FURTHER RESULTS ON SINGULAR PERTURBATIONS AND SINGULAR ARCS*

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ABSTRACT

An asymptotic solution is obtained for the linear regulator problem with "cheap control". The essential new element is the singular perturbation solution to the matrix Riccati equation.

INTRODUCTION AND OUTLINE OF RESULTS

Consider the optimal control problem where

$$\dot{x} = Ax + Bu, \quad 0 \le t \le T, \quad x(0) \text{ prescribed,} \tag{1}$$

with the scalar performance index

$$J(\epsilon) = \frac{1}{2}\int_0^t [x'Qx + \epsilon^2 u'Ru]dt \tag{2}$$

is to be minimized for ϵ a small, positive parameter (control is "cheap"). Here, A, B, Q, and R are smooth functions of t, with $Q \ge 0$ and $R > 0$ being symmetric matrices. The state x is an n vector, while the control u is an r vector. Asymptotic solutions of this problem were previously given by O'Malley and Jameson ([4], [8], [9], [10]) using singular perturbation techniques for the appropriate canonical equations. We refer to those papers for discussion of practical control applications leading to problems like (1)-(2) and interpretation of the results, noting only that (1)-(2) is a singular problem when $\epsilon = 0$ (cf. Bryson and Ho [2] and Ho [3]). Here we shall obtain a matrix Riccati solution, which has independent significance due to its feedback interpretation, numerical implementation, and extendability to the infinite interval problem (cf. Jameson and O'Malley [5]).

We first transform the problem through a change of variables previously used by Moylan and Moore [7] for the singular problem. Defining

$$u_1 = \int_0^t u(s)ds \quad \text{and} \quad x_1 = x - Bu_1, \tag{3}$$

we have the higher dimensional regulator problem

$$\begin{pmatrix} \dot{x}_1 \\ \dot{u}_1 \end{pmatrix} = \mathcal{A}\begin{pmatrix} x_1 \\ u_1 \end{pmatrix} + \mathcal{B}u, \quad \begin{pmatrix} x_1(0) \\ u_1(0) \end{pmatrix} = \begin{pmatrix} x(0) \\ 0 \end{pmatrix}, \tag{4}$$

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$$J(\epsilon) = \frac{1}{2}\int_0^T [\begin{pmatrix} x_1 \\ u_1 \end{pmatrix}' \mathcal{Q} \begin{pmatrix} x_1 \\ u_1 \end{pmatrix} + \epsilon^2 u'Ru]dt \tag{5}$$

where

$$\mathcal{A} = \begin{pmatrix} A & B_1 \\ 0 & 0 \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} 0 \\ I \end{pmatrix}, \quad \text{and} \quad \mathcal{Q} = \begin{pmatrix} Q & QB \\ B'Q & B'QB \end{pmatrix}$$

for $B_1 = AB - \dot{B}$.

Using the standard technique (cf. Kalman [6] and e.g., Anderson and Moore [1]) of a feedback law

$$u = -\frac{1}{\epsilon^2}R^{-1}\mathcal{B}'\hat{k}\begin{pmatrix} x_1 \\ u_1 \end{pmatrix}, \tag{6}$$

the symmetric, positive semi-definite matrix $\hat{k}$ must satisfy the terminal value problem

$$\frac{d\hat{k}}{dt} = -\hat{k}\mathcal{A} - \mathcal{A}'\hat{k} + \frac{1}{\epsilon^2}\hat{k}\mathcal{B}R^{-1}\mathcal{B}'\hat{k} - \mathcal{Q}, \quad \hat{k}(T) = 0 \tag{7}$$

and there remains the initial value problem

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ u_1 \end{pmatrix} = [\mathcal{A} - \frac{1}{\epsilon^2}\mathcal{B}R^{-1}\mathcal{B}'\hat{k}]\begin{pmatrix} x_1 \\ u_1 \end{pmatrix}, \quad \begin{pmatrix} x_1(0) \\ u_1(0) \end{pmatrix} = \begin{pmatrix} x(0) \\ 0 \end{pmatrix}. \tag{8}$$

