Numerical Computation of Transonic Flows with Shock Waves*

Antony Jameson

Courant Institute of Mathematical Sciences
New York University

Summary

Some recent developments in numerical methods for calcuting solutions to the transonic potential flow equation are reviewed, including (1) the construction of stable coordinate independent difference schemes; (2) the use of conservatiation form to insure proper shock jump conditions; (3) analysis of the relaxation method by the time dependent analogy; (4) accelerated iterative schemes.

1 Introduction

Finite difference methods have been quite widely used for the calculation of transonic flows. The purpose of this paper is to review some recent improvements in these methods. The discussion will be limited to the solution of the transonic potential flow equation, which may be derived from the Euler equations for the inviscid compressible flow introducing the assumption that the flow is irrotational, so that a velocity potential ϕ can be defined. Shock waves can be expected to

*Work supported by NASA under Grants 33-016-167 and NGR 33-016-201. Computations performed at the ERDA Mathematics and Computing Laboratory, New York University, under Contract E(11-1)-3077.

appear unless the profile is specially designed to prevent their formation [1]. Since an irrotational flow is isentropic, the introduction of a potential implies that shock waves are to be replaced by discontinuities across which the entropy is conserved.

Consider a two dimensional flow past a profile, Let u, v and q be the velocity components and speed

$$u = \phi_x \quad v = \phi_y \quad q = \sqrt{u^2 + v^2} \tag{1.1}$$

and let a be the local speed of sound. Then ϕ is to be determined from the quasilinear equation

$$\left(a^2 - u^2\right)\phi_{xx} - 2uv\phi_{xy} + \left(a^2 - v^2\right)\phi_{yy} = 0 \tag{1.2}$$

which is hyperbolic if the local Mach number $M = q/a > 1$, and elliptic if $M < 1$. At the profile the solution should satisfy the Neumann boundary condition

$$\frac{\partial \phi}{\partial n} = 0 \tag{1.3}$$

where n denotes the normal direction. At infinity the solution should approach a uniform free stream with a speed q_∞ and Mach number M_∞. The local speed of sound can be determined from the energy equation

$$a^2 + \frac{\gamma - 1}{2} q^2 = \left(\frac{1}{M_\infty^2} + \frac{\gamma - 1}{2}\right) q_\infty^2 \tag{1.4}$$

where γ is the ratio of specific heats. The density ρ and pressure p follow from the relations

$$\rho^{\gamma - 1} = M_\infty^2 a^2 \tag{1.5}$$

and

$$p = \frac{\rho^\gamma}{\gamma M_\infty^2} \tag{1.6}$$

In the absence of discontunities equation (1.2) implies the conservation of both mass and

momentum. Multiplied by ρ/a^2 it can be reduced to the equation for conservation of mass

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0 \tag{1.7}$$

Multiplied by $\rho u/a^2$ or $\rho v/a^2$, on the other hand, it can be reduced to the equations for conservation of the x or y components of momentum. In an isentropic flow in which the energy is conserved it is not possible, however, to conserve both mass and momentum across a discontinuity. The approximation to a shock wave preferred here is one in which the mass is conserved. The moment deficiency then provides an estimate of the wave drag [2]. Thus the desired solution should have the properties that ϕ is continuous, and that the velocity components are piecewise continuous, satisfying the conservation law (1.7) at points where the flow is smooth, together with the jump condition

$$[\rho v] - \frac{dy}{dx}[\rho u] = 0 \tag{1.8}$$

across a discontinuity, where [ ] denotes the jump, and dy/dx is the slope of the discontinuity. That is to say ϕ should be a weak solution [3] of the conservation law (1.7), satisfying the condition that

$$\iint \rho\left(\phi_x w_x + \phi_y w_y\right) dx\, dy = 0 \tag{1.9}$$

for any smooth test function w which vanishes in the far field. In a finite region the conservation law (1.7) is equivalent to the Bateman variational principle that

$$I = \iint p\, dx\, dy \tag{1.10}$$

is stationary [4].

The solution is to be determined by constructing a finite difference approximation to the differential equation. Since the quasilinear form (1.2) does not distinguish between conservation of mass and momentum, difference approximations to (1.2) will not necessarily yield a solution which

satisfies the jump condition (1.8) unless shock waves are detected and special difference formulas are used in their vicinity. If we treat the conservation law (1.7), on the other hand, and preserve the conservation form in the difference approximation, then we can expect the jump condition (1.8) to be satisfied in the limit as the mesh width tends to zero. The method of constructing difference approximations to equations (1.2) or (1.7) is based on the idea, first introduced by Murman and Cole, of using central difference formulas in the subsonic zone and upwind difference formulas in the supersonic zone [5,6]. This gives the numerical scheme a directional bias which corresponds to the upwind region of dependence in the supersonic zone. If no directional bias were introduced in the numerical scheme, any solution which could be computed for the transonic flow past a body with fore and aft symmetry such as an ellipse, would be symmetric under a reversal of the flow direction, and must therefore contain improper discontinuities.

2 Rotated Difference Scheme for the Quasilinear Form

The construction of an upwind difference scheme for the transonic potential flow equation (1.2) is complicated by the fact that the local flow direction is not known in advance and may vary substantially. The simplest approach is to rely on the use of a coordinate system such that one set of coordinate lines is roughly aligned with the flow in the supersonic zone. Then the upwind differencing can be restricted to one coordinate, as in the case of the small disturbance equation. A curvilinear coordinate system matched to the profile can be used to ensure good alignment in the region near the body. Schemes of this type have proved quite succesful in the treatment of flows with small supersonic zones [7,8].

The treatment of flows with large supersonic zones in a curvilinear coordinate system suited to the geometry of the problem requires the use of a more elaborate difference scheme, in which the direction of upwind differencing is independent of the coordinate system, and is instead rotated to conform with the local flow direction [9,10]. To illustrate the construction of such a scheme

consider the potential flow equation flow equation (1.2) in Cartesian coordinates. The required rotation of the upwind differencing at any particular point can be accomplished by introducing an auxiliary Cartesian coordinate system which is locally aligned with the flow at that point. If s and n denote the local streamwise and normal directions, then equation (1.2) becomes

$$\left(a^2 - q^2\right) \phi_{ss} + a^2 \phi_{nn} = 0 \tag{2.1}$$

Since u/q and v/q are the local direction cosines, ϕ_{ss} and ϕ_{nn} can be expressed in the original coordinate system as

$$\phi_{ss} = \frac{u^2 \phi_{xx} + 2uv\, \phi_{xy} + v^2 \phi_{yy}}{q^2} \tag{2.2}$$

and

$$\phi_{nn} = \frac{v^2 \phi_{xx} + 2uv\, \phi_{xy} + u^2 \phi_{yy}}{q^2} \tag{2.3}$$

Then at subsonic points central difference formulas are used for both ϕ_{ss} and ϕ_{nn}. At supersonic points central difference formulas are used for ϕ_{nn}, but upwind difference formulas are used for the second derivatives contributing to ϕ_{ss}. At a supersonic point at which $u > 0$ and $v > 0$, for example, ϕ_{ss} is constructed from the formulas

$$\begin{aligned} \phi_{xx} &= \frac{\phi_{ij} - 2\phi_{i-1,j} + \phi_{i-2,j}}{\Delta x^2} \\ \phi_{xy} &= \frac{\phi_{ij} - \phi_{i-1,j} - \phi_{i,j-1} + \phi_{i-1,j-1}}{\Delta x \Delta y} \\ \phi_{yy} &= \frac{\phi_{ij} - 2\phi_{i,j-1} + \phi_{i,j-2}}{\Delta y^2} \end{aligned} \tag{2.4}$$

It can be seen that the scheme reduces to a form simular to the scheme of Murman and Cole [5] for the small disturbance equation if either $u = 0$ or $v = 0$. The upwind difference formulas can be regarded as approximations to $\phi_{xx} - \Delta\phi_{xxx}, \phi_{xy} - (\Delta x/2)\, \phi_{xxy} - (\Delta y/2)\, \phi_{xyy}$ and $\phi_{yy} - \Delta y_{yyy}$.

