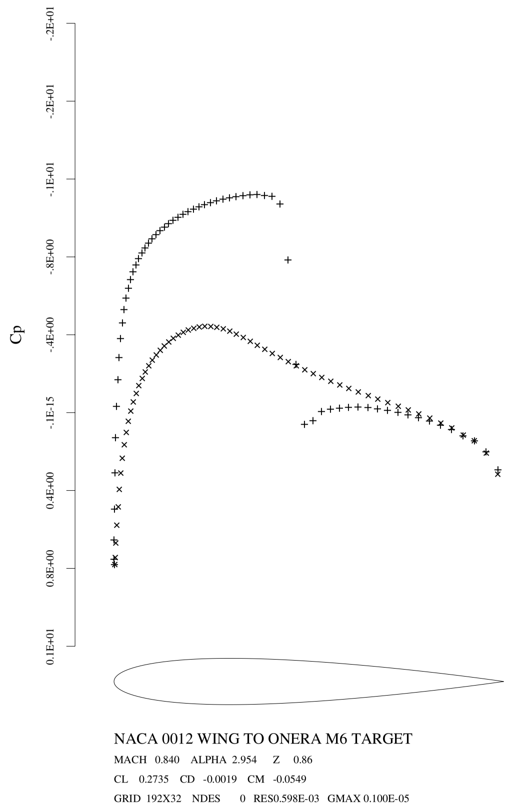
Starting C_p



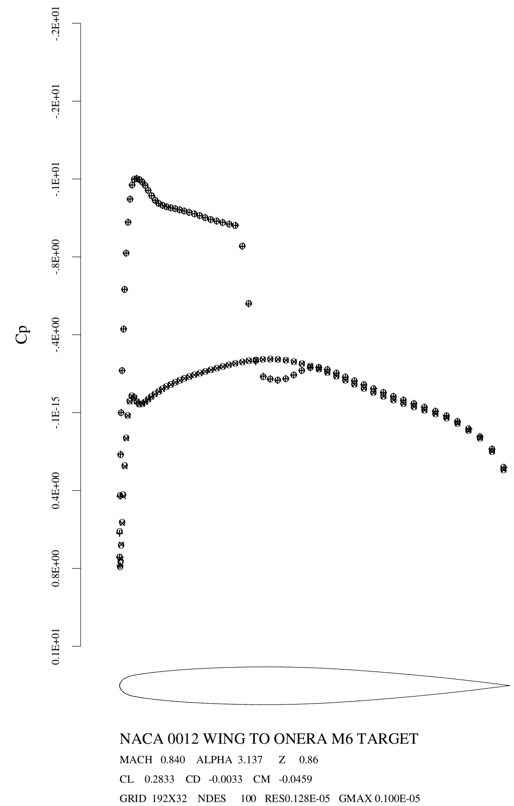
Target C_p

The pressure distribution of the final design match the specified target, even inside the shock.

Pressure Profiles at 86% span



Starting C_p



Target C_p

The pressure distribution of the final design match the specified target, even inside the shock.

SOLUTION OF THE INVERSE PROBLEM OF LINEAR OPTIMAL CONTROL WITH POSITIVENESS CONDITIONS AND RELATION TO SENSITIVITY

Formulation

Let

$$\dot{x} = Ax + Bu \qquad x \in R^m;\ u \in R^n, \tag{48}$$

$$u = Dx\ , \tag{49}$$

be a given control. It is desired to find a performance index

$$J = x^T(t_f)Fx(t_f) + \int_{t_0}^{t_f}(x^TQx + u^TRu)dt\ , \tag{50}$$

with $R = R^T > 0$, $Q = Q^T \geq 0$, which is minimized by u.

The solution of this problem without positiveness conditions on Q is given in references[*,+]. If a performance index (50) exists which is minimized by (49), then

$$-RD = B^TP\ , \tag{51}$$

where P is a symmetric matrix satisfying

$$-\dot{P} = A^TP + PA - D^TRD + Q, \quad P(t_f) = F\ . \tag{52}$$

* *Optimality of Linear Control Systems* (with E. Kreindler), IEEE Trans. on Automatic Control, Vol. AC-17, 1972, pp. 349-351.

