Eigenvalues, Eigenvectors and Symmetrization of the Magneto-Hydrodynamic (MHD) Equations

Antony Jameson

June 30, 2006

Presented at AFOSR Grantees and Contractors Meeting, August 7, 2006, Atlanta, GA.

Let ρ, u_i, p, E, B_i and μ denote the density, velocity components, pressure, energy, magnetic field components and permeability. Using the convention that a repeated index i denotes summation over $i = 1$ to 3, the eight wave MHD equations proposed by Powell [1] and also studied by Roe [2,3] can be written as

$$\begin{aligned}
\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_j) &= 0 \qquad (1)\\
\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_i u_j - \frac{B_i B_j}{\mu}) + \frac{\partial P}{\partial x_i} &= -\frac{B_i}{\mu}\frac{\partial B_j}{\partial x_j}\\
\frac{\partial}{\partial t}(\rho Z) + \frac{\partial}{\partial x_j}((\rho Z + p)u_j - u_i\frac{B_i B_j}{\mu}) &= -\frac{u_i B_i}{\mu}\frac{\partial B_j}{\partial x_j}\\
\frac{\partial B_i}{\partial t} + \frac{\partial}{\partial x_j}(u_j B_i - u_i B_j) &= -u_i\frac{\partial B_j}{\partial x_j}
\end{aligned}$$

Here Z and P are the total energy and pressure allowing for the magnetic field.

$$\begin{aligned}
Z &= E + \frac{B_i^2}{2\rho\mu} \qquad (2)\\
P &= p + \frac{B_i^2}{2\mu}
\end{aligned}$$

while for a perfect gas,

$$p = (\gamma - 1)\rho(E - \frac{u^2}{2}) \ , \ c^2 = \frac{\gamma p}{\rho} \tag{3}$$

where γ is the ratio of specific heats and c is the speed of sound.

The source terms on the right are proportional to Div **B** and should be zero in a true solution. In terms of the conservative variables w, the MHD equations can be written as

$$\frac{\partial w}{\partial t} + \frac{\partial}{\partial x_i} F_i(w) + S(w) = 0$$

where

$$w = \begin{bmatrix} \rho \\ \rho u_1 \\ \rho u_2 \\ \rho u_3 \\ \rho Z \\ B_1 \\ B_2 \\ B_3 \end{bmatrix}, \quad F_i = \begin{bmatrix} \rho u_i \\ \rho u_i u_1 + P\delta_{i1} - \frac{B_i B_1}{\mu} \\ \rho u_i u_2 + P\delta_{i2} - \frac{B_i B_2}{\mu} \\ \rho u_i u_3 + P\delta_{i3} - \frac{B_i B_3}{\mu} \\ \rho u_i (Z + P/\rho) - u_j \frac{B_i B_j}{\mu} \\ u_i B_1 - B_i u_1 \\ u_i B_2 - B_i u_2 \\ u_i B_3 - B_i u_3 \end{bmatrix}, \quad S = \frac{\partial B_j}{\partial x_j} \begin{bmatrix} 0 \\ \frac{B_1}{\mu} \\ \frac{B_2}{\mu} \\ \frac{B_3}{\mu} \\ \frac{u_j B_j}{\mu} \\ u_1 \\ u_2 \\ u_3 \end{bmatrix} \tag{4}$$

In smooth regions they can be expressed in quasi-linear form as

$$\frac{\partial w}{\partial t} + \frac{\partial F_i}{\partial w} \frac{\partial w}{\partial x_i} + S = 0$$

The source terms can be written as

$$S = S_i \frac{\partial w}{\partial x_i}$$

where

$$\begin{aligned} S_i &= b a_i^T \\ a_1^T &= [0 \ 0 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0] \\ a_2^T &= [0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 1 \ 0] \\ a_3^T &= [0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 1] \\ b^T &= [0, \frac{B_1}{\mu}, \frac{B_2}{\mu}, \frac{B_3}{\mu}, \frac{u_i B_i}{\mu}, u_1, u_2, u_3] \end{aligned} \tag{5}$$