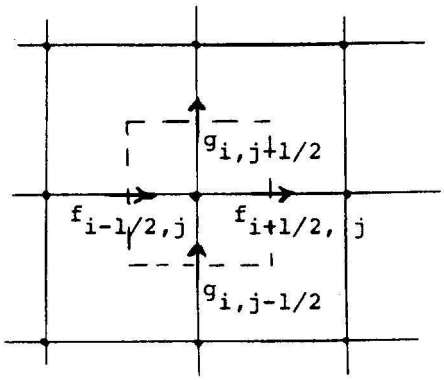


FIGURE 1
FLUX BALANCE FOR DIFFERENCE SCHEME IN CONSERVATION FORM

Thus at supersonic points the scheme introduces an effective artificial viscosity

$$\left(1-\frac{a^2}{q^2}\right)\left\{\Delta x\left(u^2u_{xx}+uvv_{xx}\right)+\Delta y\left(uvu_{yy}+v^2v_{yy}\right)\right\} \tag{2.5}$$

which is symmetric in x and y.

3 Difference Schemes in Conservation Form

The general method of constructing a difference approximation to a conservation law of the form

$$f_x+g_y=0$$

is to preserve the flux balance in each cell, as illustrated in Figure 1. This leads to a scheme of the form

$$\frac{F_{i+1/2,j}-F_{i-1/2,j}}{\Delta x}+\frac{G_{i,j+1/2}-G_{i,j-1/2}}{\Delta y}=0$$

where F and G should converge to f and g in the limit as the mesh width tends to zero. Then on multiplying by a test function w_{ij} and summing by parts, there results an approxiation to the integral (1.9). Thus the condition for a proper weak solution is satisfied [11]. Some latitude is allowed in the definition of F and G, since it is only necessary that $F=f+O\left(\Delta x\right)$ and

$G = g + O(\Delta x)$. In constructing a difference approximation to the conservation law (1.7) we can therefore introduce artificial viscosity of the form

$$\frac{\partial P}{\partial X} + \frac{\partial Q}{\partial y}$$

provided that P and Q are of the order Δx. Then the difference scheme is an approximation to the modified conservation law

$$\frac{\partial}{\partial x}(\rho u + P) + \frac{\partial}{\partial y}(\rho v + P) = 0$$

which reduces to the original conservation law (1.7) in the limit as the mesh width tends to zero. The desired upwind bias in the supersonic zone can be introduced by modelling the added terms on the artificial viscosity of the rotated difference scheme for the quasilinear form. Since equation (1.7) is equivalent to equation (1.2) multiplied by ρ/a^2, P and Q should be chosen so that $\partial P/\partial x + \partial Q/\partial y$ contains terms similar to equation (2.5) multiplied by ρ/a^2. A switch to an unbiased difference scheme in the subsonic zone can be accomplished by including in the definition of P and Q a switching function which vanishes in the subsonic zone.

The realization of these ideas can lead to a variety of differrence schemes which differ in detail. All have the general form

$$S_{ij} + T_{ij} = 0 \tag{3.1}$$

where S_{ij} is a central difference approximation to the left-hand side of equation (1.7), and T_{ij} is the artificial viscosity. The following scheme has proved successful [12]. The central difference approximation S_{ij} is constructed in the natural manner as

$$S_{ij} = \frac{(\rho u)_{i+1/2,j} - (\rho u)_{i-1/2,j}}{\Delta x} + \frac{(\rho v)_{i,j+1/2} - (\rho v)_{i,j-1/2}}{\Delta y} \tag{3.2}$$

To construct the viscosity a switching function μ which vanishes in the subsonic zone is defined

by the formula

$$\mu = \max\left\{0, \left(1 - \frac{a^2}{q^2}\right)\right\} \tag{3.3}$$

Then P and Q are defined as approximations to

$$-\mu\left\{(1-\epsilon)\,|u|\Delta x\,\rho_x + \epsilon u\,\Delta x^2 \rho_{xx}\right\}$$

and

$$-\mu\left\{(1-\epsilon)\,|v|\Delta y\,\rho_x + \epsilon v\,\Delta y^2 \rho_{yy}\right\}$$

where ϵ is a parameter controlling the accuracy. If $\epsilon = 1 - \lambda\,\Delta x$ and λ is a constant, the scheme is second order accurate. If $\epsilon = 0$, on the other hand, the scheme is first order accurate, and at a supersonic point where $u > 0$, $v > 0$, P then approximates

$$\Delta x\left(1 - \frac{a^2}{q^2}\right) u\rho_x = \Delta x\left(\frac{\rho}{a^2}\right)\left(1 - \frac{a^2}{q^2}\right)\left(u^2 u_x + uvv_x\right)$$

When this formula and the corresponding formula for Q are inserted in $\partial P/\partial x + \partial Q/\partial y$, it can be verified that the terms containing the highest derivatives of ϕ are the same as those in equation (2.5) multiplied by ρ/a^2. In the consruction of P and Q the derivatives of ρ are represented by upwind difference formulas. Thus the formula for the viscosity finally becomes

$$T_{ij} = \frac{\left(P_{i+1/2,j} - P_{i-1/2,j}\right)}{\Delta x} + \frac{\left(Q_{i,j+1/2} - Q_{i,j-1/2}\right)}{\Delta y} \tag{3.4}$$

where if $u_{i+1/2,j} > 0$

$$P_{i+1/2,j} = u_{i+1/2,j}\,\mu_{i,j}\left\{\rho_{i+1/2,j} - \rho_{i-1/2,j} - \epsilon\left(\rho_{i-1/2,j} - \rho_{i-3/2,j}\right)\right\} \tag{3.5a}$$

and if $u_{i+1/2,j} < 0$

$$P_{i+1/2,j} = u_{i+1/2,j}\,\mu_{i+1,j}\left\{\rho_{i+1/2,j} - \rho_{i+3/2,j} - \epsilon\left(\rho_{i+3/2,j} - \rho_{i+5/2,j}\right)\right\} \tag{3.5b}$$

while $Q_{i,j+1/2}$ is defined by a similar formula.

Formulas (3.2) and (3.5) call for the evaluation of the velocity components and density at the midpoint of each mesh interval. The precise method by which this is accomplished has been found to have little influence on the result. One method is to evaluate the velocity components by formulas such as

$$u_{i+1/2,j} = \frac{(\phi_{i+1,j} - \phi_{ij})}{\Delta x} \tag{3.6a}$$

and

$$v_{i+1/2,j} = \frac{(\phi_{i+1,j+1} + \phi_{i,j+1}) - (\phi_{i+1,j-1} + \phi_{i,j-1})}{4\Delta y} \tag{3.6b}$$

Then the density $\rho_{i+1/2,j}$ is determined from equation (1.5). With this scheme two densities $\rho_{i+1/2,j}$ and $\rho_{i,j+1/2}$ have to be evaluated for each mesh point.