\+ *Inverse Problem of Linear Optimal Control* (with E. Kreindler), SIAM J. on Control, Vol. 11, 1973, pp. 1-19.

Formulation

If (52) is multiplied on the left by x^T and on the right by x, then substituting from (48) and (51)

$$x^T Qx + u^T Ru = -\frac{d}{dt}(x^T Px) + (u - Dx)^T R(u - Dx) \tag{53}$$

Integrating from t_1 to t_2

$$\int_{t_1}^{t_2}(x^T Qx+u^T Ru)dt+x^T(t_2)P(t_2)x(t_2) = x^T(t_1)P(t_1)x(t_1)+\int_{t_1}^{t_2}((u-Dx)^T R(u-Dx))dt\ . \tag{54}$$

Setting $t_1 = t_0$, $t_2 = t_f$, $P(t_f) = F$, it is seen that

$$J \geq x^T(t_0)P(t_0)x(t_0)\ , \tag{55}$$

since the final term is non-negative.

Note also that on setting $t_2 = t_f$ and $u = Dx$ it follows that if $Q \geq C$ and $F \geq 0$, then $P(t_1) \geq 0$ for all $t_1 < t_f$ because the left side is non-negative. Also multiplying (51) on the right by B, the symmetry of P is seen to imply the symmetry of RDB.

Formulation

The condition for the existence of $R = R^T > 0$ and $P = P^T > 0$ satisfying (51) are given in reference*.
They are:

- DB has n independent real eigen-vectors (**A1**)
- The eigen-values of DB are non-positive (**A2**)
- $r_{DB} = r_D$, where r_D denotes the rank of D, etc. (**A3**)

For $P > 0$ **A3** is replaced by

- $r_{DB} = r_D = r_B$ (**A3***)

If $R = R^T > 0$ is given, then the conditions for a solution $P \geq 0$ to (51) are

- RDB is symmetric (**B1**)
- $RDB \leq 0$ (**B2**)
- $r_{RDB} = r_{RD}$ (**B3**)

For $P > 0$ **B3** is replaced by

- $r_{RDB} = r_{RD} = r_B$ (**B3***)

Here (**B3**) and (**B3***) are simply restatements of (**A3**) and (**A3***), but (**B3**) and (**B3***) would still be needed for a case where R is only non-negative.

* *Inverse Problem of Linear Optimal Control* (with E. Kreindler), SIAM J. on Control, Vol. 11, 1973, pp. 1-19.

Sensitivity Inequality

The property that the feedback control minimizes some performance index (50) with $Q \geq 0$ is of considerable interest because of its connection with the ability of the control to reduce the sensitivity of the system for parameter variations*
Let Δx_c be the trajectory deviation resulting from plant variations, ΔA and ΔB, when the feedback control (49) is used, and let Δx_o be the deviation when (49) is replaced by an open loop control which would give the same trajectory in the absence of parameter deviations. Also let

$$\Delta A = \epsilon \delta A \,, \Delta B = \epsilon \delta B \,, \tag{56}$$

and define $\delta x = \lim_{\epsilon \to 0} \frac{\Delta x}{\epsilon}$.
Using the equivalence of controls when $\delta A = \delta B = 0$ we have

$$\delta \dot{x}_c = (A + BD)\delta x_c + (\delta A + \delta BD)x \,, \tag{57}$$

$$\delta \dot{x}_0 = A\delta x_o + (\delta A + \delta BD)x \,. \tag{58}$$

Whence

$$\delta x_c = \delta x_o + \delta \tag{59}$$

where

$$\dot{\delta} = A\delta + BD\delta x_c \tag{60}$$

* *On Criteria for Closed Loop Sensitivity Reduction* (with E. Kreindler), J. Math. Analysis and Applications, Vol. 37, 1972, pp. 457-466.

