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THE CONNECTION BETWEEN SINGULAR PERTURBATIONS AND SINGULAR ARCS: PART 2: A THEORY FOR THE LINEAR REGULATOR*

by

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THE CONNECTION BETWEEN SINGULAR PERTURBATIONS AND SINGULAR ARCS: PART 2: A THEORY FOR THE LINEAR REGULATOR *

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ABSTRACT

Singular perturbation theory is applied to obtain the asymptotic solution for nearly singular optimal control of a constant linear system in a finite time interval. In the limit as the control cost is reduced to zero, the initial control is found to have an impulse-like behavior, while the outer solution yields a singular arc.

INTRODUCTION

O'Malley's paper at these proceedings discusses several examples of nearly singular optimal control. The cases where all components of the control are equally 'cheap', and unbounded, can be incorporated within the framework of the theory for the constant linear regulator sketeched in the following paragraphs. A more detailed treatment, using a preliminary transformation to canonical form, is given in a forthcoming paper (O'Malley and Jameson, 1974).

Consider the constant linear system

$$\dot{x} = Ax + Bu \ , \quad x(0) \text{ specified}, \tag{1}$$

where the n dimensional vector x represents the state, and the r dimensional vector u is the control. Let u be chosen to minimize

$$J = \frac{1}{2}\int_0^1 (x^TQx + \varepsilon^2 u^TRu)\, dt$$

where Q is a nonnegative definite constant matrix, and R is a positive definite constant matrix. Then

$$u = -\frac{1}{\varepsilon^2} R^{-1}B^Tp \tag{3}$$

where x and p satisfy the Hamiltonian equations (Bryson and Ho, 1969)

$$\left.\begin{aligned} \varepsilon^2\dot{x} &= \varepsilon^2 Ax - BR^{-1}B^Tp \ , && x(0) \text{ specified} \\ \dot{p} &= -Qx - A^Tp \ , && p(1) = 0 \ . \end{aligned}\right\} \tag{4}$$

We shall examine the asymptotic behavior of the solution as $\varepsilon \to 0$. The system of equations (4) is singularly perturbed because its order is reduced from 2n to n when $\varepsilon = 0$. It proves convenient to distinguish the following hierarchy of cases:

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CASE 0: $r = n$, $Q > 0$;

CASE 1: $r < n$, $B^T QB > 0$;

CASE k: $r < n$, $B^T A^{T^m} QA^m B = 0$, $m<k-1$, $B^T A^{T^{k-1}} QA^{k-1} B>0$.

This list is not exhaustive, but serves to illustrate the principal variations of behavior. Case 0 is unusual in practice, so we shall concentrate on CASE 1 and higher cases.

CASE 1

In CASE 1 let us seek an asymptotic expansion in the form

$$\left.\begin{aligned} x &= X(t,\varepsilon) + m(\tau,\varepsilon) + \varepsilon\, n(\sigma,\varepsilon) \\ p &= P(t,\varepsilon) + \varepsilon f(\tau,\varepsilon) + \varepsilon^2 g(\sigma,\varepsilon) , \end{aligned}\right\} \tag{5}$$

where each term is to be represented by a series in powers of ε. Here (X,P) is the outer solution, while $(m,\varepsilon f)$ and $(n,\varepsilon^2 g)$ are initial and terminal boundary layer corrections in the stretched time coordinates

$$\tau = \frac{t}{\varepsilon} , \quad \sigma = \frac{1-t}{\varepsilon}$$

The boundary layer corrections are required because it is not possible to satisfy all the boundary conditions using the outer outer expansion by itself, as will appear.

