Application of Dual Time Stepping to Fully Implicit Runge Kutta Schemes for Unsteady Flow Calculations

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This paper presents the formulation of a dual time stepping procedure to solve the equations of fully implicit Runge-Kutta schemes. In particular the method is applied to Gauss and Radau 2A schemes with either two or three stages. The schemes are tested for unsteady flows over a pitching airfoil modeled by both the Euler and the unsteady Reynolds averaged Navier Stokes (URANS) equations. It is concluded that the Radau 2A schemes are more robust and less computationally expensive because they require a much smaller number of inner iterations. Moreover these schemes seem to be competitive with alternative implicit schemes.

I. Introduction

The last three decades have seen the emergence of a variety of very fast steady state solvers for the Euler and Reynolds averaged Navier Stokes equations. While these may be based on time marching with explicit Runge-Kutta schemes, implicit alternating direction schemes, or symmetric Gauss Seidel schemes, convergence to a steady state is typically accelerated by the use of techniques such as variables local time stepping or multigrid. Accordingly time accuracy is completely abandoned in these calculations.

While fast and accurate steady state solvers are sufficient for most of the computational simulations needed for the preliminary design of a fixed wing aircraft, time accurate simulations are also needed for a variety of important applications, such as flutter analysis, or the analysis of the flow past a helicopter in forward flight.

If an explicit scheme is used to calculate an unsteady flow, the permissible time step for numerical stability may be much smaller than that needed to attain reasonable accuracy, with the consequence that an excessively large number of time steps may be required. On computational grids with very large variations in cell size, for example, the time step limit is set by the smallest cell. Implicit schemes which are A- or L-stable allow arbitrarily large time steps, so one may choose the largest possible step that will yield the desired accuracy, but the work required in each time step may be excessively large.

In 1991 the present author proposed the use of a dual time stepping technique in which the equations for each implicit time step are treated as a modified steady state problem which is solved by marching in a pseudo time variable[1]. The Euler equations for gas dynamics can be expressed in conservation law form.

$$\mathbf{w} = \left[\begin{array}{c} \rho \\ \rho u_1 \\ \rho u_2 \\ \rho u_3 \\ \rho E \end{array} \right],$$

where ρ is the density, u_i are the velocity components and E is the total energy. The flux vectors are

$$\mathbf{f}_i = u_i \mathbf{w} + p \left[\begin{array}{c} 0 \\ \delta_{i1} \\ \delta_{i2} \\ \delta_{i3} \\ u_i \end{array} \right]$$

where the pressure is

$$p = (\gamma - 1)\rho(E - u_i u_i).$$

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A semi-discrete finite volume scheme is obtained directly approximating the integral form on each computational cell

$$\frac{d}{dt}\int_{\text{cell}} \mathbf{w}\, dV + \int_{\text{cell boundary}} \mathbf{n}_i \mathbf{f}_i \, dS = 0,$$

where $\mathbf{n}_i$ are the components of the unit normal to the cell boundary. This leads to a semi-discrete equation with the general form

$$V\frac{d\mathbf{w}}{dt} + R(\mathbf{w}) = 0 \tag{1}$$

where $\mathbf{w}$ now denotes the average value of the state in the cell. V is the cell volume, or in the two dimensional case, the cell area. $R(\mathbf{w})$ is the residual resulting from the space discretization. Applications to external aerodynamic typically use grids in which the cell area or volume varies by many orders of magnitude between the body and the far field, and this is a principal reason for using implicit schemes for time accurate simulations. Introducing superscripts n to denote the time level, the second order backward difference formula (BDF2) for time integration is

$$\frac{3V}{2\Delta t}\mathbf{w}^{n+1} - \frac{2V}{\Delta t}\mathbf{w}^{n} + \frac{V}{2\Delta t}\mathbf{w}^{n-1} + R(\mathbf{w}^{n+1}) = 0 \tag{2}$$

In the dual time stepping scheme this equation is solved by marching the equation

$$\frac{d\mathbf{w}}{d\tau} + R^*(\mathbf{w}) = 0 \tag{3}$$

to a steady state, where the modified residual is

$$R^*(\mathbf{w}) = R(\mathbf{w}) + \frac{3V}{2\Delta t}\mathbf{w} - \frac{2V}{\Delta t}\mathbf{w}^{n} - \frac{V}{2\Delta t}\mathbf{w}^{n-1} \tag{4}$$

In solving equation 4 one is free to use every available acceleration technique for fast steady state solutions without regard for time accuracy. In the author's work it was shown that multigrid techniques can be very effective for this purpose.

The dual time stepping approach has been quite widely adopted, particularly in conjunction with the BDF2 scheme, which is both A and L-stable. Dahlquist has proved that A-stable linear multi-step schemes are at most second order accurate[2]. In the works of Butcher and other specialists in the numerical solution of ordinary differential equations it has been shown that it is possible to design A and L-stable implicit Runge- Kutta schemes which yield higher order accuracy[3–5].

Recently there has been considerable interest in whether implicit Runge-Kutta schemes can achieve better accuracy for a given computational cost than the backwards difference formulas. Most of the studies to date have focused on diagonal implicit Runge-Kutta (DIRK) schemes, sometimes called semi-implicit schemes, in which the stages may be solved successively. These schemes, however, need a large number of stages. For example, the scheme of Kennedy and Carpenter uses one explicit and five implicit stages to attain fourth order accuracy[6].

The purpose of the present study is to investigate the feasibility of using dual time stepping to solve the equations of fully implicit Runge-Kutta schemes, in which the stage equations are fully coupled. The investigation focuses in particular on the two and three stage Gauss schemes[5] and the two and three stage Radau 2A schemes[5,7]. For equation 1, the two stage Gauss scheme takes the form

$$\begin{aligned}
\boldsymbol{\xi}_1 &= \mathbf{w}^n - \frac{\Delta t}{V}\left(a_{11}R(\boldsymbol{\xi}_1) + a_{12}R(\boldsymbol{\xi}_2)\right)\\
\boldsymbol{\xi}_2 &= \mathbf{w}^n - \frac{\Delta t}{V}t\left(a_{21}R(\boldsymbol{\xi}_1) + a_{22}R(\boldsymbol{\xi}_2)\right)\\
\mathbf{w}^{n+1} &= \mathbf{w}^n - \frac{\Delta t}{2V}\left(R(\boldsymbol{\xi}_1) + R(\boldsymbol{\xi}_2)\right)
\end{aligned} \tag{5}$$

where the matrix A of coefficients is

$$A = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} - \frac{\sqrt{3}}{6} \\ \frac{1}{4} + \frac{\sqrt{3}}{6} & \frac{1}{4} \end{bmatrix} \tag{6}$$

and the stage values correspond to Gauss integration points inside the time step with the values $\left(\frac{1}{2} - \frac{\sqrt{3}}{6}\right)\Delta t$ and $\left(\frac{1}{2} + \frac{\sqrt{3}}{6}\right)\Delta t$.