Sensitivity Inequality

Then

$$\int_{t_0}^{t} \delta x_c^T D^T R D \delta x_c dt \leq \int_{t_0}^{t} \delta x_0^T D^T R D \delta x_o dt - \int_{t_0}^{t} \delta^T Y \delta dt \ , \tag{61}$$

for all t if the following condition is satisfied

C: **The sensitivity inequality**

$$S_y(t) = \int_{t_0}^{t} \left\{ (u - Dx)^T R(u - Dx) - u^T Ru \right\} dt - \int_{t_0}^{t} x^T Y x dt \geq 0 \tag{62}$$

holds, where x is the solution of (48) with $x(t_0) = 0$ under an arbitrary input u.

This follows on setting $u = D\delta x_c$ and interpreting x as δ. Now setting $t_1 = t_0$, $x(t_0) = 0$, and $t_2 = t$ in (54) it is seen that C holds with $Y = Q$ when the control (49) minimizes the performance index (50), provided that $P(t) \geq 0$. This is in turn ensured if $Q \geq 0$ and $F \geq 0$. The non-negativeness of Q and F thus guarantees a reduction in sensitivity to parameter variations in the sense of (61).

Solution of the Inverse Problem with $Q \geq 0$

It was remarked earlier that $Q \geq 0$ implies $P \geq 0$. If R is not specified, conditions (**A1**), (**A2**), and (**A3**) are necessary for a solution of the inverse problem. Also (**A1**), (**A2**), and (**A3***) are necessary and sufficient for $P > 0$, and it will now be shown that the existence of a solution $P > 0$ to 51) is sufficient for a solution with $Q \geq 0$. We thus have

Theorem:
Conditions (A1),(A2) and (A3*) are necessary for a solution of the inverse problem with $Q \geq 0$. Conditions (A1), (A2) and (A3*) are also sufficient.

Proof Necessity has already been established. To prove sufficiency observe that from (48)

$$\frac{d}{dt}(xe^{-\alpha t}) = (A - \alpha I)xe^{-\alpha t} + Bue^{-\alpha t} \ . \tag{63}$$

Thus the control (49) minimizes the performance index

$$J = \int_{t_0}^{t_f} e^{-2\alpha t}(x^T Q_o x + u^T R_o u)dt + e^{-2\alpha t_f} x^T(t_f) F_o x(t_f) \ . \tag{64}$$

If (51) holds together with

$$-\dot{P} = (A - \alpha I)^T P + P(A - \alpha I) - D^T R_o D + Q_o \ , P(t_f) = F_o \ . \tag{65}$$

Solution of the Inverse Problem with $Q \geq 0$

Under conditions (**A1**), (**A2**) and(**A3***) it is possible to construct $R = R^T > 0$ and $P = P^T > 0$ satisfying (51). Then Q_o may be constructed as

$$Q_o = Q_1 + 2\alpha P \ , \tag{66}$$

where

$$Q_1 = D^T R_o D - A^T P - PA - \dot{P} \ . \tag{67}$$

Since $P > 0$ and Q_1 is a fixed function of t, it is always possible to choose $\alpha > 0$ sufficiently large that $Q_o \geq 0$. Also comparison of (65) and (66) with (65) shows that the control (49) minimizes (64) and hence (50) on setting $Q = Q_o e^{-2\alpha t}$, $R = R_o e^{-2\alpha t}$, $F = F_o e^{-2\alpha t_f}$.

If the control (49) satisfies conditions A1, A2 and A3* then it satisfies the criterion (61) for sensitivity reduction for some $Y \geq 0$.

Observe that the procedure of the above Theorem gives a joint solution for Q and R, but cannot be used to construct $Q \geq 0$ when R is given. In particular, if the system is constant it leads to a performance with an exponential time weighting factor $e^{-2\alpha t}$, $\alpha > 0$, and establishes the sensitivity criterion (61) with a similar factor.