Suppose that the outer solution is represented as

$$X(t,\varepsilon) = \sum_{j=0}^{\infty} X_j \varepsilon^j , \quad P(t,\varepsilon) = \sum_{j=0}^{\infty} P_j \varepsilon^j \quad . \tag{6}$$

Then, equating like powers of ε, we find that

$$B\, R^{-1} B^T P_0 = 0 , \quad B\, R^{-1} B^T P_1 = 0 .$$

Multiplying on the left by B^T, since R is positive definite, it follows that

$$B^T P_0 = 0 , \quad B^T P_1 = 0 \tag{7}$$

Then we find that

$$\dot{X}_j = AX_j + BU_j \tag{8a}$$

$$\dot{P}_j = -QX_j - A^T P_j \tag{8b}$$

where

$$U_j = - R^{-1} B^T P_{j+2} ., \quad U_{-2} = U_{-1} = 0 .$$

Thus for each j, U_j must be such that the constraint

$$B^T P_j = - RU_{j-2} \tag{9}$$

is satisfied. This constraint confines the solution of (8) to a subspace, so that the outer solution cannot satisfy all the boundary conditions of (4). The unsatisfied conditions are eliminated by the boundary layer corrections.

In order to express the constraint (9) in a convenient manner we define the matrix

$$E = I - B(B^TQB)^{-1}B^TQ \qquad (10)$$

Then

$$E\,B = 0\ , \quad B^TQ\,E = 0\ , \quad E^2 = E\ . \qquad (11)$$

We can now set

$$X_j = EX_j + BU_j\ , \quad P_j = E^TP_j + QBZ_j \qquad (12)$$

where

$$V_j = (B^TQB)^{-1}B^TQX_j\ , \quad Z_j = -(B^TQB)^{-1}R\dot{U}_{j-2}$$

Multiplying (8b) on the left by B^T, it follows from (9) that

$$R\dot{U}_{j-2} = B^TQBV_j + B^TA^TP_j$$

whence

$$V_j = -(B^TQB)^{-1}(B^TA^TE^TP_j + B^TA^TQBZ_j - R\dot{U}_{j-2}) \qquad (13)$$

Substituting in (8) we find that

$$\left.\begin{aligned} \frac{d}{dt}(E\,X_j) &= \hat{A}EX_j - \hat{G}\,E^TP_j - H_j \\ \frac{d}{dt}(E^TP_j) &= -\hat{Q}EX_j - \hat{A}^TE^TP_j - K_j \end{aligned}\right\} \qquad (14)$$

where

$$\hat{A} = EAE\ , \quad \hat{G} = EAB(B^TQB)^{-1}B^TA^TE^T\ ,$$

$$\hat{Q} = E^TQE \geq 0$$

and the forcing terms are

$$H_j = EAB(B^TQB)^{-1}(B^TA^TQBZ_j - R\dot{U}_{j-2})$$

$$K_j = E^TA^TQBZ_j$$

Equations (14) form a Hamiltonian system which can be solved for EX, E^TP with $EX(0)$, $E^TP(1)$ prescribed (Bucy, 1967). Then since V_j is determined by (13), it follows from (12) that X_j satisfies the constraint

$$B^TQX_j = B^TQBV_j \qquad (15)$$

It follows from the linearity of the system that the boundary layer corrections must also satisfy (4). Thus

$$\left.\begin{aligned} \frac{dm}{d\tau} &= \varepsilon Am - BR^{-1}B^Tf \\ \frac{df}{d\tau} &= -Qm - \varepsilon A^Tf\ , \end{aligned}\right\} \qquad (16)$$

where m and f $\to 0$ as $\tau \to \infty$. Suppose that m and f are expanded as

$$m(\tau,\varepsilon) = \sum_{j=0}^{\infty} m_j \varepsilon^j \,, \quad f(\tau,\varepsilon) = \sum_{j=0}^{\infty} f_j \varepsilon^j \tag{17}$$