The three stage Gauss scheme takes the form

$$
\begin{aligned}
\boldsymbol{\xi}_1 &= \mathbf{w}^n - \frac{\Delta t}{V}\left(a_{11}R(\boldsymbol{\xi}_1) + a_{12}R(\boldsymbol{\xi}_2) + a_{13}R(\boldsymbol{\xi}_3)\right) \\
\boldsymbol{\xi}_2 &= \mathbf{w}^n - \frac{\Delta t}{V}\left(a_{21}R(\boldsymbol{\xi}_1) + a_{22}R(\boldsymbol{\xi}_2) + a_{23}R(\boldsymbol{\xi}_3)\right) \\
\boldsymbol{\xi}_3 &= \mathbf{w}^n - \frac{\Delta t}{V}\left(a_{31}R(\boldsymbol{\xi}_1) + a_{32}R(\boldsymbol{\xi}_2) + a_{33}R(\boldsymbol{\xi}_3)\right) \\
\mathbf{w}^{n+1} &= \mathbf{w}^n - \frac{\Delta t}{18V}\left(5R(\boldsymbol{\xi}_1) + 8R(\boldsymbol{\xi}_2) + 5R(\boldsymbol{\xi}_3)\right)
\end{aligned}
\tag{7}
$$

where the matrix A of coefficients is

$$
A = \begin{bmatrix}
\frac{5}{36} & \frac{2}{9} - \frac{\sqrt{15}}{15} & \frac{5}{36} - \frac{\sqrt{15}}{30} \\
\frac{5}{36} + \frac{\sqrt{15}}{30} & \frac{2}{9} & \frac{5}{36} - \frac{\sqrt{15}}{24} \\
\frac{5}{36} + \frac{\sqrt{15}}{30} & \frac{2}{9} + \frac{\sqrt{15}}{15} & \frac{5}{36}
\end{bmatrix}
\tag{8}
$$

and the stage values correspond to the intermediate times $\left(\frac{1}{2} - \frac{\sqrt{15}}{10}\right)\Delta t$, $\frac{1}{2}\Delta t$, and $\left(\frac{1}{2} + \frac{\sqrt{15}}{10}\right)\Delta t$ within the time step.

The two stage Gauss scheme is fourth order accurate, while the three stage Gauss scheme is sixth order accurate. Both schemes are A-stable but not L-stable. The Radau 2A schemes include the end of the time interval as one of the integration points, corresponding to Radau integration. Consequently they have an order of accuracy $2s - 1$ for s stages. They have the advantages, on the other hand, that the last stage value is the final value, eliminating the need for an extra step to evaluate w^{n+1}, and that they are L-stable.

For the solution of equation 1, the two stage Radau 2A scheme takes the form

$$
\begin{aligned}
\xi_1 &= \mathbf{w}^n - \frac{\Delta t}{V}\left(a_{11}R(\xi_1) + a_{12}R(\xi_2)\right) \\
\xi_2 &= \mathbf{w}^n - \frac{\Delta t}{V}\left(a_{21}R(\xi_1) + a_{22}R(\xi_2)\right) \\
\mathbf{w}^{n+1} &= \xi_2
\end{aligned}
\tag{9}
$$

where the matrix A of coefficients is

$$
A = \begin{bmatrix}
\frac{5}{12} & -\frac{1}{12} \\
\frac{3}{4} & \frac{1}{4}
\end{bmatrix}
$$

and the stage values correspond to Radau integration points at $\frac{1}{3}\Delta t$ and Δt.

The three stage Radau 2A scheme takes the form

$$
\begin{aligned}
\xi_1 &= \mathbf{w}^n - \frac{\Delta t}{V}\left(a_{11}R(\xi_1) + a_{12}R(\xi_2) + a_{13}R(\xi_3)\right) \\
\xi_2 &= \mathbf{w}^n - \frac{\Delta t}{V}\left(a_{21}R(\xi_1) + a_{22}R(\xi_2) + a_{23}R(\xi_3)\right) \\
\xi_3 &= \mathbf{w}^n - \frac{\Delta t}{V}\left(a_{31}R(\xi_1) + a_{32}R(\xi_2) + a_{33}R(\xi_3)\right) \\
\mathbf{w}^{n+1} &= \xi_3
\end{aligned}
\tag{10}
$$

where the matrix A of coefficients is

$$
A = \begin{bmatrix}
\frac{88-7\sqrt{6}}{360} & \frac{296-169\sqrt{6}}{1800} & \frac{-2+3\sqrt{6}}{225} \\
\frac{296+169\sqrt{6}}{1800} & \frac{88+7\sqrt{6}}{360} & \frac{-2-3\sqrt{6}}{225} \\
\frac{16-\sqrt{6}}{36} & \frac{16+\sqrt{6}}{36} & \frac{1}{9}
\end{bmatrix}
$$

and the stage values correspond to the Radau integration points $\frac{4-\sqrt{6}}{10}\Delta t$, $\frac{4+\sqrt{6}}{10}\Delta t$ and Δt.

It is shown in the next section that dual time stepping can be used to solve the equations for all these schemes with an inexpensive preconditioner. The last section presents results of the application of this scheme to the flow over a pitching airfoil.

II. Formulation of the dual time stepping scheme

In order to clarify the issues it is useful to consider first the application of the two stage Gauss scheme to the scalar equation

$$\frac{du}{dt} = au \tag{11}$$

where a is a complex coefficient lying in the left half plane. A naive application of dual time stepping would simply add derivatives in pseudo time to produce the scheme

$$\begin{aligned} \frac{d\xi_1}{d\tau} &= a(a_{11}\xi_1 + a_{12}\xi_2) + \frac{u^n - \xi_1}{\Delta t} \\ \frac{d\xi_2}{d\tau} &= a(a_{21}\xi_1 + a_{22}\xi_2) + \frac{u^n - \xi_2}{\Delta t} \end{aligned} \tag{12}$$

which may be written in vector form as

$$\frac{d\xi}{d\tau} = B\xi + c \tag{13}$$

where

$$B = \begin{bmatrix} a_{11}a - \frac{1}{\Delta t} & a_{12}a \\ a_{21}a & a_{22}a - \frac{1}{\Delta t} \end{bmatrix}, \; c = \frac{1}{\Delta t}\begin{bmatrix} u^n \\ u^n \end{bmatrix}$$

For equation (13) to converge to a steady state the eigenvalues of B should lie in the left half plane. These are the roots of

$$\det(\lambda I - B) = 0$$

or

$$\lambda^2 - \lambda\left((a_{11} + a_{22})a - \frac{2}{\Delta t}\right) + a_{11}a_{22}a^2 - (a_{11} + a_{22})\frac{a}{\Delta t} + \frac{1}{\Delta t^2} - a_{12}a_{21}a^2 = 0$$

Substituting the coefficient values for the Gauss scheme given in equation (6), we find that

$$\lambda = \frac{1}{4}a - \frac{1}{\Delta t} \pm ia\sqrt{\frac{1}{48}}$$

Then if $a = p + iq$

$$\lambda = \frac{1}{4}p \pm q\sqrt{\frac{1}{48}} - \frac{1}{\Delta t} + i\left(\frac{1}{4}q \pm p\sqrt{\frac{1}{48}}\right)$$

and for small Δt one root could have a positive real port even when a lies in the left plane.

In order to prevent this we can modify equation (12) by multiplying the right hand side by a preconditioning matrix. It is proposed here to take the inverse of the Runge-Kutta coefficient array A as the preconditioning matrix. Here

$$A^{-1} = \frac{1}{D}\begin{bmatrix} a_{22} & -a_{12} \\ a_{21} & a_{11} \end{bmatrix}$$

where the determinant of A is

$$D = a_{11}a_{22} - a_{12}a_{21}$$

Setting

$$\begin{aligned} r_1 &= a(a_{11}\xi_1 + a_{12}\xi_2) + \frac{u^n - \xi_1}{\Delta t} \\ r_2 &= a(a_{21}\xi_1 + a_{22}\xi_2) + \frac{u^n - \xi_2}{\Delta t} \end{aligned}$$

the preconditioned dual time stepping scheme now takes the form

$$\begin{aligned} \frac{d\xi_1}{d\tau} &= (a_{22}r_1 - a_{12}r_2)D \\ &= a\xi_1 + \frac{a_{22}}{D\Delta t}(u^n - \xi_1) - \frac{a_{12}}{D\Delta t}(u^n - \xi_2) \\ \frac{d\xi_2}{d\tau} &= (a_{11}r_2 - a_{21}r_1)/D \\ &= a\xi_2 + \frac{a_{11}}{D\Delta t}(u^n - \xi_2) - \frac{a_{21}}{D\Delta t}(u^n - \xi_1) \end{aligned}$$

which may be written in the vector form (13) where now

$$B = \begin{bmatrix} a - \frac{a_{22}}{D\Delta t} & \frac{a_{12}}{D\Delta t} \\ \frac{a_{21}}{D\Delta t} & a - \frac{a_{11}}{D\Delta t} \end{bmatrix}, \; c = \frac{1}{D\Delta t} \begin{bmatrix} (a_{22} - a_{12})u^n \\ (a_{11} - a - 21)u^n \end{bmatrix}$$