An alternative method is to determine first the density $\rho_{i+1/2,j+1/2}$ at the center of each cell. For this purpose the squared velocity components are evaluated as

$$u^2_{i+1/2,j+1/2} = \frac{u^2_{i+1/2,j} + u^2_{i+1/2,j+1}}{2} \tag{3.7a}$$

and

$$v^2_{i+1/2,j+1/2} = \frac{v^2_{i,j+1/2} + v^2_{i+1,j+1/2}}{2} \tag{3.7b}$$

The densities $\rho_{i+1/2,j}$ and $\rho_{i,j+1/2}$ are then formed by averaging as

$$\rho_{i+1/2,j} = \frac{\rho_{i+1/2,j+1/2} + \rho_{i+1/2,j-1/2}}{2} \tag{3.8a}$$

and

$$\rho_{i,j+1/2} = \frac{\rho_{i+1/2,j+1/2} + \rho_{i-1/2,j+1/2}}{2} \tag{3.8b}$$

This method has the advantage that the formula (1.5), which requires the evaluation of a fractional power, has to be used to calculate only one density $\rho_{i+1/2,j+1/2}$ for each mesh point. In the case of a flow in a finite region it results in a formula for S_{ij} which could also be obtained from the Bateman variational principle. Let the integral (1.10) be approximated by

$$I = \sum_i \sum_j p_{i+1/2,j+1/2}\, \Delta x \Delta y$$

where $p_{i+1/2,j+1/2}$ is determined from $\rho_{i+1/2,j+1/2}$ by equation (1.6). The $\partial I/\partial \phi_{ij} = S_{ij}$, where the averaged densities are used in the formula (3.2) for S_{ij}, and the condition that I is stationary is expressed by $S_{ij} = 0$.

If the profile coincides with a coordinate line, the Neumann boundary condition (1.3) can be treated in a simple manner suggested by Murman[1]. Equation (1.3) is equivalent to requiring that $\rho v = 0$ at the boundary. The second term of equation (3.2) is therefore replaced by $2\left(\rho v\right)_{i,j+1/2}/\Delta y$, corresponding to the use of a one-sided difference formula for $\left(\partial/\partial y\right)\left(\rho v\right)$. When the densities are calculated by equations (3.7) and (3.8), this is again just the formula which would be obtained from the variational principle.

4 Solution by Relaxation

The introduction of one of the difference approximations of the two preceding sections produces a set of nonlinear difference equations. In devising an iterative scheme to solve these equations the error reduction in each cycle has to be balanced against the number of computer operations required to perform the cycle. Suppose that in $(n+1)^{st}$ cycle the residual R_{ij} at the point

[1]Private communication.

$i\,\Delta x, j\,\Delta y$ is evaluated by inserting the result $\phi_{ij}^{(n)}$ of the n^{th} cycle in the difference approximation. Then in the schemes to be considered, the correction $C_{ij} = \phi_{ij}^{(n+1)} - \phi_{ij}^{(n)}$ is calculated by solving an equation of the form

$$NC + \sigma R = 0 \tag{4.1}$$

Where N is a discrete linear operator, and σ is a scaling function.

The simplest schemes of this class are relaxation methods, in which N is restricted to a lower triangular or block triangular form, so that the elements of C can be determined sequentially. These methods are easy to program, and generally give reliable but slow convergence. A useful method of analyzing a relaxation scheme is to regard it as a discrete approximation to an artificial time dependent process, with the iterations as time steps. It was shown by Garabedian that this method can be used to estimate the best over-relaxation factor for an elliptic problem [13]. Let Δt be the time step at each iteration. Then the correction C is an approximation to $\Delta t\,\phi_t$. The vector R is an approximation to $L\phi$, where L is the differential operator appearing in the differential equation. Therefore, if we regard N as an approximation to a differential operator $(1/\Delta x)\,F$, equation (4.1) is an approximate to

$$F\phi_t + \sigma\frac{\Delta x}{\Delta t}L\phi = 0 \tag{4.2}$$

Thus N should be chosen so that this is a convergent time dependent process.

To illustrate the application of this consider the standard difference scheme for Laplace's equation. Typically in a point over-relaxation scheme a provisional calut $\tilde{\phi}_{ij}$ is calculated by solving

$$\frac{1}{\Delta x^2}\left(\phi_{i-1,j}^{(n+1)} - 2\tilde{\phi}_{ij} + \phi_{i+1,j}^{(n)}\right) + \frac{1}{\Delta y^2}\left(\phi_{i,j-1}^{(n+1)} - 2\tilde{\phi}_{ij} + \phi_{i,j+1}^{(n)}\right) = 0$$

Then the new value $\phi_{i+1,j}^{(n+1)}$ is determined by the formula

$$\phi_{i,j}^{(n+1)} = \phi_{i,j}^{(n)} + \omega\left(\tilde{\phi}_{ij} - \phi_{i,j}^{(n)}\right)$$

where ω is the over-relaxation factor. Eliminating $\tilde{\phi}_{ij}$, this is equivalent to calculating the correction C_{ij} by solving

$$\tau_1 \left(C_{ij} - C_{i-1,j}\right) + \tau_2 \left(C_{ij} - C_{i,j-1}\right) + \tau_3 C_{ij} = R_{ij} \tag{4.3}$$

Where R_{ij} is the residual and

$$\tau_1 = \frac{1}{\Delta x^2}, \qquad \tau_2 = \frac{1}{\Delta y^2}$$
$$\tau_3 = \left(\frac{2}{\omega} - 1\right)\left(\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2}\right)$$

Equation (4.3) is an approximation to the wave equation

$$\tau_1 \, \Delta t \, \Delta x \, \phi_{xt} + \tau_2 \, \Delta t \, \Delta y \, \phi_{yt} + \tau_3 \, \Delta t \, \phi_t = \phi_{xx} + \phi_{yy}$$

This is damped if $\tau_3 > 3$, and to maximize the rate of convergence the relaxation factor ω should be chosen to give an optimal amount of damping.

In the case of a rectangular region with periodic boundary conditions it is easily verified by a von Neumann test that schemes with

$$\tau_1 \geq \frac{1}{\Delta x^2} \quad , \qquad \tau_2 \geq \frac{1}{\Delta y^2} \quad , \qquad \tau_3 > 0$$

will converge. Suppose that

$$\phi^{(n)} = G^n \, e^{ipx} \, e^{iqy}$$

Where G is the growth factor. Then substituting this expression in equation (4.3) gives

$$G = \frac{\left(2 - \tau_1 \Delta x^2\right) A - \left(2 - \tau_2 \Delta y^2\right) B - iC}{\tau_1 \Delta x^2 A + \tau_2 \Delta y^2 B + \tau_3 + iC}$$

where

$$A = \frac{1 - \cos p\,\Delta x}{\Delta x^2} \geq 0\ , \quad B = \frac{1 - \cos q\,\Delta y}{\Delta y^2} \geq 0\ ,$$

and

$$C = \tau_1 \sin p\,\Delta x + \tau_2 \sin q\,\Delta y.$$

If we consider the potential flow equation (1.2) at a subsonic point, these considerations suggest that the scheme (4.3), where the residual R_{ij} is evaluated from the difference approximation described in Section 2, will converge if

$$\tau_1 \geq \frac{a^2 - u^2}{\Delta x^2}\ , \quad \tau_2 \geq \frac{a^2 - v^2}{\Delta y^2}\ , \quad \tau_3 > 0.$$

Similarly the scheme

$$\tau_1 \left(C_{ij} - C_{i-1,j}\right) - \tau_2 \left(C_{i,j+1} - 2C_{ij} + C_{i,j-1}\right) + \tau_3 C_{ij} = R_{ij} \tag{4.4}$$

which requires the simultaneous solution of the corrections on each vertical line, can be expected to converge if

$$\tau_1 \geq \frac{a^2 - u^2}{\Delta x^2}\ , \quad \tau_2 = \frac{a^2 - v^2}{\Delta y^2}\ , \quad \tau_3 > 0.$$

At supersonic points schemes smilar to (4.3) or (4.4) are not necessarily convergent [9]. If we introduce a locally aligned Cartesian coordinate system and divide through by a^2, the general form of the equivalent time dependent equation is

$$\left(M^2 - 1\right) \phi_{ss} - \phi_{nn} + 1\alpha\phi_{st} + 2\beta\phi_{nt} + \gamma\phi_t = 0 \tag{4.5}$$

Where M is the local Mach number, and s and n are the streamwise and normal directions. The coefficients α, β and γ depend on the coefficients of the elements of C on the left-hand side of