Thus the quasi-linear form is

$$\frac{\partial w}{\partial t} + A_i \frac{\partial w}{\partial x_i} = 0$$

where the Jacobian matrices are

$$A_i = \frac{\partial F_i}{\partial w} + S_i$$

Under a transformation to the primitive variables

$$\tilde{w} = [\rho, u_1, u_2, u_3, p, B_1, B_2, B_3]^T$$

the equations become

$$\frac{\partial w}{\partial \tilde{w}} \frac{\partial \tilde{w}}{\partial t} + A_i \frac{\partial w}{\partial \tilde{w}} \frac{\partial \tilde{w}}{\partial x_i} = 0 \tag{6}$$

or

$$\frac{\partial \tilde{w}}{\partial t} + \tilde{A}_i \frac{\partial \tilde{w}}{\partial x_i} = 0 \tag{7}$$

where

$$\tilde{A}_i = \tilde{M}^{-1} A_i \tilde{M}, \quad A_i = \tilde{M}_i \tilde{A}_i \tilde{M}^{-1}$$

and

$$\tilde{M} = \frac{\partial w}{\partial \tilde{w}}, \quad \tilde{M}^{-1} = \frac{\partial \tilde{w}}{\partial w}$$

The primitive equations in full are

$$\begin{aligned}
\frac{\partial \rho}{\partial t} + u_j \frac{\partial \rho}{\partial x_j} + \rho \frac{\partial u_j}{\partial x_j} &= 0 \qquad (8)\\
\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} + \frac{1}{\rho} \frac{\partial p}{\partial x_i} + \frac{B_j}{\rho \mu} \left(\frac{\partial B_j}{\partial x_i} - \frac{\partial B_i}{\partial x_j} \right) &= 0 \\
\frac{\partial p}{\partial t} + u_j \frac{\partial p}{\partial x_j} + \gamma p \frac{\partial u_j}{\partial x_j} &= 0 \\
\frac{\partial B_i}{\partial t} + u_j \frac{\partial B_i}{\partial x_j} + B_i \frac{\partial u_j}{\partial x_j} - B_j \frac{\partial u_i}{\partial x_j} &= 0
\end{aligned}$$

Also,

$$\tilde{M} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ u_1 & \rho & 0 & 0 & 0 & 0 & 0 & 0 \\ u_2 & 0 & \rho & 0 & 0 & 0 & 0 & 0 \\ u_3 & 0 & 0 & \rho & 0 & 0 & 0 & 0 \\ \frac{u^2}{2} & \rho u_1 & \rho u_2 & \rho u_3 & \frac{1}{\gamma - 1} & \frac{B_1}{\mu} & \frac{B_2}{\mu} & \frac{B_3}{\mu} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{9}$$

and,

$$\tilde{M}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{u_1}{\rho} & \frac{1}{\rho} & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{u_2}{\rho} & 0 & \frac{1}{\rho} & 0 & 0 & 0 & 0 & 0 \\ -\frac{u_3}{\rho} & 0 & 0 & \frac{1}{\rho} & 0 & 0 & 0 & 0 \\ \bar{\gamma}\frac{u^2}{2} & -\bar{\gamma}u_1 & \bar{\gamma}u_2 & -\bar{\gamma}u_3 & \bar{\gamma} & -\bar{\gamma}\frac{B_1}{\mu} & -\bar{\gamma}\frac{B_2}{\mu} & -\bar{\gamma}\frac{B_3}{\mu} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{10}$$

where $\bar{\gamma} = (\gamma - 1)$

Since $\gamma p = \rho c^2$, the Jacobian matrices can be written as

$$\tilde{A}_1 = \begin{bmatrix} u_1 & \rho & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & u_1 & 0 & 0 & \frac{1}{\rho} & 0 & \frac{B_2}{\rho\mu} & \frac{B_3}{\rho\mu} \\ 0 & 0 & u_1 & 0 & 0 & 0 & -\frac{B_1}{\rho\mu} & 0 \\ 0 & 0 & 0 & u_1 & 0 & 0 & 0 & -\frac{B_1}{\rho\mu} \\ 0 & \rho c^2 & 0 & 0 & u_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & u_1 & 0 & 0 \\ 0 & B_2 & -B_1 & 0 & 0 & 0 & u_1 & 0 \\ 0 & B_3 & 0 & -B_1 & 0 & 0 & 0 & u_1 \\ & & & & & & & \end{bmatrix}$$

$$
\tilde{A}_2 = \begin{bmatrix}
u_2 & 0 & \rho & 0 & 0 & 0 & 0 & 0 \\
0 & u_2 & 0 & 0 & 0 & -\frac{B_2}{\rho\mu} & 0 & 0 \\
0 & 0 & u_2 & 0 & \frac{1}{\rho} & \frac{B_1}{\rho\mu} & 0 & \frac{B_3}{\rho\mu} \\
0 & 0 & 0 & u_2 & 0 & 0 & 0 & -\frac{B_2}{\rho\mu} \\
0 & 0 & \rho c^2 & 0 & u_2 & 0 & 0 & 0 \\
0 & -B_2 & B_1 & 0 & 0 & u_2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & u_2 & 0 \\
0 & 0 & B_3 & -B_2 & 0 & 0 & 0 & u_2 \\
\end{bmatrix}
$$