Now the dual time stepping scheme will reach a steady state if the roots of

$$\det(\lambda I - B) = 0$$

lie in the left half plane. Substituting the coefficients of B the roots satisfy

$$\lambda^2 - \lambda \left(2a - \frac{a_{11} + a_{22}}{D\Delta t} \right) + a^2 - a\frac{a_{11} + a_{22}}{D\Delta t} + \frac{1}{D}\Delta t^2 = 0$$

and using the coefficient values of the Gauss scheme, we now find that

$$\lambda^2 - \lambda \left(2a - \frac{6}{\Delta t} \right) + a^2 - \frac{6a}{\Delta t} + \frac{12}{\Delta t^2} = 0$$

yielding

$$\lambda = a - \frac{3}{\Delta t} \pm i\frac{\sqrt{3}}{\Delta t}$$

Accordingly both roots lie in the left half plane whenever a lies in the left half plane, establishing the feasibility of the dual time stepping scheme.

In the case of the linear system

$$\frac{du}{dt} = Au$$

a similar analysis may be carried out if A can be reduced to diagonal form by a similarity transformation

$$A = V\Lambda V^{-1}$$

Then setting $v = V^{-1}u$,

$$\frac{dv}{dt} = \Lambda v$$

and the same preconditioning scheme can be used separately for each component of v.

Following this approach, the proposed dual time stepping scheme for the nonlinear equations (5) is

$$\begin{aligned} \mathbf{r}_1 &= \frac{V}{\Delta t}(\mathbf{w}^n - \boldsymbol{\xi}_1) - a_{11}R(\boldsymbol{\xi}_1) - a_{12}R(\boldsymbol{\xi}_2) \\ \mathbf{r}_2 &= \frac{V}{\Delta t}(\mathbf{w}^n - \boldsymbol{\xi}_2) - a_{21}R(\boldsymbol{\xi}_1) - a_{22}R(\boldsymbol{\xi}_2) \end{aligned} \tag{14}$$

and

$$\begin{aligned} \frac{d\boldsymbol{\xi}_1}{d\tau} &= (a_{22}\mathbf{r}_1 - a_{12}\mathbf{r}_2)/D \\ \frac{d\boldsymbol{\xi}_2}{d\tau} &= (a_{11}\mathbf{r}_2 - a_{21}\mathbf{r}_1)/D \end{aligned} \tag{15}$$

A more general approach, which facilitates the analysis of dual time stepping for implicit Runge-Kutta schemes in general, is presented in the next paragraphs. Using vector notation a naive application of dual time stepping yields the equations

$$\frac{d\xi}{d\tau} = aA\xi + \frac{1}{\Delta t}\left(w^n - \xi\right) \tag{16}$$

and the eigenvalues of the matrix

$$B = aA - \frac{1}{\Delta t}I \tag{17}$$

do not necessarily lie in the left half plane. Introducing A^{-1} as a preconditioning matrix the dual time stepping equations become

$$\frac{d\xi}{d\tau} = a\xi + \frac{1}{\Delta t}A^{-1}(w^n - \xi) \tag{18}$$

so we need the eigenvalues of

$$B = aI - \frac{1}{\Delta t}A^{-1} \tag{19}$$

to lie in the left half plane for all values of a in the left half plane.

The eigenvalues of B are

$$a - \frac{1}{\Delta t}\frac{1}{\lambda_k}\ ,\ k = 1, 2, 3$$

where λ_k are the eigenvalues of A. Thus they will lie in the left half plane for all values of a in the left half plane if the eigenvalues of A lie in the right half plane. The characteristic polynomials of A for the two-stage Gauss and Radau 2A schemes are

$$\lambda^2 - \frac{1}{2}\lambda + \frac{1}{12} = 0$$

and

$$\lambda^2 - \frac{2}{3}\lambda + \frac{1}{6} = 0$$

with roots

$$\lambda = \frac{1}{4} \pm i\sqrt{\frac{1}{48}}$$

and

$$\lambda = \frac{1}{3} \pm i\sqrt{\frac{1}{18}}$$

respectively, which in both cases lie in the right half plane. It may be determined by a rather lengthy calculation that the characteristic polynomials for the three stage Gauss and Radau 2A schemes are

$$\lambda^3 - \frac{1}{2}\lambda^2 + \frac{1}{10}\lambda - \frac{1}{120} = 0 \tag{20}$$

and

$$\lambda^3 - \frac{6}{10}\lambda^2 + \frac{3}{20}\lambda - \frac{1}{60} = 0. \tag{21}$$

Rather than calculating the roots directly, it is simpler to use the Routh-Hurwitz criterion which states that the roots of

$$a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0$$

lie in the left half plane if all the coefficients are positive and

$$a_2a_1 > a_3a_0.$$

The roots of A will lie in the right half plane if the roots of $-A$ be in the left half plane. Here, the characteristic polynomials of $-A$ for the two-three stage schemes are

$$\lambda^3 + \frac{1}{2}\lambda^2 + \frac{1}{10}\lambda + \frac{1}{120}$$

and

$$\lambda^3 + \frac{6}{10}\lambda^2 + \frac{3}{20}\lambda + \frac{1}{60}$$

and it is easily verified that the Routh-Hurwitz condition is satisfied in both cases. Thus, it may be concluded that the preconditioned dual time stepping scheme will work for all four of the implicit Runge-Kutta schemes under considerations.

In the present analysis it has been directly proved that for each of the four schemes the eigenvalues of A lie in the right half plane. It may be noted, however, that for an implicit Runge-Kutta scheme with coefficient matrix A and row vector b^T of the final stage coefficients, the stability function is

$$r(z) = 1 + b^T(I - zA)^{-1}e,$$

where

$$e^T = [1, 1, \dots].$$

When this is expanded as

$$r(z) = \frac{N|z|}{D|z|},$$

$\det(I - zA)$ appears in the denominator, for an A-stable scheme. $D(z)$ cannot have any zeros in the left half plane. Accordingly it may be concluded the eigenvalues of A must lie in the right half plane for any A-stable implicit Runge-Kutta schemes.

The proposed dual time stepping scheme for the three stage schemes is finally

$$\begin{aligned}
\mathbf{r}_1 &= \frac{V}{\Delta t}(\mathbf{w}^n - \boldsymbol{\xi}_1) - a_{11}R(\boldsymbol{\xi}_1) - a_{12}R(\boldsymbol{\xi}_2) - a_{13}R(\boldsymbol{\xi}_3) \\
\mathbf{r}_2 &= \frac{V}{\Delta t}(\mathbf{w}^n - \boldsymbol{\xi}_2) - a_{21}R(\boldsymbol{\xi}_1) - a_{22}R(\boldsymbol{\xi}_2) - a_{23}R(\boldsymbol{\xi}_3) \\
\mathbf{r}_3 &= \frac{V}{\Delta t}(\mathbf{w}^n - \boldsymbol{\xi}_3) - a_{31}R(\boldsymbol{\xi}_1) - a_{32}R(\boldsymbol{\xi}_2) - a_{33}R(\boldsymbol{\xi}_3)
\end{aligned}$$

and

$$\begin{aligned}
\frac{d\boldsymbol{\xi}_1}{d\tau} &= d_{11}\mathbf{r}_1 + d_{12}\mathbf{r}_2 + d_{13}\mathbf{r}_3 \\
\frac{d\boldsymbol{\xi}_2}{d\tau} &= d_{21}\mathbf{r}_1 + d_{22}\mathbf{r}_2 + d_{23}\mathbf{r}_3 \\
\frac{d\boldsymbol{\xi}_3}{d\tau} &= d_{31}\mathbf{r}_1 + d_{32}\mathbf{r}_2 + d_{33}\mathbf{r}_3
\end{aligned}$$

where the coefficients d_{jk} are the entries of A^{-1}.