Solution of the Riccati equation (7) (as in the singular problem) will differ considerably in a hierarchy of cases. We shall restrict attention here to the simplest case when

$$B'QB > 0, \tag{9}$$

while further cases will be discussed elsewhere. Note that (9) implies that $\mathcal{Q} \geq 0$ since

$$Q - QB(B'QB)^{-1}B'Q = E'QE \geq 0 \tag{10}$$

for $E = I - B(B'QB)^{-1}B'Q$. Putting $\hat{k}$ into its 2×2 block form, equation (7) can be written as three coupled equations for the independent elements. Analyzing them, one finds that the unique solution to (7) is of the form

$$\hat{k} = \begin{pmatrix} k & kB \\ B'k & B'kB \end{pmatrix} \tag{11}$$

where the $n \times n$ symmetric matrix k is the positive semi-definite solution of the singularly perturbed Riccati equation

$$\epsilon^2 \frac{dk}{dt} = -\epsilon^2(kA + A'k + Q) + kBR^{-1}B'k, \quad k(T) = 0. \tag{12}$$

The equations for the other components of $\hat{k}$ follow from

$$\epsilon^2 \frac{d}{dt}(kB) = -\epsilon^2(kB_1 + A'kB + QB) + kBR^{-1}B'kB \tag{13}$$

and

$$\epsilon^2 B' \frac{d}{dt}(kB) = -\epsilon^2(B'kB + B'A'kB + B'QB) + B'kBR^{-1}B'kB. \quad (14)$$

Our principal result will be the asymptotic solution of the Riccati equation (12). For $B'QB > 0$, we will obtain an asymptotic solution of the form

$$k(t,\epsilon) = K(t,\epsilon) + \epsilon\ell(\sigma,\epsilon) \quad (15)$$

where the outer solution K has an asymptotic expansion

$$K(t,\epsilon) \sim \sum_{j\geq 0} K_j(t)\epsilon^j \quad (16)$$

and the terminal boundary layer correction ℓ has an expansion

$$\ell(\sigma,\epsilon) \sim \sum_{j\geq 0} \ell_j(\sigma)\epsilon^j \quad (17)$$

whose terms decay to zero as the stretched variable

$$\sigma = \frac{T-t}{\epsilon} \quad (18)$$

tends to infinity. Such expansions have been obtained for other singularly perturbed control problems by Yackel and Kokotović [12] and O'Malley and Kung [11]. Their direct approach doesn't seem to work here, so a more roundabout procedure (as in [5]) will be employed.

Once the expansions (15) are determined, the state equations (8) will have the asymptotic solution

$$\begin{aligned} x_1(t,\epsilon) &= X_1(t,\epsilon) + \epsilon m_1(\tau,\epsilon) + \epsilon^2 n_1(\sigma,\epsilon) \\ u_1(t,\epsilon) &= U_1(t,\epsilon) + v_1(\tau,\epsilon) + \epsilon w_1(\sigma,\epsilon) \end{aligned} \quad (19)$$

consisting of an outer solution (X_1,U_1), a right boundary layer correction $(\epsilon^2 n_1(\sigma,\epsilon),\ \epsilon w_1(\sigma,\epsilon))$ which tends to zero as $\sigma \to \infty$, and a left boundary layer correction $(\epsilon m_1(\tau,\epsilon), v_1(\tau,\epsilon))$ which tends to zero as the (left) stretched variable

$$\tau = \frac{t}{\epsilon} \quad (20)$$

tends to infinity. The transformation (3) then implies that the optimal control u and the corresponding trajectory x will be asymptotically represented in the form

$$\begin{aligned} u(t,\epsilon) &= \dot{u}_1 = U(t,\epsilon) + \frac{1}{\epsilon}v(\tau,\epsilon) + w(\sigma,\epsilon) \\ x(t,\epsilon) &= x_1 + Bu_1 = X(t,\epsilon) + m(\tau,\epsilon) + \epsilon n(\sigma,\epsilon) \quad . \end{aligned} \quad (21)$$

We note, in particular, that the optimal control will be unbounded at $t = 0$ as $\epsilon \to 0$ and that the optimal cost $J^*(\epsilon)$ will have an asymptotic series expansion. Further, the limiting solution within $(0,T)$ will agree with the singular arc solution given in [3] and elsewhere (cf. [11]).

THE RICCATI GAIN

Since the outer solution K represents the gain k asymptotically for $t < T$ $(\sigma \sim \infty)$, K must satisfy equations (12)-(14). When $\epsilon = 0$, note that (12) implies $K_0BR^{-1}B'K_0 = 0$, so the singular arc condition (cf. [2])

$$B'K_0 = K_0B = 0 \tag{22}$$

follows. (When B is invertible, then, $K_0 = 0$). Since $B'QB > 0$, (14) implies that $B'KB = O(\epsilon)$ is positive definite and, by (13),

$$KB = \epsilon^2[KB_1 + QB + \frac{d}{dt}(KB) + A'KB](B'KB)^{-1}R. \tag{23}$$

Substituting back into (12), then,

$$\begin{aligned}\frac{dK}{dt} + KA + A'K + Q = {} & [KB_1 + QB + \frac{d}{dt}(KB) + A'KB] \\ & \cdot [B'QB + B'\frac{d}{dt}(KB) + B'KB_1 + B'A'KB]^{-1} \\ & \cdot [B_1'K + B'Q + \frac{d}{dt}(B'K) + B'KA].\end{aligned} \tag{24}$$