$$
\tilde{A}_3 = \begin{bmatrix}
u_3 & 0 & 0 & \rho & 0 & 0 & 0 & 0 \\
0 & u_3 & 0 & 0 & 0 & -\frac{B_3}{\rho\mu} & 0 & 0 \\
0 & 0 & u_3 & 0 & 0 & 0 & -\frac{B_3}{\rho\mu} & 0 \\
0 & 0 & 0 & u_3 & \frac{1}{\rho} & \frac{B_1}{\rho\mu} & \frac{B_2}{\rho\mu} & \\
0 & 0 & 0 & \rho c^2 & u_3 & 0 & 0 & 0 \\
0 & -B_3 & 0 & B_1 & 0 & u_3 & 0 & 0 \\
0 & 0 & -B_3 & B_2 & 0 & 0 & u_3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & u_3 \\
\end{bmatrix}
$$

In a finite volume scheme the flux across a face with normal vector $\mathbf{n}$ and area S is $F = n_i F_i S$. The corresponding Jacobian matrices for the conservative and primitive forms area

$$A = n_i A_i, \quad \tilde{A} = n_i \tilde{A}_i$$

where

$$\tilde{A} = \tilde{M}^{-1} A \tilde{M}, \quad A = \tilde{M} \tilde{A} \tilde{M}^{-1}$$

Define the normal components of $\mathbf{u}$ and $\mathbf{B}$ as

$$u_n = \mathbf{u} \cdot \mathbf{n}, \quad B_n = \mathbf{B} \cdot \mathbf{n}$$

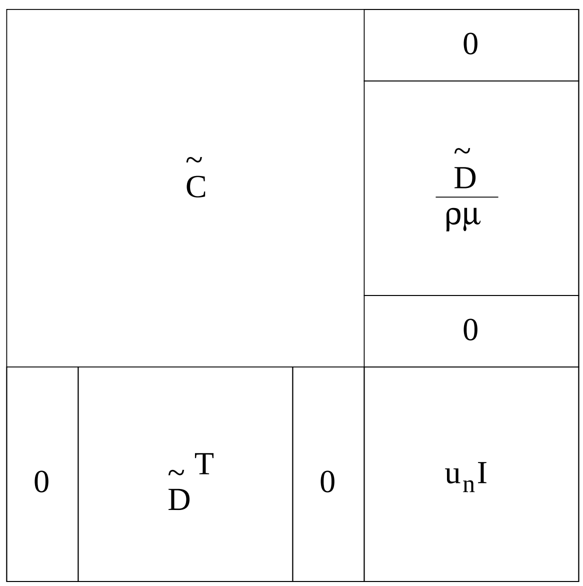
and the magnitudes of $\mathbf{u}$ and $\mathbf{B}$ as

$$u = \sqrt{u_i^2}, \quad B = \sqrt{B_i^2}$$

The Jacobian matrix for the primitive variables can now be written as

$$\tilde{A} = \begin{bmatrix} u_n & n_1\rho & n_2\rho & n_3\rho & 0 & 0 & 0 & 0 \\ 0 & u_n & 0 & 0 & \frac{n_1}{\rho} & \frac{n_1B_1-B_n}{\rho\mu} & \frac{n_1B_2}{\rho\mu} & \frac{n_1B_3}{\rho\mu} \\ 0 & 0 & u_n & 0 & \frac{n_2}{\rho} & \frac{n_2B_1}{\rho\mu} & \frac{n_2B_2-B_n}{\rho\mu} & \frac{n_2B_3}{\rho\mu} \\ 0 & 0 & 0 & u_n & \frac{n_3}{\rho} & \frac{n_3B_1}{\rho\mu} & \frac{n_3B_2}{\rho\mu} & \frac{n_3B_3-B_n}{\rho\mu} \\ 0 & n_1\rho c^2 & n_2\rho c^2 & n_3\rho c^2 & u_n & 0 & 0 & 0 \\ 0 & (n_1B_1-B_n) & n_2B_1 & n_3B_1 & 0 & u_n & 0 & 0 \\ 0 & n_1B_2 & (n_2B_2-B_n) & n_3B_2 & 0 & 0 & u_n & 0 \\ 0 & n_1B_3 & n_2B_3 & (n_3B_3-B_n) & 0 & 0 & 0 & u_n \end{bmatrix}$$