These schemes have been applied to solve the Euler equations for unsteady flow past a pitching airfoil with these implicit Runge-Kutta schemes. The spatial discretization uses the Jameson-Schmidt-Turkel (JST) scheme[8]. The dual time stepping equations (15) are solved by a preconditioned Runge-Kutta scheme similar to that proposed by Rossow[9] and Swanson *et al*[10]. Writing the dual time stepping equations as

$$\frac{d\xi}{d\tau} + R^*(\xi) = 0$$

the scheme takes the general form

$$\begin{aligned}
\xi^{(1)} &= \xi^{(0)} - \alpha_1 \Delta\tau P^{-1} R^{*(0)} \\
\xi^{(2)} &= \xi^{(0)} - \alpha_2 \Delta\tau P^{-1} R^{*(1)} \\
\xi^{(3)} &= \xi^{(0)} - \alpha_3 \Delta\tau P^{-1} R^{*(2)}
\end{aligned}$$

where P^{-1} denotes a single sweep of an LUSGS scheme in each direction.

In evaluating the residuals, the convective and dissipative parts are treated separately. Thus if R^* is split as

$$R^{*(k)} = Q^{*(k)} + D^{*(k)},$$

then

$$Q^{*(0)} = Q^*\left(\xi^{(0)}\right), \quad D^{*(0)} = D^*\left(\xi^{(0)}\right)$$

and

$$\begin{aligned}
Q^{*(k)} &= Q^*\left(\xi^{(k)}\right) \\
D^{*(k)} &= \beta_k D^*\left(\xi^{(k)}\right) + (1 - \beta_k)\, D^{*(k-1)}.
\end{aligned}$$

Both two and three stage schemes have been used. In Euler simulations, the coefficients of the two stage scheme are

$$\alpha_1 = 0.30, \quad \beta_1 = 1,$$

$$\alpha_2 = 1, \quad \beta_2 = 2/3,$$

while those of the three stage scheme are

$$\alpha_1 = 0.15, \quad \beta_1 = 1,$$

$$\alpha_2 = 0.40, \quad \beta_2 = 0.5,$$

$$\alpha_3 = 1, \quad \beta_3 = 0.5.$$

The LUSGS scheme uses a first order Roe discretization scheme on the left hand side, and it is stabilized by rounding out the absolute values of the eigenvalues above zero, and the addition of a diagonal load. The scheme is further accelerated by an over-relaxation factor in each LUSGS step, and under-relaxing the dissipative terms by a factor of S in the second and third stages. Finally the preconditioned Runge-Kutta scheme is combined with the full approximation multigrid time stepping scheme proposed by the author[11] to yield convergence rates in the range of 0.5 per multigrid W cycle.

III. Results

The flow past a pitching airfoil has been used as a test case for the new dual time stepping implicit Runge-Kutta schemes. The selected case is the A GARD case CT-6. This is a pitching NACA 64A010 airfoil at a Mach number of 0.796. The airfoil is symmetric and the mean angle of attack is zero, leading to a flow in which shock waves appear alternately on the upper and lower surface. The pitching amplitude is ± 1.01 degrees, and the reduced frequency, defined as

$$k = \frac{\omega_{\text{chord}}}{2 q_\infty}$$

where ω is the pitching rate, has a value of 2.02. Calculations were performed on an O-mesh with 160×32 cells (displayed in Figure 1), which has a very tight spacing at the trailing edge. An initial steady state was established using 50 multigrid cycles. These were sufficient to reduce the density residual to a value less than 10^{-12}. Then 6 pitching cycles were calculated with the dual time stepping scheme. This is sufficient to reach an almost steady periodic state. The implicit time step was selected such that each pitching cycle was calculated with 18 steps, corresponding to a shift of 20 degrees in the phase angle per step.

It was found that the Gauss implicit Runge-Kutta schemes require the equations to be solved to a very small tolerance in each step. In the case of the two stage scheme the required tolerance is typically smaller than 10^{-6} as measured by the density residual. This tolerance was achieved with 15 inner iterations of the dual time stepping scheme in each implicit step. This is the absolute minimum number of inner iterations required for this case, and the calculation fails when 14 inner iterations are used. The failure mode is a failure to satisfy the Kutta condition leading to the sudden appearance of a low pressure spike and possibly a vacuum state at the trailing edge. The convergence history of the inner iterations of the last step is displayed in Figure 2.

Figure III. displays a snapshot of the solution calculated by the two stage Gauss scheme at several phase angles during the fifth pitching cycle. The pressure distribution is displayed by the pressure coefficient C_p with the negative axis upward, using $+$ symbols for the lower surface. Figure 4 displays the lift coefficient C_L against the angle of attack α. these values lie on a slanting oval curve because of the phase lag between the lift coefficient and the angle of attack. These results show excellent agreement with calculations using the second or third order backward difference formulas.

Very similar results were obtained with the three stage Gauss scheme. However, it was found necessary to converge the inner iterations to an even smaller tolerance, typically smaller than 10^{-7} as measured by the density residual, and the number of inner iterations needed to be increased from 15 to 25 in order to reliably solve the AGARD CT6 test case. Since three coupled equations have to be solved instead of two, the computational cost of the three stage sixth order accurate scheme is about $2\frac{1}{2}$ that of the two stage fourth order scheme for this application.

The L-stability of the Radau schemes might be expected to improve their capability to treat stiff equations. In this application, however, they proved to be more robust in a different sense. They did not require anywhere near such a high level of convergence of the inner iterations for each implicit time step. It appears that the most dangerous operation in the Gauss schemes is the calculation of the final updated value from the stage values, which involves multiplication by $\Delta t/V$. Here Δt may be quite large, while the cell area V becomes very small in the vicinity of the trailing edge. The Radau schemes eliminate this operation because the last stage value is the final updated value. It was found, in fact, that as few as 5 inner iterations at each implicit time step were sufficient for the three stage Radau scheme. Moreover the two stage preconditioned Runge-Kutta scheme could be used for these iterations. The final result is indistinguishable from those obtained with the two and three stage Gauss schemes. Actually the calculated lift and drag coefficients during the last pitching cycle agree to four digits to those obtained with the Gauss schemes, and hence the results are not displayed separately.

The advantage of the Radau schemes becomes even more strongly apparent when the flow is modeled by the unsteady Reynolds averaged Navier-Stokes (URANS) equations. Figures 5-8 show the AGARD CT6 case calculated on a 512×64 C-mesh at a Reynolds number of 6 million with a Baldwin-Lomax turbulence model, using the three stage Radau scheme. In this case 25 inner iterations at each implicit time step were found to be sufficient.

IV. Conclusion

Dual time stepping is a feasible approach for solving the coupled residual equations of a fully implicit Runge-Kutta schemes for unsteady flow simulations. The required number of inner iterations is significantly smaller for the Radau 2A schemes in comparison with the Gauss schemes. Since the two and three stage Radau schemes are third and fifth order accurate respectively, and they are also L-stable, they appear to be competitive with alternative implicit schemes. It seems, therefore, that they merit further study.