(4.3) or (4.4). The substitution

$$T = t - \frac{\alpha s}{M^2 - 1} + \beta n$$

reduces this equation to the diagonal form

$$\left(M^2 - 1\right) \phi_{ss} - \phi_{nn} - \left(\frac{\alpha^2}{M^2 - 1} - \beta^2\right) \phi_{TT} + \gamma \phi_T = 0$$

Since the coefficients of ϕ_{nn} and ϕ_{ss} have opposite signs when $M > 1$, T cannot be the timelike direction at a supersonic point. Instead either s or n is timelike depending on the sign of the coefficient ϕ_{TT}. Since s is the timelike direction of the steady state problem, it ought also to be the timelike direction of the unsteady problem. Thus when $M > 1$ the relaxation scheme should be designed so that α and β satisfy the compatibility condition

$$\alpha > \beta \sqrt{M^2 - 1} \tag{4.6}$$

The characteristic cone of equation (4.5) touches the $s - n$ plane. As long as condition (4.6) holds with $\alpha > 0$ and $\beta > 0$, it slants upstream in the reverse time direction as illustrated in Figure 2. To ensure that the iterative scheme has the proper region of dependence, the flow field should be swept in a direction such that the updated region always includes the upwind line of tangency between the characteristic cone and the $s - n$ plane. The characteristic cone at a supersonic point does not contain the t axis. Thus if the coefficients of equation (4.5) were constant with $M > 1$, the region of dependence would cease to intersect the initial data after a sufficient time interval. Instead it would intersect a surface containing the Cauchy data of the steady state problem. Consequently no damping term $\gamma \phi_t$ is required for the convergence in the supersonic zone. In fact a von Neumann test based on the local values of the coefficients indicates that the coefficient γ should be zero when $M > 1$ [9].

Relaxation schemes which are dereived simply by substituting the latest available values of ϕ in the difference equations do not necessarily satisfy the compatibility condition (4.6) at supersonic

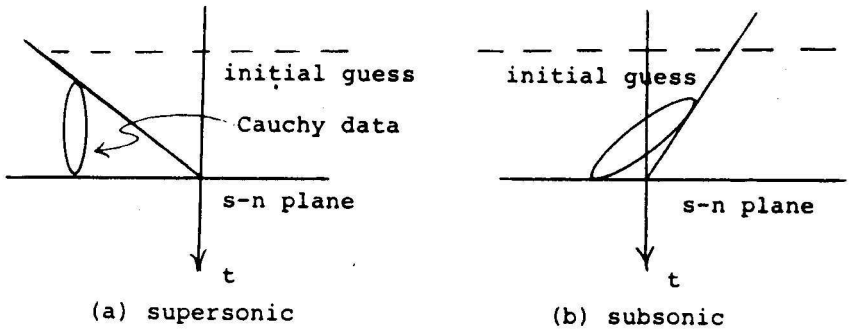


Figure 2
Characteristic cone of equivalent time dependent equation

points. The following line relaxation scheme has proved successful in the calculation of a wide range of flows, including flows with supersonic free streams. At subsonic points the correction C_{ij} is determined by solving an equation similar to (4.4). At supersonic points the equation for the correction is derived by substituting in the upwind difference formulas for ϕ_{ss} a combination of new and old values such that the operator N is diagonally dominant. If $u > 0$, ϕ_{xx} is represented by

$$\frac{2\phi_{ij}^{(n+1)} - \phi_{ij}^{(n)} - 2\phi_{i-1,j}^{(n+1)} + \phi_{i-2,j}^{(n)}}{\Delta x^2}$$

This is an approximation to $\phi_{xx} - 2\left(\Delta t/\Delta x\right)\phi_{xt}$. Similar formulas are used for ϕ_{xy} and ϕ_{yy}. Thus the approximation to ϕ_{ss} introduces a term

$$2\left(M^2-1\right)\left\{(u/q)\,/\,(\Delta t/\Delta x) + (v/q)\,(\Delta t/\Delta y)\right\}\phi_{st}$$

in the equivalent time dependent equation. To make sure that (4.6) is satisfied when the local Mach number is close to unity, the coefficient of ϕ_{st} is further augmented by adding an upwind approximation to ϕ_{st} at both supersonic and subsonic points. If $u > 0$ and $v > 0$ this term is

$$\omega_s \left\{ \frac{u}{\Delta x}\left(C_{ij} - C_{i-1,j}\right) + \frac{v}{\Delta y}\left(C_{ij} - C_{i,j-1}\right) \right\}$$

where ω_s is relaxation factor with a value ≥ 0. The best rate of convergence is obtained by using

the smallest possible value of ω_s, and in fact it often suffices to $\omega_s = 0$.

The same ideas can be applied to the solution of the difference equations which result from approximations to conservation form (1.7). Since the conservation form is equivalent to the quasi-linear form multiplied by ρ/a^2, we can produce a time dependent process for the conservation form which converges at about the same rate as the process for the quasilinear form simply by multiplying the operator N by ρ/a^2 [12]. This procedure has the advantage that the iterative scheme does not have to be modified to reflect detailed variations in the difference equations. The same operator N can be used for all values of the viscosity parameter ϵ in equation (3.5), for example, since the dominant terms of the equivalent time dependent equation are independent of ϵ.

5 Fast Iterative Scheme

The iterative scheme will converge rapidly if the operator N in equation (4.1) is a good approximation to the operator used to evaluate the residual R. If N is restricted to a triangular form, however, as in a relaxation method, it cannot be a very good approximation. Thus we can expect to improve the rate of convergence by introducing more general forms for N which allow the approximation to be improved, but still produce a set of equations which can easily be solved at each cycle. One approach is to generate N as a product of factors

$$N = P_1 \, P_2 \, \cdots \, P_n$$

each of which is easily invertible. The transonic small disturbance equation can be solved by a rapidly convergent alternating direction method constructed along these lines [14].

Another approach takes advantage of the fact that fast methods are available for inverting the discret Laplacian [15,16]. The potential equation can be scaled so that the Laplacian represents

its linear part by dividing the quasilinear form (1.2) by a^2 or the conservation form (1.7) by ρ. This suggests the use of an iteration in which N is the Laplacian, so that at each cycle we have to solve a discrete Poisson's equation with the nonlinear terms as forcing terms. A scheme of this type was proposed for the small disturbance equation by Martin and Lomax [17]. A similar procedure has also been used by Periaux for subsonic flow calculations using the finite element method [18].

In order to estimate the rate of convergence which might be obtained consider the linearized small disturbance equation

$$\left(1 - M_\infty^2\right) \phi_{xx} + \phi_{yy} = 0 \tag{5.1}$$

with $M_\infty < 1$. If we take $\sigma = 1$, equation (4.1) becomes a central difference approximation to

$$\phi_{xx}^{(n+1)} + \phi_{yy}^{(n+1)} = M_\infty^2 \phi_{xx}^n$$

where $\phi^{(n)}$ is the result of the n^{th} cycle. In the case of a rectangular region with periodic boundary conditions we can estimate the amplification factor G by setting

$$\phi^{(n)} = G^n e^{ipx} e^{iqy} \tag{5.2}$$

Then we find that

$$G = M_\infty^2 \frac{\sin^2\left(p\Delta x/2\right)}{\sin^2\left(p\Delta x/2\right) + \left(\Delta x/\Delta y\right)^2 \sin^2\left(q\Delta y/2\right)}$$

Thus the error should be reduced by a factor bounded by M_∞^2 at every cycle, independent of the mesh size, leading to very rapid convergence in subsonic flow.