$\tilde{A}$ can be partitioned as



where $\tilde{D}$ and $\tilde{D}^T$ can be written in dyadic form as

$$\tilde{D} = \mathbf{nB} - B_nI, \quad \tilde{D}^T = \mathbf{Bn} - B_nI$$

The Jacobian matrix can now be reduced to symmetric form by a further transformation to the symmetrizing variables, which can be written in differential form as

$$d\tilde{w} = [\frac{dp}{\rho c}, du_1, du_2, du_3, \frac{dp - c^2 d\rho}{\rho c}, \frac{dB_1}{\sqrt{\rho\mu}}, \frac{dB_2}{\sqrt{\rho\mu}}, \frac{dB_3}{\sqrt{\rho\mu}}]^T$$

Here the fifth variable corresponds to entropy and all the variables are scaled so that they have the dimensions of velocity. The transformation matrices are

$$\frac{\partial \bar{w}}{\partial \tilde{w}} = \left[\begin{array}{cccccccc} 0 & 0 & 0 & 0 & \frac{1}{\rho c} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -\frac{c}{\rho} & 0 & 0 & 0 & \frac{1}{\rho c} & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{\mu\rho}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{\mu\rho}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{\mu\rho}} \\ & & & & & & & \end{array} \right]$$

and

$$\frac{\partial \tilde{w}}{\partial \bar{w}} = \left[\begin{array}{cccccccc} \frac{\rho}{c} & 0 & 0 & 0 & -\frac{\rho}{c} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ c\rho & 0 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & \sqrt{\mu\rho} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{\mu\rho} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{\mu\rho} \\ & & & & & & & \end{array} \right]$$

Then

$$\bar{A} = \frac{\partial \bar{w}}{\partial \tilde{w}} \tilde{A} \frac{\partial \tilde{w}}{\partial \bar{w}}$$

where

$$\frac{\partial \bar{w}}{\partial \tilde{w}} \tilde{A} = \begin{bmatrix} 0 & n_1 c & n_2 c & n_3 c & \frac{u_n}{\rho c} & 0 & 0 & 0 \\ 0 & u_n & 0 & 0 & \frac{n_1}{\rho} & \frac{n_1 B_1 - B_n}{\rho\mu} & \frac{n_1 B_2}{\rho\mu} & \frac{n_1 B_3}{\rho\mu} \\ 0 & 0 & u_n & 0 & \frac{n_2}{\rho} & \frac{n_2 B_1}{\rho\mu} & \frac{n_2 B_2 - B_n}{\rho\mu} & \frac{n_2 B_3}{\rho\mu} \\ 0 & 0 & 0 & u_n & \frac{n_3}{\rho} & \frac{n_3 B_1}{\rho\mu} & \frac{n_3 B_2}{\rho\mu} & \frac{n_3 B_3 - B_n}{\rho\mu} \\ -\frac{c}{\rho} u_n & 0 & 0 & 0 & \frac{u_n}{\rho c} & 0 & 0 & 0 \\ 0 & \frac{n_1 B_1 - B_n}{\sqrt{\rho\mu}} & \frac{n_2 B_1}{\sqrt{\rho\mu}} & \frac{n_3 B_1}{\sqrt{\rho\mu}} & 0 & \frac{u_n}{\sqrt{\rho\mu}} & 0 & 0 \\ 0 & \frac{n_1 B_2}{\sqrt{\rho\mu}} & \frac{n_2 B_2 - B_n}{\sqrt{\rho\mu}} & \frac{n_3 B_2}{\sqrt{\rho\mu}} & 0 & 0 & \frac{u_n}{\sqrt{\rho\mu}} & 0 \\ 0 & \frac{n_1 B_3}{\sqrt{\rho\mu}} & \frac{n_2 B_3}{\sqrt{\rho\mu}} & \frac{n_3 B_3 - B_n}{\sqrt{\rho\mu}} & 0 & 0 & 0 & \frac{u_n}{\sqrt{\rho\mu}} \end{bmatrix}$$