Solution of the Inverse Problem with $Q \geq 0$ for given R

We now consider methods of solving for Q when R is a given matrix satisfying conditions (**B1**), (**B2**) and (**B3**). To find additional requirements on R for $Q \geq 0$ multiply (52) on the left by B^T. Then using (51)

$$-B^T \dot{P} = B^T A^T P + RDA - B^T D^T RD + B^T Q \ . \tag{68}$$

But differentiating (51)

$$-B^T \dot{P} - \dot{B}^T P = \frac{d}{dt}(RD) \ . \tag{69}$$

Thus

$$B^T Q = L \ , \tag{70}$$

where

$$L = B^T D^T RD + RDA - (B^T A^T - \dot{B}^T)P + \frac{d}{dt}(RD) \ . \tag{71}$$

Also multiplying (71) on the right by B and again using (51)

$$B^T QB = M \ , \tag{72}$$

where

$$M = B^T D^T RDB + RDAB + B^T A^T D^T R + \frac{d}{dt}(RDB) - RD\dot{B} - \dot{B}^T D^T R \ . \tag{73}$$

Solution of the Inverse Problem with $Q \geq 0$ for given R

Since M depends only on the system matrices and R, and $B^T Q B \geq 0$ if $Q \geq 0$, the necessary condition for $Q \geq 0$ is:

- $M \geq 0$ where M is defined in (73) (**B4**)

In order to construct $Q \geq 0$ we shall assume the stronger condition:

- $M > 0$ (**B4***)

As long as (51) holds

$$B^T L^T = M \ . \tag{74}$$

Thus if (70) is regarded as an equation for Q it has a symmetric solution

$$Q_o = L^T M^{-1} L \ . \tag{75}$$

Moreover if Q is any other solution then

$$B^T (Q - Q_o) = 0 \ . \tag{76}$$

Thus the general symmetric solution for Q is

$$Q = L^T M^{-1} L + Y \ , \tag{77}$$

where Y is any symmetric matrix such that

$$B^T Y = 0 \tag{78}$$

Solution of the Inverse Problem with $Q \geq 0$ for given R

Condition (**B4***) ensures that $Q \geq 0$ if $Y \geq 0$. Also let

$$x = (I - BM^{-1}L)z \; , \tag{79}$$

where z is an arbitrary vector. Then

$$x^T Q x = z^T Y z \; . \tag{80}$$

Thus $Y \geq 0$ is also necessary for $Q \geq 0$.

Solution of the Inverse Problem with $Q \geq 0$ for given R

Consider the differential equation that is obtained when (77) is substituted for Q in (52):

$$-\dot{P} = A^T P + PA - D^T RD + L^T M^{-1} L + Y \ . \tag{81}$$

Since M is given and L is linear in P, this is a Ricatti equation which can be integrated to determine P and hence L. We shall verify that (81) has solutions that satisfy (51). Define K by

$$K = B^T P + RD \tag{82}$$

so that $K = 0$ if (51) holds. Then when (51) is no longer assumed to hold (71) and (73) yield

$$B^T L^T = M - K(AB - \dot{B}) \tag{83}$$

instead of (74). Also using (81)

$$\begin{aligned} -\dot{K} &= -B^T \dot{P} - \dot{B}^T P - \frac{d}{dt}(RD) && (84) \\ &= B^T A^T P + B^T PA - B^T D^T RD + B^T L^T M^{-1} L + B^T Y - \dot{B}^T P - \frac{d}{dt}(RD) \ , && (85) \end{aligned}$$

whence in view of (71), (78) and (83)

$$\begin{aligned} -\dot{K} &= -L + KA + \left[M - K(AB - \dot{B})\right] M^{-1} L && (86) \\ &= K\left[A - (AB - \dot{B})M^{-1} L\right] \ . \end{aligned}$$

Solution of the Inverse Problem with $Q \geq 0$ for given R

Under conditions B1, B2 and B3 it is possible to choose $P(t_f) \geq 0$ satisfying (51). Then $K(t_f) = 0$ and the solution of (86) when integrated backwards is $K = 0$. The corresponding solution of (81) therefore satisfies (51).