A similar argument suggests that the poisson scheme will not converge in supersonic flow. Consider equation (5.1) with $M_\infty > 1$, and suppose that an upwind difference formula is used to evaluate ϕ_{xx} in calculating the residual, while N is still a central difference approximation to the

Laplacian. Then the substitution (5.2) results in the formula

$$G = \frac{\left\{(M^2-1)\, e^{ip\Delta x} + 1\right\} \sin^2(p\Delta x/2)}{\sin^2(p\Delta x/2) + (\Delta x/\Delta y)^2 \sin^2(q\Delta y/2)}$$

for the growth factor. if we consider a harmonic with a low frequency in y and a moderate frequency in x, we find thet $|G|$ exceeds unity, indicating divergence. This conclusion is confirmed by an analysis which includes the effect of the boundary conditions [19].

One method of stabilizing the scheme for transonic flow calculations is to desymmetrize the operator N by adding an upwind approximation to $(\alpha/\Delta x)\,\partial/\partial x$, where α is a sufficiently large positive coefficient. As long as α is a function of x only, independent of the local type of the flow, the equations can still be solved by a fast method such as the Buneman algorithm [15]. Martin has reported good results for calculations in which the transonic small disturbance theory was formulated as a system of first order equations, and an iterative scheme of this type was used to obtain the solution of the resulting difference equations [20].

An alternative method is to use the Poisson scheme in combination with some other method designed to remove the errors from the supersonic zone. The relaxation method is effective for this purpose [19,21]. It turns out that if a suffucient number of relaxation sweeps is used after each Poisson step the combined scheme converges in both the supersonic and subsonic zones, and usually at a rate much faster than can be obtained by using relaxation sweeps alone. In such a scheme, which can easily be applied to the transonic potential flow equation in either quasilinear or conservation form, the Poisson steps are the principle source of convergence in the subsonic zone, while the relaxation sweeps are dominant in the supersonic zone. The best number p of relaxation sweeps to be used after each Poisson step is most easily determined by numerical experiemnts. Typically, when the rotated difference schemes of Section 2 and 3 are used, the best rate of convergence is obtained with $p \sim 5 - 8$.

6 Mapped Coordinate System

The choice of a coordinate system can have a substantial influence on the accuracy of the result. A favorable coordinate system for the treatment of the flow past a two dimensional profile can be generated by mapping the exterior of the profile conformally onto the interior of a unit circle [7,8,9]. The introduction of polar coordinates r and θ in the circle then leads to a regular and finite mesh, in which the profile becomes the coordinate line $r = 1$. Thus the Neumann boundary condition can be simply represented in the manner indicated in Section 3. Also, since the Laplacian is invariant under a conformal transformation, we can use a fast solver for Poisson's equation in polar coordinates to perform the Poisson steps of the fast iterative method proposed in Section 5.

The far field boundary condition has to be applied at $r = 0$, where the potential becomes infinite. This singularity can be removed by defining a reduced potential

$$G = \phi - \frac{\cos(\theta + \alpha)}{r} + E(\theta + \alpha) \tag{6.1}$$

where $2\pi E$ is the circulation, and α is the angle of attack. Then G is finite and single valued. The modulus of the mapping function also becomes infinite at $r = 0$, and the use of finite difference formulas to represent derivatives of quantities depending on the mapping function can lead to large errors. Thus it is best to calculate the mapping to the exterior of the circle, and to perform an explicit inversion to the interior of the circle. Equation (1.2) then becomes

$$\begin{aligned}\left(a^2 - u^2\right) G_{\theta\theta} - 2uvrG_{r\theta} + \left(a^2 - v^2\right) r\frac{\partial}{\partial r}\left(rG_r\right) \\ - 2uv\left(G_\theta - E\right) + \left(u^2 - v^2\right) rG_r + \left(u^2 - v^2\right)\left(\frac{u}{r}H_\theta + vH_r\right) = 0\end{aligned} \tag{6.2}$$

where H is the modulus of the derivative of the transformation to the exterior of the circle, and

u and v are the velocity components in the θ and r directions.

$$u = \frac{r\,(G_\theta - E) - \sin(\theta + \alpha)}{H}\,, \quad v = \frac{r^2 G_r - \cos(\theta + \alpha)}{H} \tag{6.3}$$

The conservation form (1.7) becomes

$$\frac{\partial}{\partial\theta}\left(\rho\frac{Hu}{r}\right) + r\frac{\partial}{\partial r}\left(\rho\frac{Hv}{r}\right) = 0 \tag{6.4}$$

The Neumann boundary condition (1.3) reduces to

$$G_r = \cos(\theta + \alpha) \quad \text{at} \quad r = 1$$

while the far field boundary condition becomes

$$G = E\left\{\theta + \alpha - \arctan\left[\sqrt{1 - M_\infty^2}\,\tan(\theta + \alpha)\right]\right\} \quad \text{at} \quad r = 0$$

Finally the circulation constant E is determined by the Kutta condition, which requires the velocity to be finite at the trailing edge, where $H = 0$. Thus ϕ_θ must also vanish, giving

$$E = G_\theta - \sin\alpha \quad \text{at} \quad r = 1, \quad \theta = 0.$$

The derivative of the mapping function can be represented as a power series. Let $z = x + iy$ and $\sigma = (1/r)\,e^{-i\theta}$ be corresponding points exterior to the profile and the unit circle. Then $H = |dz/d\sigma|$ where, if ϵ is the included angle at the trailing edge and we use a series with N terms, we can set

$$\frac{dz}{d\sigma} = \left(1 - \frac{1}{\sigma}\right)^{1-\epsilon/\pi} \exp\sum_{n=0}^{N}\frac{c_n}{\sigma^n} \tag{6.5}$$

This method of representing the transformation has the advantage that it allows a profile with an

open tail to be mapped to a closed circle. Expanding (6.5), the coefficient of $1/\sigma$ is

$$\tilde{c} = (c_1 - 1 + \epsilon/\pi) \exp(c_0) \tag{6.6}$$

Then according to the Cauchy integral theorem, integration of $dz/d\sigma$ around any closed curve exterior to the unit circle in the σ plane results in a fixed gap

$$z_2 - z_1 = 2\pi i \tilde{c} \tag{6.7}$$

This gap can be used to represent a wake of constant thickness.

The mapping coefficients can be calculated by a simple iterative procedure [22]. Let β and s be the tangent angle and arc length of the profile, and let $c_n = a_n - ib_n$. Then, seperating the real and imaginary parts of (6.5) when $r = 1$, we obtain

$$\log \frac{ds}{d\theta} - \left(1 - \frac{\epsilon}{\pi}\right) \log \left(2 \sin \frac{\theta}{2}\right) = \sum_{n=0}^{N} (a_n \cos n\theta + b_n \sin n\theta) \tag{6.8}$$

and

$$\beta + \theta + \frac{\pi}{2} - \left(1 - \frac{\epsilon}{\pi}\right) \frac{\theta - \pi}{2} = \sum_{n=0}^{N} (a_n \sin n\theta - b_n \cos n\theta) \tag{6.9}$$

$\beta(s)$ is known from the geometry of the profile. Given an estimate $s(\theta)$ of the arc length as a function of the angle θ in the circle plane, we can therefore calculate the Fourier coefficients a_n and b_n from equation (6.9), and hence we can construct the conjugate Fourier series (6.8). The resulting value of $ds/d\theta$ can be integrated to provide a new estimate $s(\theta)$, and the process can then be repeated. The iterations usually converge quite rapidly, provided that the closure condition (6.7) is used to freeze the coefficient c_1. It is convenient to use a series with K terms to represent the mapping function at $2K$ equally spaced mesh points around the circle. Following a suggestion of Ives,[2] the Fourier coefficients can then be evaluated with the aid of the fast Fourier transform,

[2]Private communication

with the result that the number of operations required to perform each iteration can be reduced to $O(K \log K)$.

A mapping method similar to this was proposed by Timman [23]. In its present form the method has proved fast and accurate in numerous calculations [8], allowing the construction of the coordinate system in a time which is negligible compared with the time required for the subsequent calculation of the flow by relaxation.