and finally

$$\bar{A} = \begin{bmatrix} u_n & n_1 c & n_2 c & n_3 c & 0 & 0 & 0 & 0 \\ n_1 c & u_n & 0 & 0 & 0 & n_1 \bar{B}_1 - \bar{B}_n & n_1 \bar{B}_2 & n_1 \bar{B}_3 \\ n_2 c & 0 & u_n & 0 & 0 & n_2 \bar{B}_1 & n_2 \bar{B}_2 - \bar{B}_n & n_2 \bar{B}_3 \\ n_3 c & 0 & 0 & u_n & 0 & n_3 \bar{B}_1 & n_3 \bar{B}_2 & n_3 \bar{B}_3 - \bar{B}_3 \\ 0 & 0 & 0 & 0 & u_n & 0 & 0 & 0 \\ 0 & n_1 \bar{B}_1 - \bar{B}_n & n_2 \bar{B}_1 & n_3 \bar{B}_1 & 0 & u_n & 0 & 0 \\ 0 & n_1 \bar{B}_2 & n_2 \bar{B}_2 - \bar{B}_n & n_3 \bar{B}_2 & 0 & 0 & u_n & 0 \\ 0 & n_1 \bar{B}_3 & n_2 \bar{B}_3 & n_3 \bar{B}_3 - \bar{B}_n & 0 & 0 & 0 & u_n \end{bmatrix}$$

where the magnetic field is represented by the scaled variables

$$\bar{B}_i = \frac{B_i}{\sqrt{\rho\mu}}, \quad \bar{B}_n = \frac{B_n}{\sqrt{\rho\mu}}$$

which have the dimensions of velocity so that all entries in $\bar{A}$ have this dimension. It is also useful to introduce the component $\bar{\mathbf{B}}_\perp$ of $\bar{\mathbf{B}}$ perpendicular to $\mathbf{n}$.

$$\bar{\mathbf{B}}_\perp = \bar{\mathbf{B}} - \bar{B}_n \mathbf{n}$$

The transformation between the conservative and symmetrizing variables is

$$\bar{M} = \frac{\partial w}{\partial \tilde{w}}\frac{\partial \tilde{w}}{\partial \bar{w}} = \begin{bmatrix} \frac{\rho}{c} & 0 & 0 & 0 & -\frac{\rho}{c} & 0 & 0 & 0 \\ \frac{\rho}{c}u_1 & \rho & 0 & 0 & -\frac{\rho}{c}u_1 & 0 & 0 & 0 \\ \frac{\rho}{c}u_2 & 0 & \rho & 0 & -\frac{\rho}{c}u_2 & 0 & 0 & 0 \\ \frac{\rho}{c}u_3 & 0 & 0 & \rho & -\frac{\rho}{c}u_3 & 0 & 0 & 0 \\ \frac{\rho}{c}H & \rho u_1 & \rho u_2 & \rho u_3 & -\frac{\rho}{c}\frac{u^2}{2} & \sqrt{\frac{\rho}{\mu}}B_1 & \sqrt{\frac{\rho}{\mu}}B_2 & \sqrt{\frac{\rho}{\mu}}B_3 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{\rho\mu} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{\rho\mu} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{\rho\mu} \end{bmatrix}$$

where H is the total enthalpy

$$H = \frac{c^2}{\gamma - 1} + \frac{u^2}{2}$$

The reverse transformation is

$$\bar{M}^{-1} = \frac{\partial \bar{w}}{\partial \tilde{w}}\frac{\partial \tilde{w}}{\partial w} = \begin{bmatrix} \bar{\gamma}\frac{u^2}{c} & -\bar{\gamma}u_1 & -\bar{\gamma}u_2 & -\bar{\gamma}u_3 & \bar{\gamma} & -\bar{\gamma}\frac{B_1}{\mu} & -\bar{\gamma}\frac{B_2}{\mu} & -\bar{\gamma}\frac{B_3}{\mu} \\ -\frac{u_1}{\rho} & \frac{1}{\rho} & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{u_2}{\rho} & 0 & \frac{1}{\rho} & 0 & 0 & 0 & 0 & 0 \\ -\frac{u_3}{\rho} & 0 & 0 & \frac{1}{\rho} & 0 & 0 & 0 & 0 \\ \bar{\gamma}(u^2 - H) & -\bar{\gamma}u_1 & -\bar{\gamma}u_2 & -\bar{\gamma}u_3 & \bar{\gamma} & -\bar{\gamma}\frac{B_1}{\mu} & -\bar{\gamma}\frac{B_2}{\mu} & -\bar{\gamma}\frac{B_3}{\mu} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{\rho\mu}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{\rho\mu}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{\rho\mu}} \end{bmatrix}$$

where $\bar{\gamma} = \frac{\gamma-1}{\rho c}$.