Setting $\epsilon = 0$, we finally have the parameter-free Riccati equation

$$\begin{aligned}\frac{dK_0}{dt} + K_0(A - B_1(B'QB)^{-1}B'Q) + (A' - QB(B'QB)^{-1}B_1')K_0 \\ - K_0B_1(B'QB)^{-1}B_1'K_0 + (Q - QB(B'QB)^{-1}B'Q) = 0, \quad K_0(T) = 0.\end{aligned} \tag{25}$$

By (10), a unique K_0 exists throughout $0 \le t \le T$ (cf. [1]). Moreover, (22) will hold since $\frac{d}{dt}(K_0B) = 0$ whenever $K_0B = 0$. We observe that (25) does not follow from setting $\epsilon = 0$ in (12); that only implies (22).

Higher order terms in the outer expansion K are solutions of non-homogeneous linear equations obtained from equating coefficients of higher powers of ϵ in (12)-(14). Specifically, for K_1, (13) and (14) imply that $B'K_1B > 0$ satisfies

$$(B'K_1B)R^{-1}(B'K_1B) = B'QB \tag{26}$$

while

$$K_1B = (K_0B_1 + QB)(B'K_1B)^{-1}R. \tag{27}$$

Since $K_1(T)$ cannot then be zero, a boundary layer correction $\ell_0(\sigma)$ to K_1 is needed to satisfy $k(T,\epsilon) = 0$. Further, the coefficient of ϵ in (24) implies a problem of the form

$$\frac{dK_1}{dt} + K_1[A - B_1(B'QB)^{-1}(B'Q + B_1'K_0)] +$$

$$+ [A' - (QB + K_0B)(B'QB)^{-1}B_1']K_1 = \alpha_1, \quad K_1(T) = -\ell_0(0) \tag{28}$$

where α_1 is determined by K_0 and $\ell_0(0)$ is, as yet, unspecified. Since (27)-(28) is a linearized version of the problem for K_0, its solution is unique.

Because the outer solution K satisfies (12), the representation (15) implies that the boundary layer correction $\ell(\sigma,\epsilon)$ must satisfy

$$\frac{d\ell}{d\sigma} = -\frac{1}{\epsilon}(\ell BR^{-1}B'K + KBR^{-1}B'\ell) - \ell BR^{-1}B'\ell + \epsilon(\ell A + A'\ell) \tag{29}$$

for $\sigma \geq 0$ and it must tend to zero as $\sigma \to \infty$. Introducing $C(T,\epsilon) = \epsilon^{-1}R^{-1/2}B'KBR^{-1/2}$, $L(\sigma,\epsilon) = R^{-1/2}B'\ell BR^{-1/2}$, and $L(\sigma,\epsilon) = \ell BR^{-1/2}$, and setting $\epsilon = 0$ in (29) implies

$$\frac{dL_0}{d\sigma} = -L_0C_0(T) - C_0(T)L_0 - L_0^2 \tag{30}$$

$$\frac{dL_0}{d\sigma} = -L_0(C_0(T) + L_0(\sigma)) + L_0(0)L_0(\sigma) \tag{31}$$

and

$$\frac{d\ell_0}{d\sigma} = L_0L_0'(0) + L_0(0)L_0' - L_0(\sigma)L_0 \tag{32}$$

while the initial condition $\ell_0(0) = -K_1(T)$ is unknown but completely determines

$$L_0(0) = -C_0(T) < 0 \quad \text{and} \quad L_0(0) = -K_1(T)B(T)R^{-1/2}(T). \tag{33}$$

Integrating (30) and (31), then, we have

$$L_0(\sigma) = 2L_0(0)(I + e^{2C_0(T)\sigma})^{-1}. \tag{34}$$

Since $\ell_0 \to 0$ as $\sigma \to \infty$, we have

$$\ell_0(\sigma) = -\int_\sigma^\infty \frac{d\ell_0}{d\sigma}(s)ds = -2L_0(0)(I + e^{2C_0(T)\sigma})^{-1}C_0^{-1}(T)L_0'(0). \tag{35}$$

In particular, note that this completely determines $K_1(T)$. Higher order terms follow as solutions of linear equations.

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