7 Results of Numerical Calculations

Some typical results are presented in Figures 3-9. Each figure shows the surface pressure distribution, and the lift and drag coefficients. The critical pressure at which the flow has sonic velocity is marked by a horizontally line on the axis. The solution is usually obtained first on a coarse grid. The mesh width is then halved in each coordinate direction, and the interpolated result of the calculation on the previous grid is used to provide the initial guess for the calculation on the new grid. This procedure, which may be repeated more than once, leads to substantial savings in the number of cycles required for convergence on the fine mesh. It also serves to provide an indication of the sensitivity of the numerical result to variations in the mesh width of the grid.

Figures 3 and 4 show the flow past an NACA 0012 airfoil predicted first by the quasilinear and then by the conservation form. Both calculations were performed on a grid with 128 cells in the θ direction and 32 cells in the r direction. The viscosity parameter ϵ was set equal to zero in the calculation in conservation form, giving first order accuracy. The difference in the two results is similar to that reported by Murman when a nonconservative scheme for the small disturbance equation was replaced by a conservative scheme [6]. The jump at a normal shock wave is consistently underestimated by calculations which do not use conservation form.

Figures 5 and 6 show the effect of the mesh width in two different cases. Both calculations were performed in conservation form on the same sequence of grids, with successively $64 \times 16, 128 \times 32$

and 256×64 cells. The viscosity parameter ϵ was zero. The result for the first, an NACA 64A410 airfoil, can be seen to be rather insensitive to the mesh width, but the result for the second case, a symmetric airfoil designed by Murman and Hicks to produce a low wave drag at mach .80 [24], shows an over-expansion on the coarse grid. A similar effect has been observed in other calculations in which shock free or nearly shock free flows have been treated in conservation form, and it forces the use of a very fine grid to produce an accurate result. This difficulty has not been experienced with the quasilinear form.

The accuracy can generally be improved by using values of the viscosity parameter ϵ close to unity. In the easier cases it is possible to set $\epsilon = 1$, giving second order accuracy, but this can lead to divergence when a sensitive flow has to be calculated. Figure 7 shows the result of a calculation of a shock free flow. This calculation was performed in conservation form on a 192×32 grid with $\epsilon = 1$, and the result is in excellent agreement with the design calculation for this airfoil [25]. A perfectly smooth recompression was not obtained when the same case was calculated with $\epsilon = 0$.

The rate of convergence of the iterative method can be measured by the absolute value of the largest residual anywhere in the field. Another convenient measure is the largest correction to the potential in a relaxation sweep. If the residual is normalized by multiplying by $\Delta\theta^2$, these two quantities are usually of the same order of magnitude. When the solution is obtained by relaxation, rather a large number of cycles are needed to reduce the errors to acceptable levels. The errors can be reduced quite rapidly, however, with the aid of the fast iterative method of Section 5. The results shown in Figures 3-6 were obtained by this method, using the Buneman algorithm [15] to perform the Poisson steps. All the calculations were continued until the largest normalized residual was less than 10^{-9}. The fast iterative method is particularly effective when the flow is predicted by the quasilinear form. To produce the result shown in Figure 3, for example, a preliminary calculation was performed on a 64×16 grid. Then 14 cycles, each consisting of a Poisson step followed by 6 relaxation sweeps, were sufficient to reduce the largest residual from $.22 \times 10^{-2}$ to $.80 \times 10^{-9}$ on the 128×32 grid. Each Poisson step takes about the same amount of

time as 2 relaxation sweeps, so in this case the total time required to complete the calculation on the fine mesh was about the same as the time it would take to perform 112 relaxation sweeps. The convergence was slower in the calculation of the corresponding result in conservation form, shown in Figure 4. Using 8 relaxation sweeps after each Poisson step, 57 cycles were required to reduce the largest residual from $.39 \times 10^{-2}$ to $.92 \times 10^{-9}$ on the 128×32 grid. It is hardly practicable to reduce the errors to these levels by relaxation alone. Typically it takes more than 4000 relaxation sweeps to reduce the largest residual to 10^{-9} on a 128×32 grid.

In order to simulate experimental results, it is important to allow for viscous effects in the boundary layer. It has been found that quite good agreement can be obtained simply by adding the displacement thickness of the boundary layer to the profile [26,27]. This procedure is effective in the regime where the shock waves are not strong enough to seperate the flow. Since the growth of the boundary layer, which is assumed to be turbulent downstream of a specified transition point, depends on the pressure distribution, the calculation of the boundary later correction is included in the iterative scheme. After a certain number of cycles of the flow calculation, the boundary layer displacement thikness is calculated using the current estimate of the presure distribution. The profile is then modified, and the mapping function is recalculated by the fast method of Section 6 before continuing the flow calculation. The whole process is repeated until the flow field and boundary layer have both converged. Results obtained by this procedure are compared with experimental data in Figures 8 and 9. These calculations were performed by Frances Bauerm, using conservation form on a 160×30 grid. Tunnel wall effects were not included in the calculations, but in each case the angle of attack was adjusted so that the calculated lift coefficient matched the experimental lift coefficient. Figure 8 shows the flow past an airfoil designed by Garabedian to produce a shock free flow at Mach .75 and a lift coefficient of .7 [26, p. 102]. The figure shows an off-design condition, with a shock wave which is quite well simulated by the calculation. Figure 9 shows a supercritical airfoil designed by Whitcomb [28]. In this case the calculation required a Mach number correction to move a shock wave forward to the location observed in the

experiment. When the same case was calculated using the quasilinear form, the location of the shock wave was correctly predicted without a Mach number correction, but the shock jump was less well simulated.

8 Conclusion

Relaxation methods have now been developed to point where they can be used to make reliable predictions of transonic flows. It appears that flows which shock waves of moderate strength can be quite well simulated by calculations which introduce the potential flow approximation. Difference approximations in conservation form give a better representation of shock waves than schemes in quasilinear form, and should converge to the correct jump conditions in the limit as the mesh width is reduced to zero. The schemes described in Section 3 have the disadvantage, however, of exhibiting rather large discretization errors on coarse grids. The soloution of the difference equations can be calculated quite rapidly by the fast iterative method of Section 5. Calculations in conservation form require more time than calculations in quasilinear form. Thus the use of shock fitting in combination with the quasilinear form would be attractive alternative if a sufficiently reliable shock fitting scheme could be devised.

References

[1] Morawetz, C. S.: On the nonexistence of continuous flows past profiles, Comm. Pure Appl. Math. 9 (1956) 445-468.

[2] Steger, J. L.; Baldwin, B. S.: Shock waves and drag in the numerical calculation of isentropic transonic flow, NASA TN D-6997, 1972.

[3] Lax, Peter D.: Weak solutions of nonlinear hyperbolic equations and their numerical computation, Comm. Pure Appl. Math. 7 (1954) 159-193.

[4] Bateman, H.: Notes on a differential equation which occurs in the two dimensional motion of compressible fluid and the associated variational problem, Proc. Roy. Soc. Series A. 125 (1929) 598-618.

[5] Murman, E. M.; Cole, J. D.: Calculation of plane steady transonic flows, AAIA Journal 9 (1971) 114-121.

[6] Murman, Earll M.: Analysis of embedded shock waves calculated by relaxation methods, AAIA Conference on Computational Fluid Dynamics, Palm Springs, July 1973.

[7] Garabedian, P. R.; Korn, D. G.: Analycis of transonic airfoils, Comm. Pure Appl. Math. 24 (1972) 841-851.

[8] Jameson, Antony: Transonic flow calculations for airfoils and bodies of revolution, Grumman Aerodynamics Report 391-71-1 (December 1971).

[9] Jameson, Antony: Iterative solution of transonic flows over airfoils and wings, including flows at Mach 1, Comm. Pure Appl. Math. 27 (1974) 283-309.

[10] South, J. C.; Jameson, A.: Relaxation solutions for inviscid axisymmetric flow over blunt or pointed bodies, AIAA Conference on Computational Fluid Dynamics, Palm Springs (July 1973).