If the symmetrizing variables are multiplied by a scale factor α, then all entries of $\bar{M}$ are divided by α and all entries of $\bar{M}^{-1}$ are multiplied by α. With $\alpha = \frac{\rho}{c}$ the symmetrizing variables have the dimension of density,

$$d\bar{w} = \left[\ \frac{dp}{c^2},\quad \frac{\rho}{c}du_1,\quad \frac{\rho}{c}du_2,\quad \frac{\rho}{c}du_3,\quad \frac{dp}{c^2} - d\rho,\quad \sqrt{\frac{\rho}{\mu}}\frac{d\bar{B}_1}{c},\quad \sqrt{\frac{\rho}{\mu}}\frac{d\bar{B}_2}{c},\quad \sqrt{\frac{\rho}{\mu}}\frac{d\bar{B}_3}{c}\ \right]^T$$

Correspondingly

$$
\bar{M} = \begin{bmatrix}
1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
u_1 & c & 0 & 0 & -u_1 & 0 & 0 & 0 \\
u_2 & 0 & c & 0 & -u_2 & 0 & 0 & 0 \\
u_3 & 0 & 0 & c & -u_3 & 0 & 0 & 0 \\
H & cu_1 & cu_2 & cu_3 & -\frac{u^2}{2} & \frac{c\bar{B}_1}{\sqrt{\rho\mu}} & \frac{c\bar{B}_2}{\sqrt{\rho\mu}} & \frac{c\bar{B}_3}{\sqrt{\rho\mu}} \\
0 & 0 & 0 & 0 & 0 & c\sqrt{\frac{\mu}{\rho}} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & c\sqrt{\frac{\mu}{\rho}} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & c\sqrt{\frac{\mu}{\rho}} \\
\end{bmatrix}
$$

Also,

$$
\bar{M}^{-1} = \begin{bmatrix}
\bar{\gamma}\frac{u^2}{2} & -\bar{\gamma}u_1 & -\bar{\gamma}u_2 & -\bar{\gamma}u_3 & \bar{\gamma} & -\bar{\gamma}\frac{\bar{B}_1}{\mu} & -\bar{\gamma}\frac{\bar{B}_2}{\mu} & -\bar{\gamma}\frac{\bar{B}_3}{\mu} \\
-\frac{u_1}{c} & \frac{1}{c} & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{u_2}{c} & 0 & \frac{1}{c} & 0 & 0 & 0 & 0 & 0 \\
-\frac{u_3}{c} & 0 & 0 & \frac{1}{c} & 0 & 0 & 0 & 0 \\
\bar{\gamma}(u^2 - H) & -\bar{\gamma}u_1 & -\bar{\gamma}u_2 & -\bar{\gamma}u_3 & \bar{\gamma} & -\bar{\gamma}\frac{\bar{B}_1}{\mu} & -\bar{\gamma}\frac{\bar{B}_2}{\mu} & -\bar{\gamma}\frac{\bar{B}_3}{\mu} \\
0 & 0 & 0 & 0 & 0 & \frac{1}{c}\sqrt{\frac{\rho}{\mu}} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{c}\sqrt{\frac{\rho}{\mu}} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{c}\sqrt{\frac{\rho}{\mu}} \\
\end{bmatrix}
$$

where $\bar{\gamma} = \frac{\gamma-1}{c^2}$. With this scaling all entries in the first 5×5 partition of $\bar{M}$ or its inverse depend only on the speeds u_i and c.

It is convenient to write the symmetrized Jacobian $\bar{A}$ in partitioned form as

$$\left[\begin{array}{ccc|c} & & & 0 \\ \hline & \bar{C} & & \bar{D} \\ \hline & & & 0 \\ \hline 0 & \bar{D}^T & 0 & u_n I \end{array}\right]$$

where

$$\bar{C} = \begin{bmatrix} u_n & n_1 c & n_2 c & n_3 c & 0 \\ n_1 c & u_n & 0 & 0 & 0 \\ n_2 c & 0 & u_n & 0 & 0 \\ n_3 c & 0 & 0 & u_n & 0 \\ 0 & 0 & 0 & 0 & u_n \\ \end{bmatrix}$$

and $\bar{D}$ can be expressed in dyadic form as

$$\bar{D} = \mathbf{n}\bar{\mathbf{B}} - \bar{B}_n I$$

The eigenvalues of $\bar{A}$ are $u_n, u_n, u_n + \bar{B}_n, u_n - \bar{B}_n, u_n + c_f, u_n - c_f, u_n + c_s, u_n - c_s$. The first pair correspond to advection. The second pair represent Alfven waves. The third and four pairs represent the fast and slow magneto acoustic waves where the acoustic speeds c_f and c_s satisfy

$$c_f^2 = \frac{1}{2}\left\{(c^2 + \bar{B}^2) + \sqrt{(c^2 + \bar{B}^2)^2 - 4c^2\bar{B}_n{}^2}\right\}$$

$$c_s^2 = \frac{1}{2}\left\{(c^2 + \bar{B}^2) - \sqrt{(c^2 + \bar{B}^2)^2 - 4c^2\bar{B}_n{}^2}\right\}$$

