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THE CONNECTION BETWEEN SINGULAR PERTURBATIONS AND SINGULAR ARCS:
PART 2: A THEORY FOR THE LINEAR REGULATOR*

by

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ABSTRACT

Singular perturbation theory is applied to obtain the asymptotic solution for nearly singular optimal control of a constant linear system in a finite time interval. In the limit as the control cost is reduced to zero, the initial control is found to have an impulse-like behavior, while the outer solution yields a singular arc.

INTRODUCTION

O'Malley's paper at these proceedings discusses several examples of nearly singular optimal control. The cases where all components of the control are equally 'cheap', and unbounded, can be incorporated within the framework of the theory for the constant linear regulator sketched in the following paragraphs. A more detailed treatment, using a preliminary transformation to canonical form, is given in a forthcoming paper (O'Malley and Jameson, 1974).

Consider the constant linear system

$$\dot{x} = Ax + Bu, \quad x(0) \text{ specified}, \quad (1)$$

where the n dimensional vector x represents the state, and the r dimensional vector u is the control. Let u be chosen to minimize

$$J = \frac{1}{2} \int_0^1 (x^T Q x + \epsilon^2 u^T R u) dt$$

where Q is a nonnegative definite constant matrix, and R is a positive definite constant matrix. Then

$$u = - \frac{1}{\epsilon^2} R^{-1} B^T p \quad (3)$$

where x and p satisfy the Hamiltonian equations (Bryson and Ho, 1969)

$$\left. \begin{aligned} \epsilon^2 \dot{x} &= \epsilon^2 Ax - BR^{-1} B^T p, & x(0) \text{ specified} \\ \dot{p} &= -Qx - A^T p, & p(1) = 0. \end{aligned} \right\} \quad (4)$$

We shall examine the asymptotic behavior of the solution as $\epsilon \rightarrow 0$. The system of equations (4) is singularly perturbed because its order is reduced from $2n$ to n when $\epsilon = 0$. It proves convenient to distinguish the following hierarchy of cases:

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CASE 0: $r = n$, $Q > 0$;

CASE 1: $r < n$, $B^T Q B > 0$;

CASE k: $r < n$, $B^T A^T m Q A^m B = 0$, $m < k-1$, $B^T A^T^{k-1} Q A^{k-1} B > 0$.

This list is not exhaustive, but serves to illustrate the principal variations of behavior. Case 0 is unusual in practice, so we shall concentrate on CASE 1 and higher cases.

CASE 1

In CASE 1 let us seek an asymptotic expansion in the form

$$\left. \begin{aligned} x &= X(t, \epsilon) + m(\tau, \epsilon) + \epsilon n(\sigma, \epsilon) \\ p &= P(t, \epsilon) + \epsilon f(\tau, \epsilon) + \epsilon^2 g(\sigma, \epsilon) \end{aligned} \right\} \quad (5)$$

where each term is to be represented by a series in powers of ϵ . Here (X, P) is the outer solution, while $(m, \epsilon f)$ and $(n, \epsilon^2 g)$ are initial and terminal boundary layer corrections in the stretched time coordinates

$$\tau = \frac{t}{\epsilon} , \quad \sigma = \frac{1-t}{\epsilon}$$

The boundary layer corrections are required because it is not possible to satisfy all the boundary conditions using the outer expansion by itself, as will appear.

Suppose that the outer solution is represented as

$$X(t, \epsilon) = \sum_{j=0}^{\infty} X_j \epsilon^j , \quad P(t, \epsilon) = \sum_{j=0}^{\infty} P_j \epsilon^j \quad (6)$$

Then, equating like powers of ϵ , we find that

$$B R^{-1} B^T P_0 = 0 , \quad B R^{-1} B^T P_1 = 0 .$$

Multiplying on the left by B^T , since R is positive definite, it follows that

$$B^T P_0 = 0 , \quad B^T P_1 = 0 \quad (7)$$

Then we find that

$$\dot{X}_j = A X_j + B U_j \quad (8a)$$

$$\dot{P}_j = -Q X_j - A^T P_j \quad (8b)$$

where

$$U_j = -R^{-1} B^T P_{j+2} , \quad U_{-2} = U_{-1} = 0 .$$

Thus for each j , U_j must be such that the constraint

$$B^T P_j = -R U_{j-2} \quad (9)$$

is satisfied. This constraint confines the solution of (8) to a subspace, so that the outer solution cannot satisfy all the boundary conditions of (4). The unsatisfied conditions are