Theorem

Given $R = R^T > 0$, conditions $(\mathbf{B1} - \mathbf{4})$ are necessary for a solution of the inverse problem with $Q \geq 0$. Conditions $(\mathbf{B1} - \mathbf{3})$ are sufficient for a solution over some finite time interval. Every solution with $Q \geq 0$ is then given by the solution of (71) and (81) for some $Y \geq 0$ satisfying (78)

Solution of the Inverse Problem with $Q \geq 0$ for given R

If equation (86) is unstable when integrated backwards then the integration of (81) would tend to drift away from satisfying (51). To overcome this difficulty we may introduce instead of L and M the matrices

$$L_1 = B^T D^T RD + RDA_1 + B^T P(A_1 - A) - (B^T A^T - \dot{B}^T)P + \frac{d}{dt}(RD) \; , \tag{87}$$

and

$$M_1 = B^T L_1 \tag{88}$$

where A_1 is a matrix to be selected. Then

$$L_1 = L + K(A_1 - A) \; , \tag{89}$$

and from (83)

$$\begin{aligned} M_1 &= B^T L^T + B^T(A_1^T - A^T)K^T & (90)\\ &= M - K(AB - \dot{B}) - (B^T A^T - B^T A_1^T)K^T \; . & (91) \end{aligned}$$

Solution of the Inverse Problem with $Q \geq 0$ for given R

Thus $L_1 = L$ and $M_1 = M$ when $K = 0$. We now integrate

$$-\dot{P} = A^T P + PA - D^T RD + L_1 M_1^{-1} L_1 + Y \tag{92}$$

Then from (78) and (88)

$$-\dot{K} = B^T A^T P + B^T PA - B^T D^T RD + L_1 - \dot{B}^T P - \frac{d}{dt}(RD) \tag{93}$$

whence (87) yields

$$-\dot{K} = KA_1 \tag{94}$$

Thus if A_1 is chosen as any stable matrix, backward integration of (92) will preserve $K = 0$ without danger of drift. But then $L_1 = L$ and $M_1 = M$, so that along the integration path $M_1 = M_1^T$ and under condition B4*, $M_1 > 0$.

Solution of the Inverse Problem with $Q \geq 0$ for given R

While conditions B1-B3 and B4* establish the existence of $Q \geq 0$ over some finite time interval, and every Q can then be constructed from equation (77) or (92), these conditions do not establish the existence of $Q \geq 0$ over an arbitrarily large time interval, because equation (81) or (92) which follows the same integration path, may have a finite escape time.
In particular, conditions B1-B3 and B4* are not sufficient for the existence of a solution with constant $Q \geq 0$ and $R > 0$, when A, B and D are constant. This is easily seen from an example. Consider the system

$$\dot{x} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \; , \tag{95}$$

with

$$u = \begin{bmatrix} -2 & \frac{1}{2} \end{bmatrix} x \; , \tag{96}$$

where it is desired to find a performance index

$$J = \int_0^\infty (x^T Q x + u^2) dt \; , \tag{97}$$

which is minimized by u. RDB and M are scalars,

$$RDB = -2 \; , \tag{98}$$

and

$$M = (B^T D^T)^2 + 2DAB = 5 \; . \tag{99}$$

Solution of the Inverse Problem with $Q \geq 0$ for given R

So conditions B1 - B3 and B4* all hold. On the other hand (51) may be solved for P to give

$$P = \begin{bmatrix} 2 & -\frac{1}{2} \\ -\frac{1}{2} & P_{22} \end{bmatrix}, \tag{100}$$

where P_{22} is the only undetermined element in P. Then substituting for P in (52) with $\dot{P} = 0$ gives

$$\begin{aligned} Q &= D^D - A^T P - PA \\ &= \begin{bmatrix} 5 & 1 - P_{22} \\ 1 - P_{22} & -\frac{3}{4} \end{bmatrix}, \end{aligned}$$

so that it is not possible to obtain $Q \geq 0$ by choice of P_{22} when P is constant.

Solution of the Inverse Problem with $Q \geq 0$ for given R

This example indicates the need to examine the conditions under which (81) or (92) can be integrated over an arbitrary interval. Since (92) follows the same path as (81) when integrated from a final value of P which satisfies (51), it suffices to consider (81). Substituting from (71), it may be written as

$$-\dot{P}_1 = A^T P_1 + P_1 A + D^T R D - Y - D_1^T M D_1 \ , \tag{101}$$

where

$$P_1 = -P \ , \tag{102}$$

and

$$D_1 = -M^{-1}(B^T A^T P_1 + \dot{B}^T D^T R D + R D A + \frac{d}{dt}(RD) - \dot{B}^T P_1) \tag{103}$$