Both c_f^2 and c_s^2 are roots of the equation

$$\alpha^4 - \alpha^2(c^2 + \bar{B}^2) + c^2\bar{B_n}^2 = 0$$

The eigenvectors of $\bar{A}$ corresponding to distinct eigenvalues are orthogonal because $\bar{A}$ is symmetric. It is easily checked that

$$r_1 = [0, 0, 0, 0, 1, 0, 0, 0]$$

and

$$r_2 = [0, 0, 0, 0, 0, n_1, n_2, n_3]$$

are eigenvectors corresponding to the advection speed u_n. Moreover r_1 and r_2 are orthogonal to each other and of unit length. Thus it is possible to find a complete set of orthonormal eigenvectors as long as the other wave speeds are distinct.

Let $\mathbf{l}$ be a vector orthogonal to both $\mathbf{n}$ and $\bar{\mathbf{B}}$, and thus orthogonal to the plane containing $\mathbf{n}$ and $\bar{\mathbf{B}}$ if they are not parallel. Then

$$\bar{D}\mathbf{l} = \bar{D}^T\mathbf{l} = \bar{B_n}\mathbf{l}$$

since

$$\mathbf{n}\bar{\mathbf{B}} \cdot \mathbf{l} = \mathbf{0}, \quad \bar{\mathbf{B}}\mathbf{n} \cdot \mathbf{l} = \mathbf{0}, \quad \mathbf{n} \cdot \mathbf{l} = \mathbf{0}$$

It can now be easily verified that

$$r_3 = [0, l_1, l_2, l_3, 0, -l_1, -l_2, -l_3]^T$$

and

$$r_4 = [0, l_1, l_2, l_3, 0, l_1, l_2, l_3]^T$$

are eigenvectors satisfying

$$\bar{A}r_3 = (u_n + \bar{B_n})r_3$$

and

$$\bar{A}r_4 = (u_n - \bar{B_n})r_4$$

They are of unit length if $l_1^2 + l_2^2 + l_3^2 = 1/2$.
If $\mathbf{n}$ and $\bar{\mathbf{B}}$ are not parallel one can take

$$\mathbf{l} = (\mathbf{n} \times \bar{\mathbf{B}})/\alpha$$

where the scale factor α satisfies

$$\alpha^2 = \left|\mathbf{n} \times \bar{\mathbf{B}}\right|^2 = \bar{B}^2 - \bar{B}_n^2$$

The eigenvectors corresponding to the magneto-acoustic speeds can be expressed in terms of the vectors

$$\mathbf{l_f} = c_f(\mathbf{n} - \frac{\bar{B}_n}{c_f^2 - \bar{B_n}^2}\bar{\mathbf{B}}_\perp), \mathbf{m_f} = (\frac{c_f^2}{c_f^2 - \bar{B_n}^2}\bar{B}_\perp)$$

$$\mathbf{l_s} = c_s(\mathbf{n} - \frac{\bar{B}_n}{c_s^2 - \bar{B_n}^2}\bar{\mathbf{B}}_\perp), \mathbf{m_f} = (\frac{c_s^2}{c_s^2 - \bar{B_n}^2}\bar{B}_\perp)$$

Then it may be verified that

$$\bar{A}r_5 = (u_n + c_f)r_5, \bar{A}r_6 = (u_n - c_f)r_6,$$

$$\bar{A}r_7 = (u_n + c_s)r_7, \bar{A}r_8 = (u_n - c_s)r_8$$

where

$$\begin{aligned}
r_5 &= \frac{1}{\alpha_f}[c, l_{f_1}, l_{f_2}, l_{f_3}, 0, m_{f_1}, m_{f_2}, m_{f_3}]^T \\
r_6 &= \frac{1}{\alpha_f}[-c, l_{f_1}, l_{f_2}, l_{f_3}, 0, -m_{f_1}, -m_{f_2}, -m_{f_3}]^T \\
r_7 &= \frac{1}{\alpha_s}[c, l_{s_1}, l_{s_2}, l_{s_3}, 0, m_{s_1}, m_{s_2}, m_{s_3}]^T \\
r_8 &= \frac{1}{\alpha_s}[-c, l_{s_1}, l_{s_2}, l_{s_3}, 0, -m_{s_1}, -m_{s_2}, -m_{s_3}]^T
\end{aligned} \quad (11)$$

and the scale factors α_f and α_s may be chosen to scale the eigenvectors to unit length.