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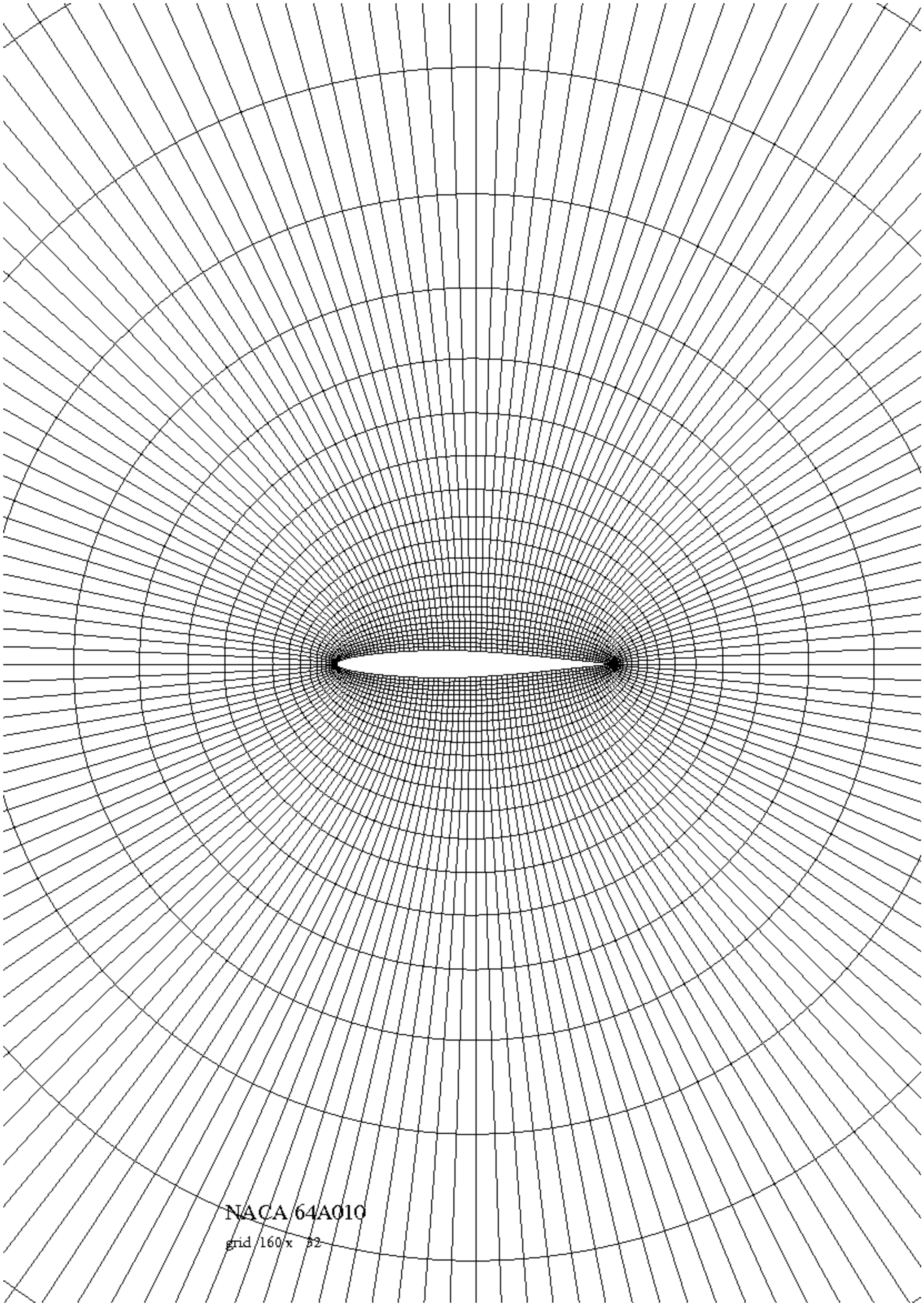


Figure 1: O-mesh used in pitching airfoil calculation

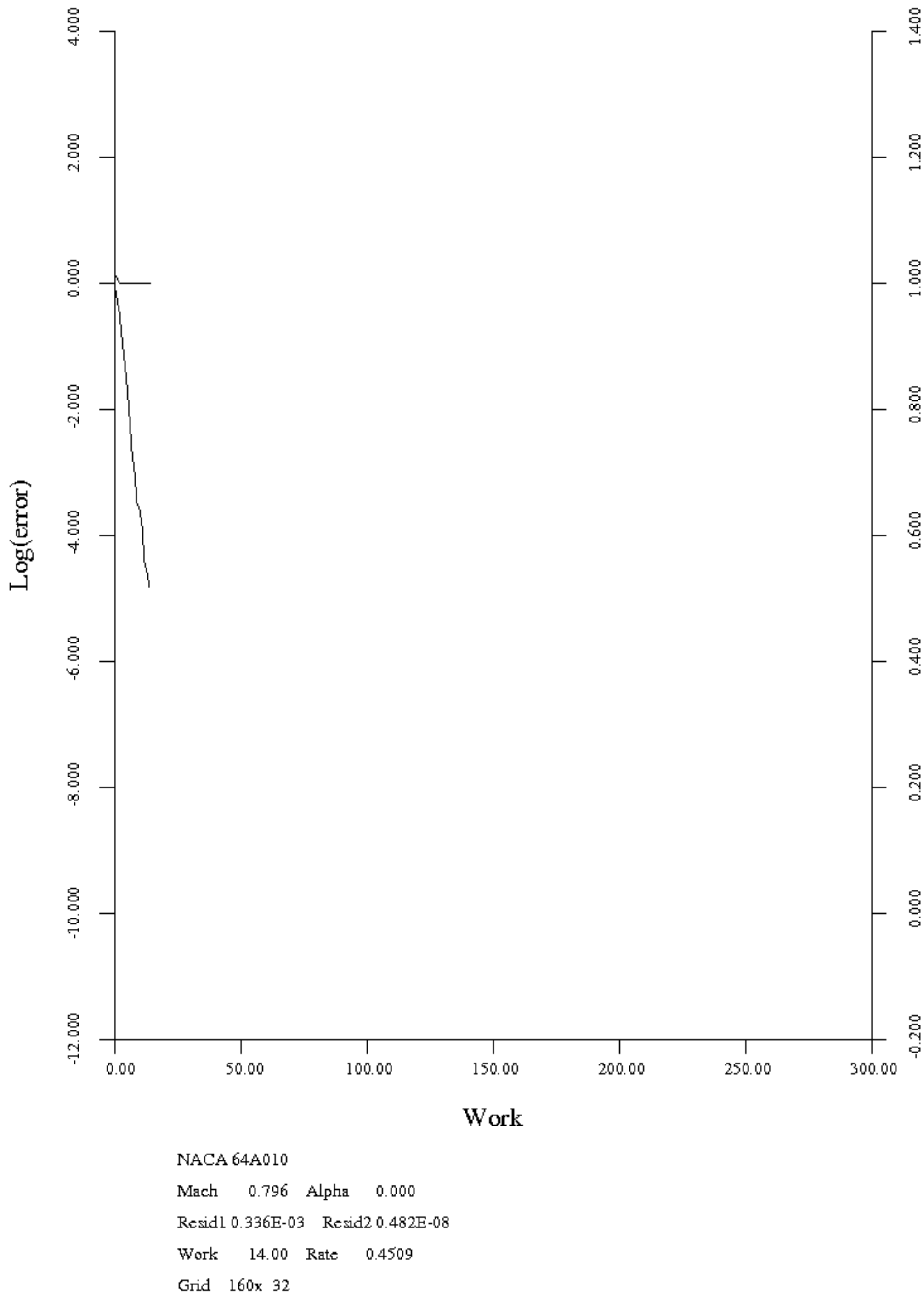
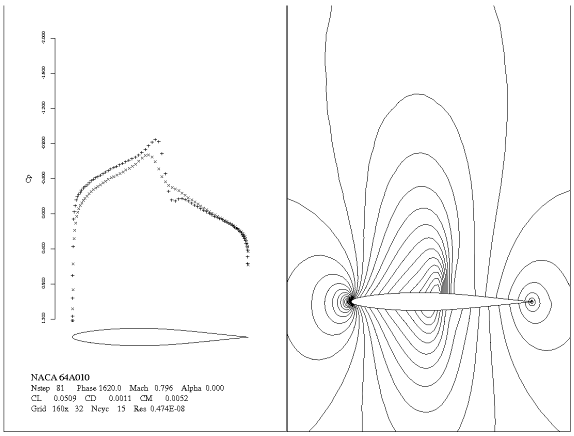
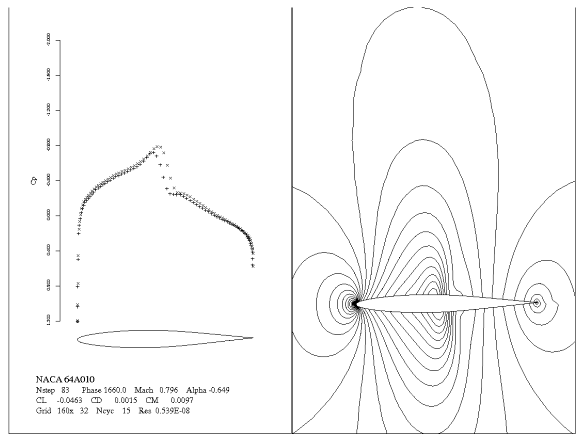