Let

$$\dot{x}_1 = A x_1 + A B u_1 - \dot{B} u_1 \ . \tag{104}$$

Then equations (101) and (103) are equations for determining the control

$$u_1 = D_1 x_1 \ , \tag{105}$$

which minimizes

$$\begin{aligned} J_1 \ &= \ \int_{t_1}^{t_f} \left\{ x_1^T (D^T R D - Y) x_1 \right. \\ &+ \ \left. 2 u_1^T (B^T D^T R D + R D A + \frac{d}{dt}(RD)) x_1 + u_1^T M u_1 \right\} dt \\ &+ \ x_1^T(t_f) P_1(t_f) x_1(t_f) \ . \end{aligned} \tag{106}$$

Solution of the Inverse Problem with $Q \geq 0$ for given R

Substituting for M from (73) and using (78), (106) becomes

$$\begin{aligned} J_1 &= \int_{t_1}^{t_f} (x_1 + Bu_1)^T (D^T RD - Y)(x_1 + Bu_1) dt \\ &+ 2\int_{t_1}^{t_f} u_1^T (RDA + \frac{d}{dt}(RD))(x_1 + Bu_1) dt \\ &- \int_{t_1}^{t_f} u_1^T \frac{d}{dt}(RDB) u_1 dt + x_1^T(t_f) P_1(t_f) x_1(t_f) \; . \end{aligned} \tag{107}$$

Set

$$x = x_1 + Bu_1 \; . \tag{108}$$

Then x satisfies (48) where $\dot{u}_1 = u$, or

$$u_1 = \int_{t_1}^{t_f} u dt \; , \tag{109}$$

there being no constant if (104) and (48) are both to be in equilibrium with zero control. The second and third terms in (107) become

$$2\int_{t_1}^{t_f} u_1^T (RDA + \frac{d}{dt}(RD))(x) dt - \int_{t_1}^{t_f} u_1^T \frac{d}{dt}(RDB) u_1 dt \tag{110}$$

$$= \int_{t_1}^{t_f} \left\{ 2u_1^T RD\dot{x} + 2u_1^T \frac{d}{dt}(RD)x - 2u_1^T RDB\dot{u}_1 - u_1^T \frac{d}{dt}(RDB) u_1 \right\} dt \tag{111}$$

$$= \left[2u_1^T RDx - u_1^T RDBu_1 \right]_{t_1}^{t_f} - 2\int_{t_1}^{t_f} uRDxdt \; . \tag{112}$$

Solution of the Inverse Problem with $Q \geq 0$ for given R

Also using (51) and (102)

$$x_1^T P_1 x_1 + 2u_1^T R D x_1 + u_1^T R D B u_1 = x^T P_1 x = -x^T P x \ . \tag{113}$$

Thus (107) becomes

$$J_1 = S_y - x^T(t_f)P(t_f)x(t_f) + I(t_1) \ , \tag{114}$$

where S_y is defined by (62) and

$$I = -2u_1 R D x_1 - u_1 R D B u_1 \ . \tag{115}$$

Now (109) shows that a non-zero value of u_1 at $t = t_1$ corresponds to an impulse at $t = t_1$ in u

$$u = u_1(t_1)\delta(t - t_1) \ . \tag{116}$$

But under such an impulse x is shifted from x_0^- to $x_0^+ = x_0^- + Bu_1(t_1)$ with a contribution to S_y exactly equal to $I(t_1)$, since x_1 is continuous so that $x_1(t_1) = x_0^-$. Thus the first and third terms in J_1 equal S_y evaluated from t_1^- in case of an initial impulse in u. It follows that $-x^T(t_1)P(t_1)x(t_1)$ is the minimum value of S_y when $x(t_1) = x_0^-$. Suppose that this quantity is not bounded. Then since the system is controllable, it can be brought from $x(t_0) = 0$ to $x(t_1) = x_0^-$ with a finite contribution to S_y, and hence over the interval (t_o, t_f) condition C would be violated. We deduce that condition C is sufficient for the existence of a solution to (81).