[11] Lax, Peter; Wendroff, Burton: Systems of conservation laws, Comm. Pure Appl. Math. 13 (1960) 217-237.

[12] Jameson, Antony: Transonic potential flow calculations using conservation form, Second AIAA Conference on Computational Fluid Dynamics, Hartford, (June 1975).

[13] Garabedian, P. R.: Estimation of the relaxation factor for small mesh size, Math. Tables Aids Comp. 10 (1956) 183-185.

[14] Jameson, Antony: An alternating direction method for the solution of the transonic small disturbance equation, New York Univ., ERDA Report COO-3077-96 (1975).

[15] Buzbee, B. L.; Golub, G. H.; Nielsen, C. W.: On direct methods of solving Poisson's equation, SIAM J. Numerical Analysis 7 (1970) 627-656.

[16] Fischer, D.; Golub, G.; Hald, O.; Leiva, C.; Widlund, O.: On Fourier Towplitz methods for separable elliptic problems, Math. Comp. 28 (1974) 349-368.

[17] Martin, E. Dale; Lomax, Harvard: Rapid finite difference computation of subsonic and transonic aerodynamics flows, AIAA Paper 74-11 (1974).

[18] Periaux, J.: Calcul tridimensionnel de fluides compressibles par la methode des elements finis, 10^e Colloque d'Aerodynamique Appliquee, Lille (November 1973).

[19] Jameson, Antony: Accelerated iteration schemes for transonic flow calculations using fasr Poisson solvers, New York Univ. ERDA Report COO-3077-82 (March 1975).

[20] Martin, E. Dale: A fast semi-direct method for computing transonic aerodynamic flows, Second AIAA Computational Fluid Dynamics Conference, Hartford (June 1975).

[21] Jameson, Antony: Numerical solution of nonlinear partial differential equations of mixed type, Third Symposium on the Numerical Solution of Partial Differential Equation (SYNSPADE), Univ. of Maryland (May 1975).

[22] James, R. M.: A new look at incompressible airfoil theory, McDonnel Douglas Report J 0198 (May 1971).

[23] Timman, R.: The direct and inverse problem of airfoil theory. A method to obtain numerical solutions, Nat. Luchtv. Lab. Amsterdam Report F 16 (1951).

[24] Hicks, R. M.; Murman, E. M.; Vanderplaats, G. N.: An Assessment of airfoil design by numerical optimization, NASA TMX-3092 (1974).

[25] Lock, R. C.: Test cases for numerical methods in two dimensional transonic flow, AGARD Report 570 (1970).

[26] Bauer, F.; Garabedian, P.; Korn, D.; Jameson, A.: Supercritical Wing Sections II, Springer-Verlag, New York (1975).

[27] Bauer, F.; Korn, D.: Computer simulation of transonic flow past airfoils with boundary layer correction, Second AIAA Conference on Computational Fluid Dynamics, Hartford (June 1975).

[28] Whitcomb, R. T.: Review of NASA supercritical airfoils, Ninth International Congress on Aeronautical Sciences, Haifa, Isarael (1974).

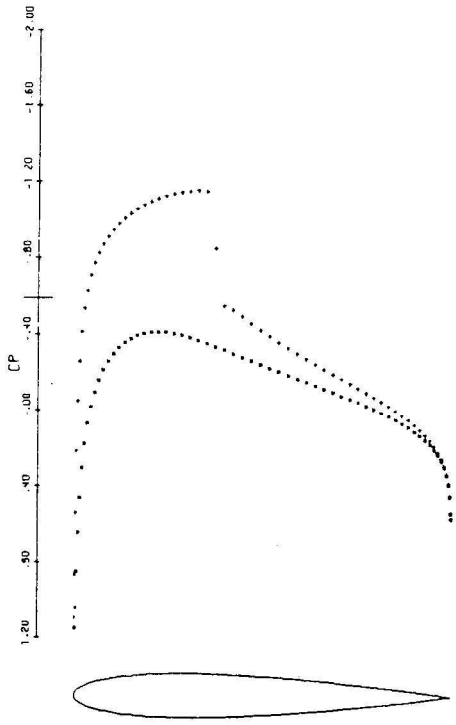


FIGURE 3
NACA 0012
MACH .750 α 1.5° CL .3286 CD .0074
QUASILINEAR FORM GRID 128 x 32

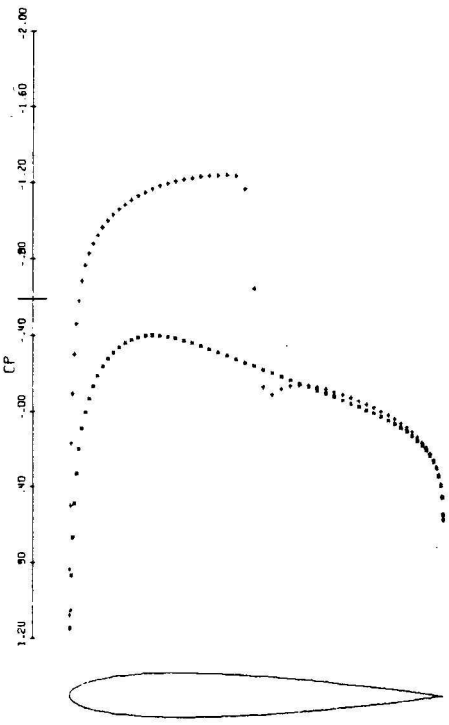


FIGURE 4
NACA 0012
MACH .750 α 1.5° CL .4005 CD .0078
CONSERVATION FORM GRID 128 x 32

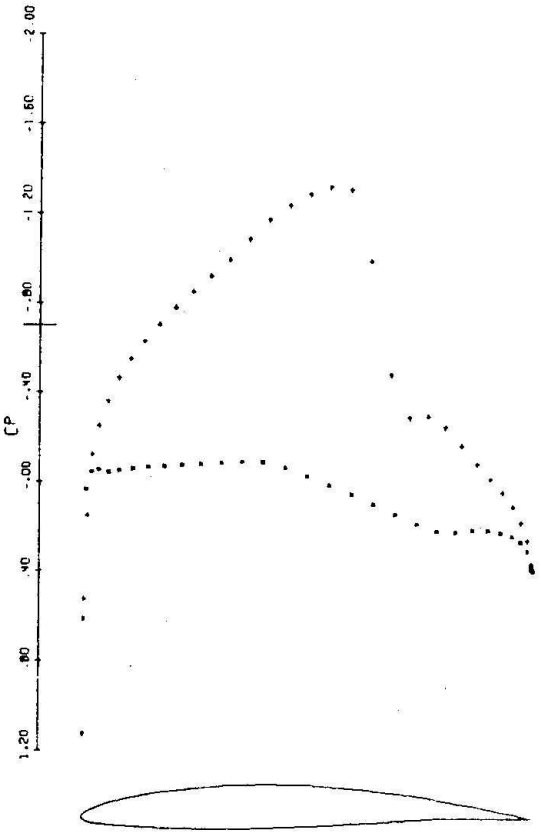
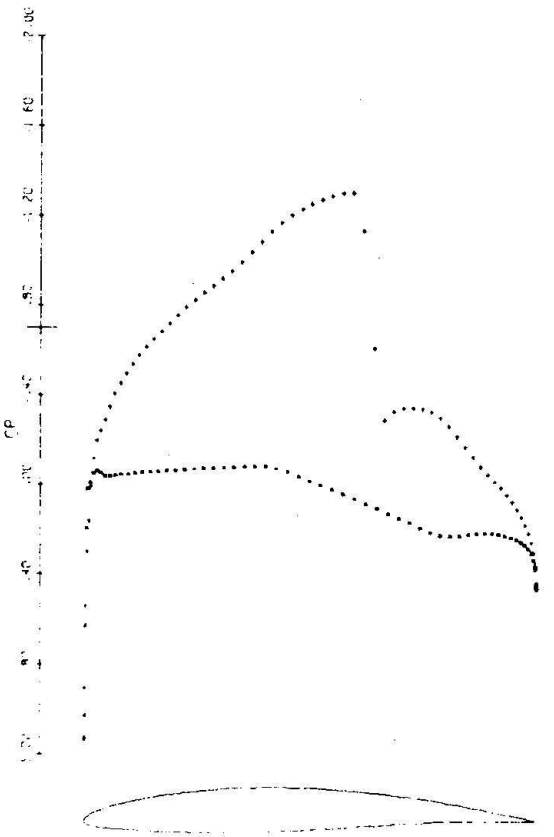