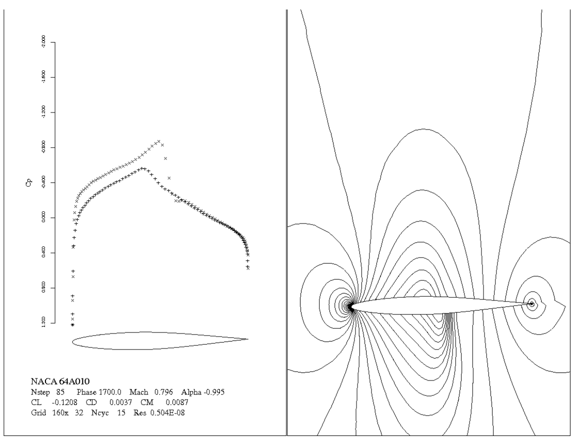
Figure 2: Convergence history of the inner iterations of the last step



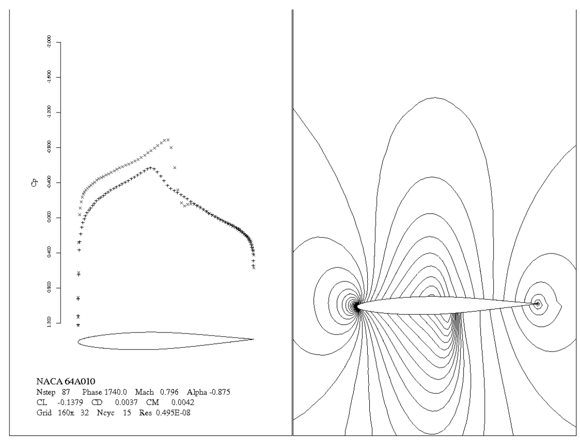
(a) Phase angle 1



(b) Phase angle 2



(c) Phase angle 3



(d) Phase angle 4

Figure 3: Snapshot of the solution at several phase angles during the fifth pitching cycle.

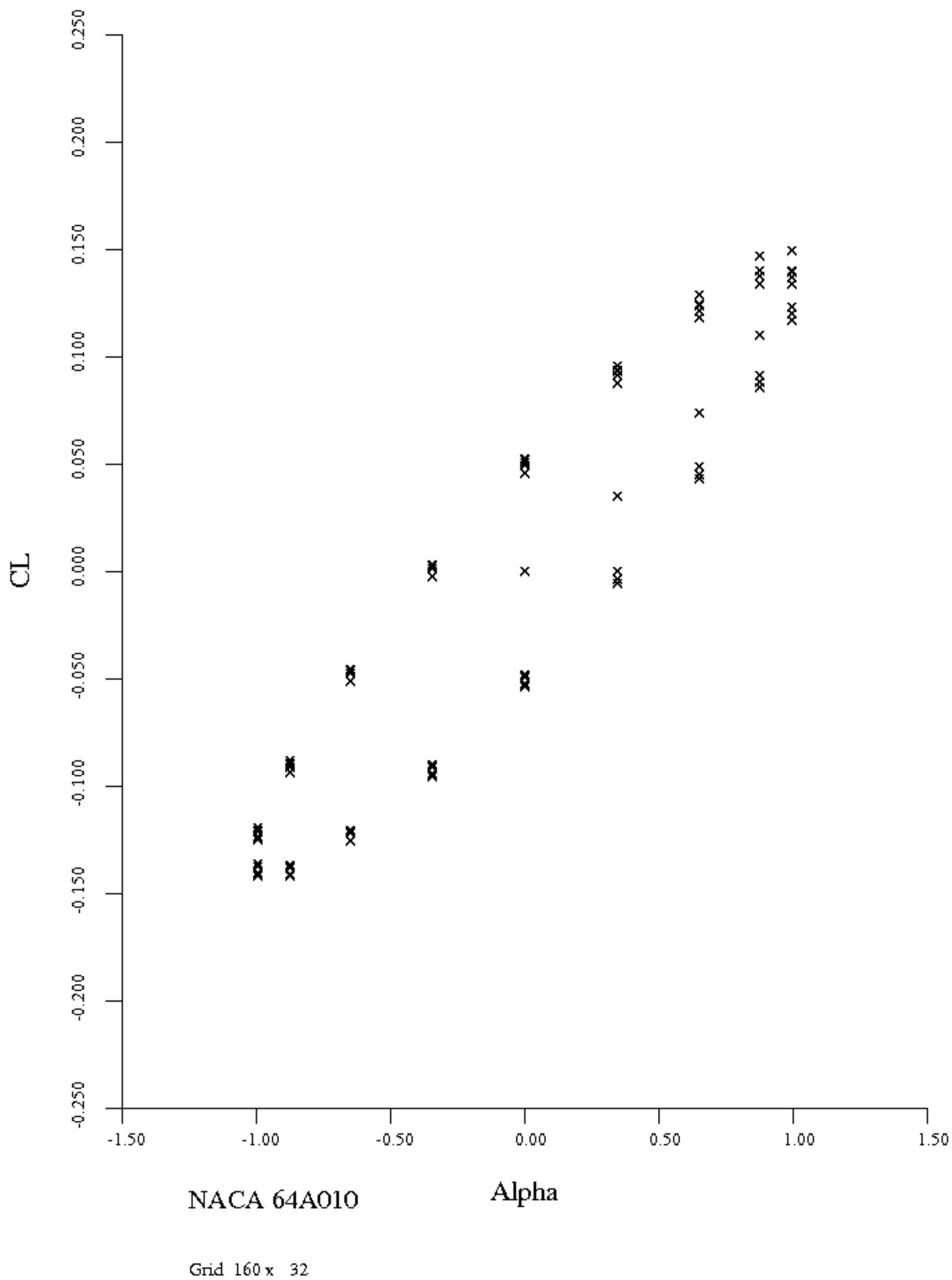


Figure 4: Lift coefficient C_L versus angle of attack α.