Then, equating like powers of ε, we find that

$$\frac{dm_j}{d\tau} = - BR^{-1}B^T f_j + A\, m_{j-1} \tag{18a}$$

$$\frac{df_j}{d\tau} = - Q\, m_j \qquad - A^T f_{j-1} \tag{18b}$$

where

$$m_{-1} = f_{-1} = 0$$

Combining (18a) and (18b), we seek decaying solutions of

$$\frac{d^2 m_j}{d\tau^2} = B\, R^{-1}B^T Q m_j + h_{j-1} \tag{19}$$

where

$$h_{j-1} = A\, \frac{dm_{j-1}}{d\tau} + B\, R^{-1}B^T A^T f_{j-1}$$

Let

$$m_j = Em_j + B\, R^{-1/2} M_j \tag{20}$$

where

$$M_j = R^{1/2}(B^T QB)^{-1} B^T Q m_j \,.$$

Then

$$\frac{d^2}{d\tau^2} Em_j = E\, h_{j-1}(\tau) \tag{21}$$

and

$$\frac{d^2}{d\tau^2} M_j = R^{-1/2}B^T Q\, B\, R^{-1/2} M_j + R^{1/2}(B^T QB)^{-1} B^T Q h_{j-1} \tag{22}$$

Equation (21) has the decaying particular integral

$$Em_j = \int_\tau^\infty dr \int_r^\infty E h_{j-1}(s)\, ds \,, \tag{23}$$

Thus $E\, m_j(0)$ is determined. In particular $Em_0(0) = 0$. The homogeneous part of equation (22) has the decaying solution

$$M_j = e^{-C\tau}\, M_j(0) \tag{24}$$

where

$$C = \sqrt{R^{-1/2}B^T QBR^{-1/2}}$$

from which decaying particular integrals can be constructed by integration for arbitrary $M_j(0)$.

Now to match the initial conditions of (4) we have

$$m_0(0) + X_0(0) = x(0)$$

whence

$$EX_0(0) = Ex(0)$$

providing the initial condition for (14). Also

$B^T Q X_0(0)$ is then determined by (15), so $m_0(0)$ must be such that

$$B^T Q m_0(0) = B^T Q(x(0) - X_0(0))$$

According to (20) this will be the case if

$$M_0(0) = R^{1/2}(B^T Q B)^{-1} B^T Q(x(0)-X_0(0))$$

providing the initial condition for the leading term of the boundary layer. For the higher terms the initial conditions are split in a similar manner. Since $E\,m_j(0)$ is uniquely determined by the requirement for a decaying boundary layer, we have

$$EX_j(0) = -\,Em_j(0)\ , \qquad j > 0\ .$$

Then $B^T QX_j(0)$ is determined, and hence the initial value $M_j(0)$ for the part of the boundary layer correction lying in the range space of B.

The terminal boundary layer correction can be determined in a similar manner. Setting

$$n(\sigma,\varepsilon) = \sum_{j=0}^{\infty} n_j \varepsilon^j\ , \qquad g(\sigma,\varepsilon) = \sum_{j=0}^{\infty} g_j \varepsilon^j \tag{25}$$

we find that

$$\left.\begin{aligned} \frac{dn_j}{d\sigma} &= BR^{-1}B^T g_j - An_{j-1} \\ \frac{dg_j}{d\sigma} &= Qn_j + A^T g_{j-1} \ . \end{aligned}\right\} \tag{26}$$

Then

$$\frac{d^2 g_j}{d\sigma^2} = Q\ B\ R^{-1}B^T g_j + k_{j-1} \tag{27}$$

where

$$h_{j-1} = A^T \frac{dg_{j-1}}{d\tau} - Q\ A\ n_{j-1}\ .$$

Now we set

$$g_j = E^T g_j + Q\ B\ R^{-1/2} G_j \tag{28}$$

where

$$G_j = R^{1/2}(B^T QB)^{-1}B^T g_j\ .$$

Then (27) implies that

$$\frac{d^2}{d\sigma^2} E^T g_j = E^T k_{j-1} \tag{29}$$

and

$$\frac{d^2 G_j}{d\sigma^2} = R^{-1/2}B^T Q\ B\ R^{-1/2} G_j + R^{1/2}(B^T Q\ B)^{-1}B^T k_{j-1} \tag{30}$$

For a decaying solution $E^T g_j(0)$ is determined, and hence $E^T P_j(1)$ is determined. Also $B^T P_j(1)$ is determined, providing the terminal value $G_j(0)$.