FIGURE 5(A)
NACA 64A410

MACH .720 α 0° CL .6991 CD .0044
CONSERVATION FORM GRID 64 X 16



FIGRE 5(B)
NACA 64A410

MACH .720 α 0° CL .6709 CD .0032
CONSERVATION FORM GRID 128 X 32

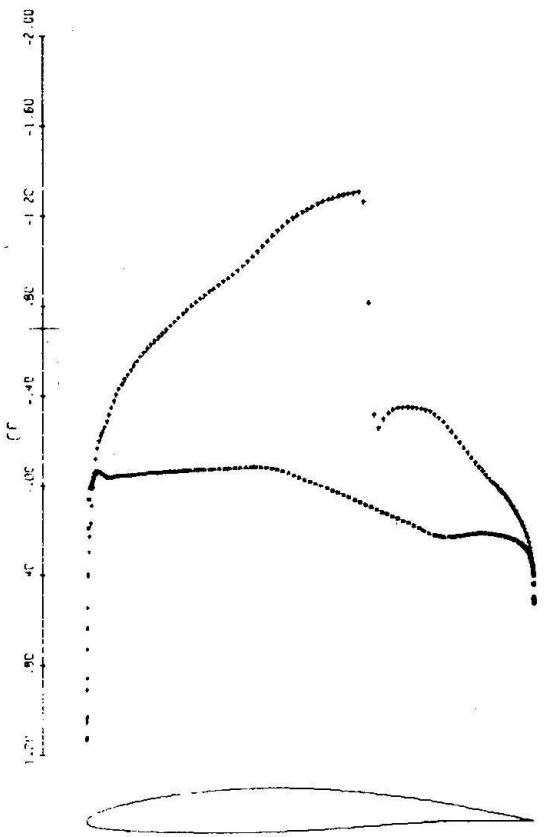


FIGURE 5(C)
NACA 64A410

MACH .720 α 0° CL .6648 CD .0031
CONSERVATION FORM GRID 256 X 64

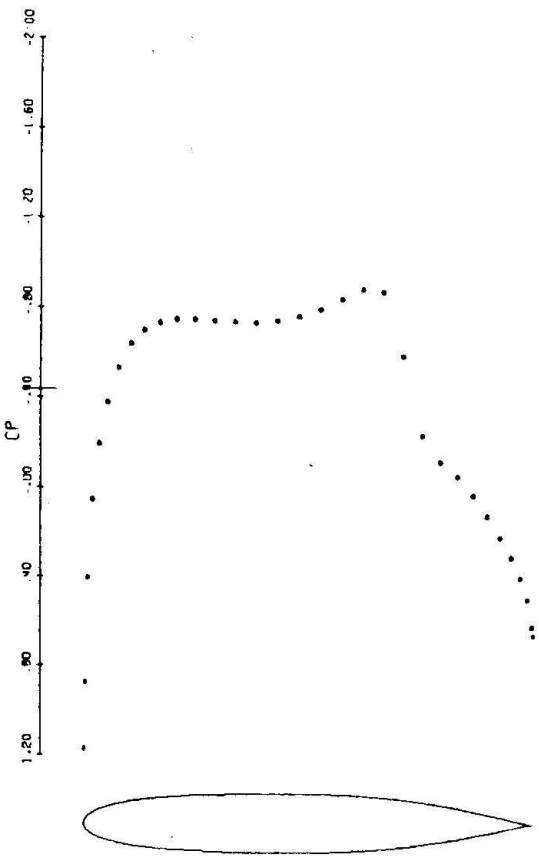


FIGURE 6(A)
HICKS-MURMAN AIRFOIL
MACH .800 α 0° CL .0000 CD .0038
CONSERVATION FORM GRID 64 X 16

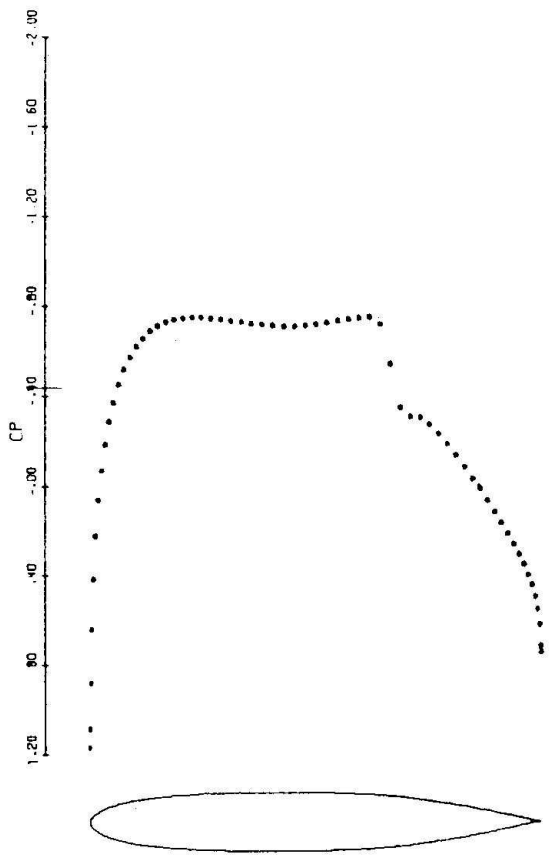


FIGURE 6(B)
HICKS-MURMAN AIRFOIL
MACH .800 α 0° CL .0000 CD .0007
CONSERVATION FORM GRID 128 X 32

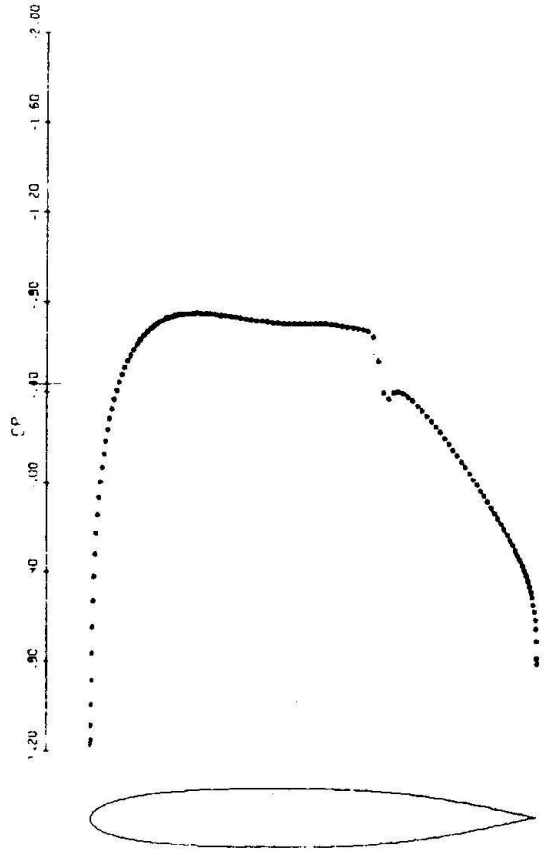


FIGURE 6(C)
HICKS-MURMAN AIRFOIL
MACH .800 α 0° CL .0000 CD .0002
CONSERVATION FORM GRID 256 X 64