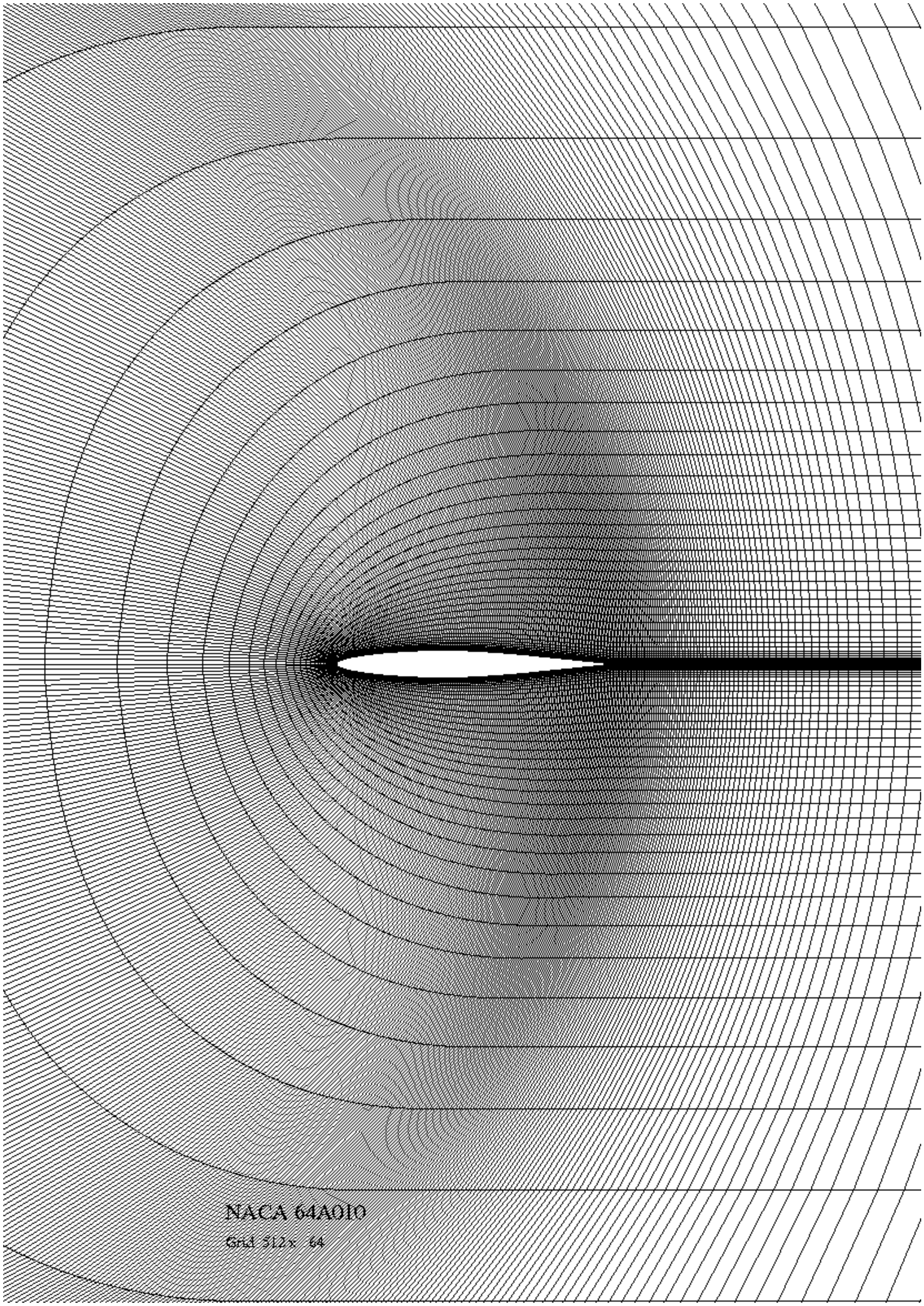


Figure 5: C-mesh for URANS calculation.

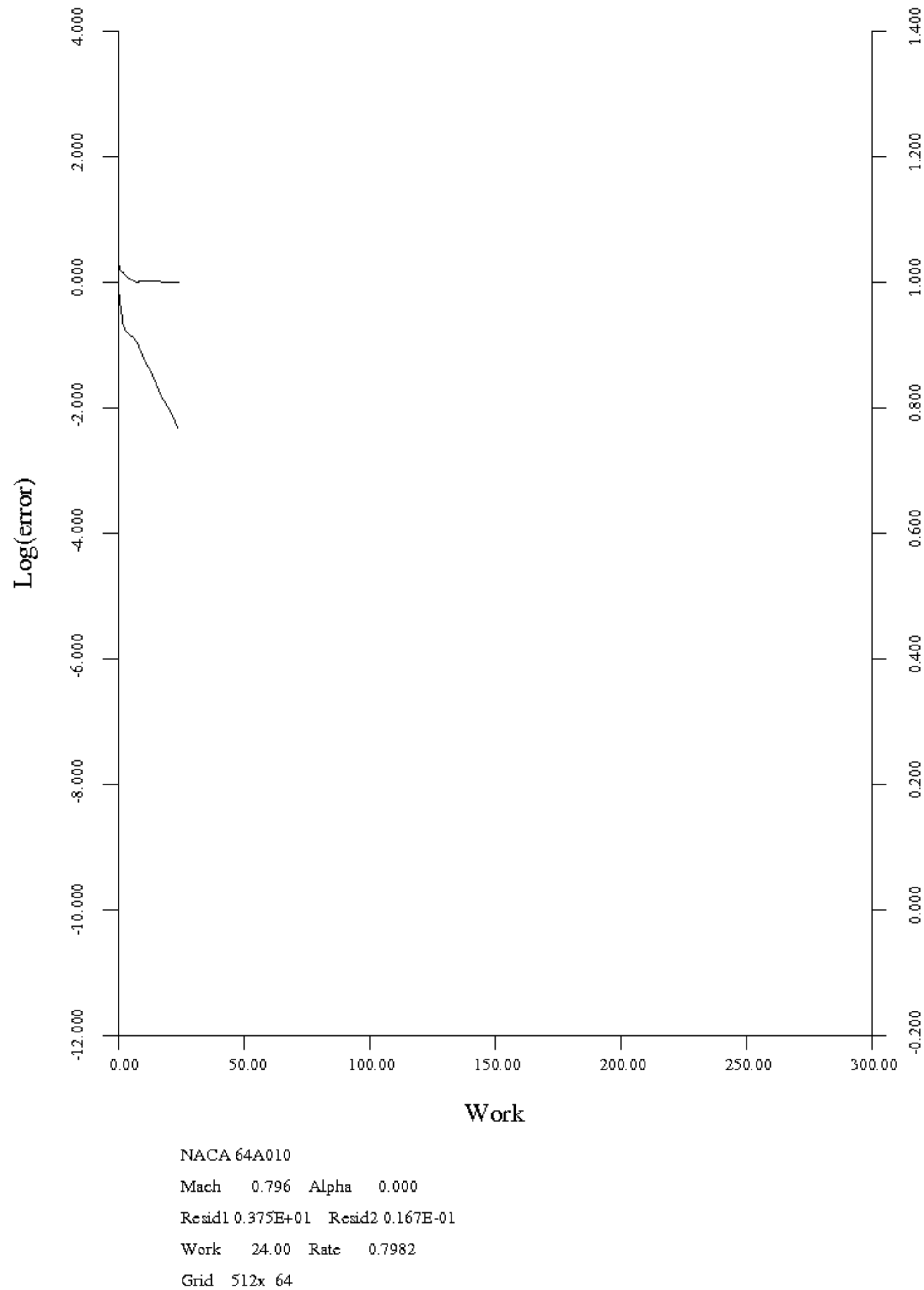
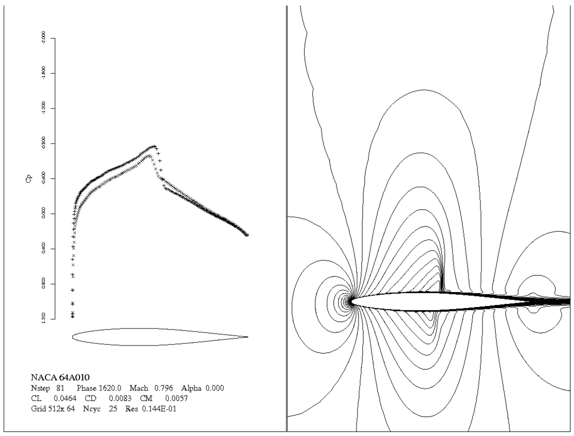
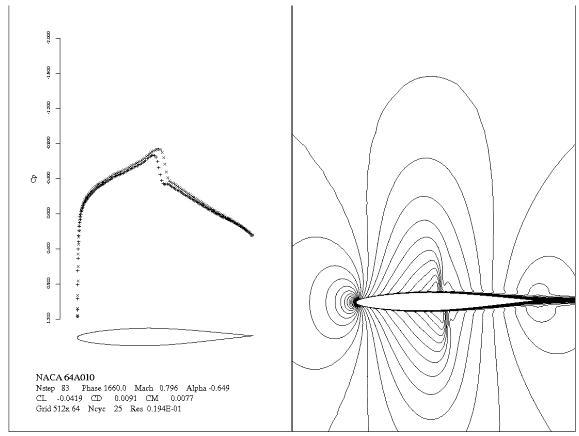