Thus the complete asymptotic solution can be determined term by term. We observe that the leading term of the outer expansion is just the familiar solution for a singular arc which occurs when $\varepsilon = 0$ (Moylan and Moore, 1971). We also note that near $t = 0$ the dominant term of the control is $(1/\varepsilon)R^{-1}B^Tf_0$, where

$$B\ R^{-1}B^Tf_0 = -\ B\ R^{-1/2}\ \frac{dM_0}{d\tau} = B\ R^{-1/2}C\ e^{-C\tau}M_0(0)\ .$$

Thus near $t = 0$ the control has the form

$$\frac{1}{\varepsilon}\ R^{-1/2}C\ e^{-C\ t/\varepsilon}\ M_0(0)\ .$$

In the limit as $\varepsilon \to 0$ this reduces to the impulse which is required to make the initial transfer to a singular arc.

HIGHER CASES

In CASE 2 we note that since $Q > 0$ the condition

$$B^TQ\ B = 0$$

implies that

$$Q\ B = 0$$

The appropriate stretching for the initial boundary layer correction proves to be $\tau = t/\mu$, where $\mu = 1/\sqrt{\varepsilon}$. Expanding the outer solution as

$$X(t,\mu) = \sum_{j=0}^{\infty} X_j\mu^j\ , \quad P(t,\mu) = \sum_{j=0}^{\infty} P_j\mu^j$$

equations (8) hold, where now

$$B^TP_j = R\ U_{j-4}\ , \qquad U_j = 0\ ,\ j < 0\ . \tag{31}$$

Then (8b) gives

$$B^TA^TP_j = R\ \dot{U}_{j-4} \tag{32}$$

The additional constraint (32) confines the outer solution to a smaller subspace. Consequently a transfer through the space spanned by the columns of B, AB must be accomplished in the initial boundary layer. If the corrections to x and p have the form

$$\frac{1}{\mu}\ m(\tau,\mu) = \frac{1}{\mu}\sum_{j=0}^{\infty} m_j\mu^j\ , \quad \mu f(\tau,\mu) = \mu\sum_{j=0}^{\infty} f_j\mu^j$$

we find that

$$\frac{d^2m_j}{d\tau^2} = B\ R^{-1}B^TA^Tf_j + A\,\frac{dm_{j-1}}{d\tau}$$

$$\frac{d^2f_j}{d\tau^2} = -\ Q\ A\ m_j - A^T\,\frac{df_{j-1}}{d\tau}$$

The leading term satisfies the homogeneous equation

$$\frac{d^4m_0}{d\tau^4} = -B\ R^{-1}B^TA^TQ\ A\ B\ m_0$$

with the decaying solution

$$m_0 = B\ R^{-1/2} Zi(e^{\rho C\tau} - e^{\bar{\rho} C\tau}) \ , \quad m_0(0) = 0$$

where Z is a constant matrix,

$$\rho = -\frac{1+i}{\sqrt{2}}$$

and

$$C = \sqrt[4]{R^{-1/2} B^T A^T Q\ A\ B\ R^{-1/2}}$$

Then $m_1(\tau)$ lies in the space spanned by the columns of B, AB and can be used to match the boundary conditions at $t = 0$. Near $t = 0$ the dominant term is $(1/\mu)m_0$. Thus x makes an excursion in the range space of B, which approaches the form of a delta function as $\mu \to 0$.

For CASE k we find similarly that the appropriate expansions are in powers of $\mu = \varepsilon^{1/k}$, where the stretching for the the initial boundary layer is $\tau = t/\mu$. Now $B^T A^{T^m} P_j$ is constrained for $m < k$, confining the outer solution to a progressively smaller subspace with increasing k. The initial boundary layer correction has to meet the boundary conditions in the space spanned by the columns of the first k partitions of the controllability matrix

$$[B, AB, \ldots\ A^{n-1}B]$$

and x makes an excursion in the space spanned by the columns of the first k-1 partitions. Thus the increasing thickness of the boundary layer corresponds to the additional work required from the initial control. In the limit the control has a form corresponding to a delta function and its first k-1 derivatives.

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