$$\alpha_f^2 = l_f^2 + m_f^2 + c^2, \alpha_s^2 = l_s^2 + m_s^2 + c^2$$

The verification of these eigenvectors requires some algebraic manipulation. The first entry of $\bar{A}r_5$ is

$$q_n c + c\mathbf{n} \cdot \mathbf{n} c_f = (q_n + c_f)c$$

because $\mathbf{n} \cdot \mathbf{B}_\perp = 0$. For the same reason the last three entries of r_5 comprising the vector $\mathbf{m_f}$ yield

$$(\bar{\mathbf{B}}\mathbf{n} \cdot \mathbf{n} - \bar{B}_n\mathbf{n})c_f + \bar{B}_n^2\frac{c_f}{c_f^2 - \bar{B}_n^2}\bar{\mathbf{B}}_\perp + q_n\frac{c_f^2}{c_f^2 - \bar{B}_n^2}\bar{\mathbf{B}}_\perp = (q_n + c_f)\mathbf{m_f}$$

The second to the fifth entries of r_5 comprising the vector $\mathbf{l_f}$ yield

$$c^2\mathbf{n} + q_n c_f \mathbf{n} - q_n \bar{B}_n \bar{\mathbf{B}}_\perp \frac{c_f}{c_f^2 - \bar{B}_n^2} + (\mathbf{n}\bar{\mathbf{B}} \cdot \bar{\mathbf{B}}_\perp - \bar{B}_n \bar{\mathbf{B}}_\perp)\frac{c_f^2}{c_f^2 - \bar{B}_n^2}$$

$$= (q_n + c_f)\left(c_f\mathbf{n} - \bar{B}_n\bar{\mathbf{B}}_\perp \frac{c_f}{c_f^2 - \bar{B}_n^2}\right) + \bar{B}_n\mathbf{B}_\perp \frac{c_f^2}{c_f^2 - \bar{B}_n^2} + (c^2 - c_f^2)\mathbf{n} + \quad (12)$$

$$\mathbf{n}\bar{\mathbf{B}} \cdot (\bar{\mathbf{B}} - \bar{B}_n\mathbf{n})\frac{c_f^2}{c_f^2 - \bar{B}_n^2} - \bar{B}_n\bar{\mathbf{B}}_\perp \frac{c_f^2}{c_f^2 - \bar{B}_n^2}$$

$$= (q_n + c_f)\mathbf{l_f} + \left\{(c^2 - c_f^2)(c_f^2 - \bar{B}_n^2) + \bar{B}^2 - \bar{B}_n^2)c_f^2\right\} \frac{\mathbf{n}}{c_f^2 - \bar{B}_n^2} \quad (13)$$

$$= (q_n + c_f)\mathbf{l_f} - \left\{c_f^4 - (c^2 + \bar{B}^2)c_f^2 + c^2\bar{B}_n^2\right\} \frac{\mathbf{n}}{c_f^2 - \bar{B}_n^2} \quad (14)$$

where the last term vanishes because c_f^2 is a root of the bracketed quadratic expression. The verification of r_6, r_7 and r_8 is similar.

Now the eigenvector matrix R with the eigenvectors r_k as its columns satisfies

$$R^T R = RR^T = I$$

and

$$R^T \bar{A} R = \Lambda, \quad \bar{A} = R\Lambda R^T$$

where the diagonal matrix Λ has the eigenvalues as its elements.

$$\Lambda = diag\left\{u_n, u_n, u_n + \bar{B}_n, u_n - \bar{B}_n, u_n + c_f, u_n - c_f, u_n + c_s, u_n - c_s\right\}$$

Finally A can be expressed as

$$A = \bar{M}\bar{A}\bar{M}^{-1} = \bar{M}R\Lambda R^T\bar{M}^{-1}$$

References

1. Kenneth G. Powell, *An Approximate Riemann Solver for Magnetohydrodynamics (That Works in More than One Dimension)*, NASA CR-194902, ICASE Report No. 94-24, April 1994.

2. Kenneth G. Powell, P.L. Roe, R.S. Myong, T. Gombosi and D. De Zeenw, *An Upwind Scheme for Magnetohydrodynamics*, AIAA-95-1704-CP.

3. P.L. Roe and D.S. Balsara, *Notes on the EigenSystem of Magnetohydrodynamics*, SIAM Journal of Applied Mathematics, Vol. 56, No. 1., pp. 57-67, Feb. 1996.