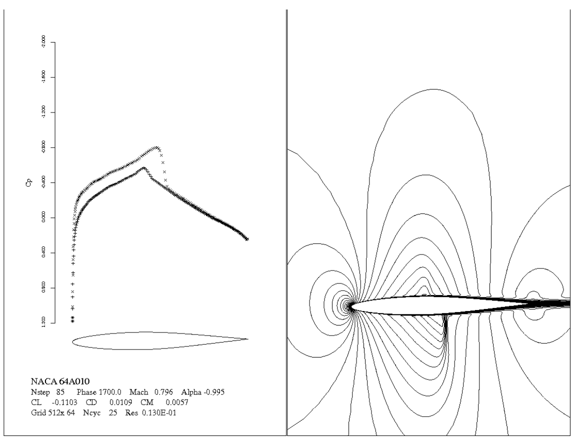
Figure 6: Convergence history of the inner iterations during the last step of the URANS calculation.



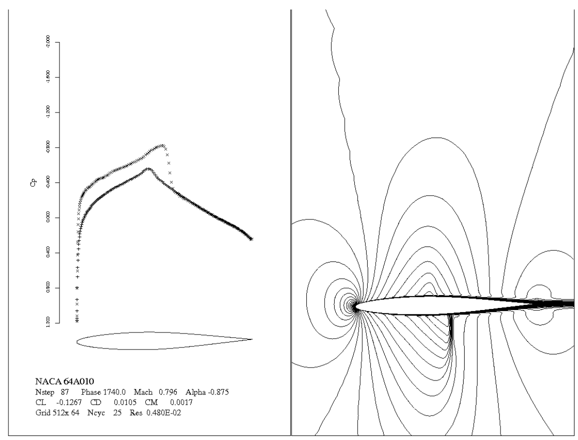
(a) Phase angle 1



(b) Phase angle 2



(c) Phase angle 3



(d) Phase angle 4

Figure 7: Snapshot of the solution during the last pitching cycle of the URANS calculation.

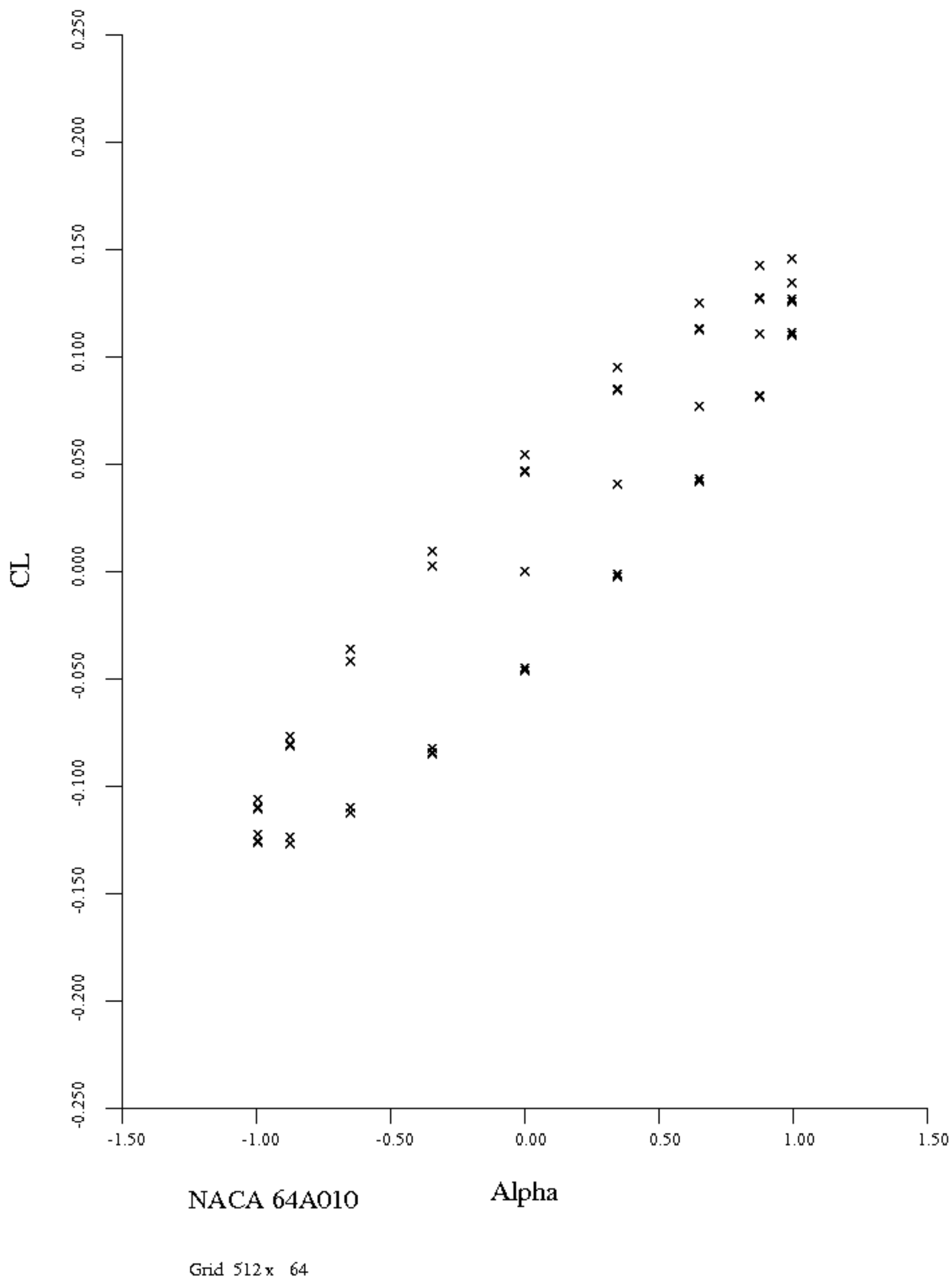


Figure 8: Lift coefficient C_L versus angle of attack α for the URANS calculation.