Design of a Single-Input System for Specified Roots Using Output Feedback

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Abstract—**A method is given for choosing feedbacks from an output vector of smaller dimension than the state vector in such a way that a subset of the roots assume desired values. A sufficient condition for arbitrary placement of as many roots as there are feedbacks is proved. It is also proved that a root corresponding to a mode which is either uncontrollable or unobservable cannot be altered. It may be desired to place more roots than there are feedbacks, and it is shown that the method can be extended to give an approximate solution of this problem.**

Introduction

It is well known that the characteristic roots of a single input system can be forced to assume arbitrarily specified values by the addition of feedbacks from a measurement, or a suitable estimate, of every state variable [1],[2]. An augmentation system can, therefore, be designed which will guarantee stability. The plant, however, may not need so many feedbacks for its stabilization. Also the response to the normal range of inputs is often largely determined by a few dominant roots. For the sake of simplicity, it may in these cases be desirable to restrict the number of feedbacks. The system may, for example, contain actuators which are fast compared with the main plant. Often the precise locations of the roots introduced by the actuators are relatively unimportant, and feedbacks around the actuators are an unwanted complication.

In this paper the existing theory is extended to treat systems in which feedbacks are only permitted from specified output variables. It is shown how to calculate feedbacks so that as many roots as there are feedbacks assume specified values. Limitations on the attainable roots are also investigated. It may be desired to choose a set of feedbacks to bring more roots than there are feedbacks as close as possible to desired values. The condition that the characteristic polynomial vanishes at each desired root then yields an overdetermined set of equations for the feedbacks. The least-squares solution of these equations represents the best approximate solution, in the sense that the deviations of the characteristic polynomial from zero will be as small as possible at the desired roots. If the roots can actually be brought close to their desired values, this method will often yield a suitable design. Its practical application would be in the design of an augmentation system of the minimum complexity necessary to assure acceptable stability.

Formulation

Consider a single input system

$$\dot{x} = Ax + bu \tag{1}$$

where the n-dimensional vector x represents the state, and u is the scalar control. Suppose that only an m-dimensional output vector

$$y = Cx \tag{2}$$

is measured and that the control is

$$u = d^T y. \tag{3}$$

The equation of the closed-loop system is

$$\dot{x} = (A + bd^T C)x. \tag{4}$$

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Also

$$(\lambda I - A)^{-1} = F(\lambda)/q(\lambda) \tag{5}$$

where $q(\lambda)$ is the open-loop characteristic polynominal $|\lambda I - A|$, and $F(\lambda)$ is the adjoint matrix, which may be expressed as a series

$$F(\lambda) = I\lambda^{n-1} + F_1\lambda^{n-2} + \cdots + F_{n-1}. \tag{6}$$

The matrices F_k can be computed jointly with the coefficients a_k of the characteristic polynominal by Leverrier's algorithm [4]:

$$A_1 = A, \quad a_1 = \operatorname{tr}(A_1), \quad F_1 = A_1 + a_1 I$$
$$A_k = AF_{k-1}, \quad a_k = -(1/k)\operatorname{tr}(A_k), \quad F_k = A_k + a_k I. \tag{7}$$

Let $r(\lambda)$ be the closed-loop characteristic polynomial $|\lambda I - A - bd^Tc|$. Removing the factor $\lambda I - A$,

$$r(\lambda) = |I - bd^TC(\lambda I - A)^{-1}|\,|\lambda I - A|$$
$$= |I - bd^TCF(\lambda)/q(\lambda)|\,q(\lambda).$$

Using the identity $|I + XY| = |I + YX|$, the matrix of the first determinant is reduced to a scalar. Thus

$$r(\lambda) = q(\lambda) - [CF(\lambda)b]^Td. \tag{8}$$

Define the $m \times n$ matrix W as

$$W = C[b, F_1b, \cdots, F_{n-1}b]. \tag{9}$$

Then substituting the series (6) for $F(\lambda)$,

$$CF(\lambda)b = W\begin{bmatrix}\lambda^{n-1}\\ \lambda^{n-2}\\ \vdots\\ 1\end{bmatrix}.$$

Let p desired roots λ_{D_i} be specified. The feedbacks should be chosen so that $r(\lambda_{D_i})$ vanishes for each λ_{D_i}. This leads to a set of linear equations for the feedbacks

$$EW^Td = e \tag{10}$$

where the $p \times n$ matrix E and p vector e are defined as

$$E = \begin{bmatrix}\lambda_{D_1}{}^{n-1} & \lambda_{D_1}{}^{n-2} & \cdots & 1\\ \lambda_{D_2}{}^{n-1} & \lambda_{D_2}{}^{n-2} & \cdots & 1\\ \vdots & & & \\ \lambda_{D_p}{}^{n-1} & \lambda_{D_p}{}^{n-2} & \cdots & 1\end{bmatrix}, \quad e = \begin{bmatrix}q(\lambda_{D_1})\\ q(\lambda_{D_2})\\ \vdots\\ q(\lambda_{D_p})\end{bmatrix}. \tag{11}$$

If two equal desired roots $\lambda_{D_1} = \lambda_{D_2}$ are specified, then $(d/d\lambda)r(\lambda)|_{\lambda=\lambda_1}$ should also vanish, so that the second row of E and e must be replaced by $[(n-1)\lambda_{D_1}{}^{n-2}, (n-2)\lambda_{D_2}{}^{n-3}, \cdots, 1, 0]$ and $(dq/d\lambda)|_{\lambda=\lambda_{D1}}$. If some of the desired roots appear in conjugate complex pairs, the equations can be reduced to a set of real equations

$$TWE^Td = Te \tag{12}$$

by multiplying on the left by a block diagonal matrix T, with elements equal to unity for each real root, and 2×2 blocks of the form

$$\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ -i & i\end{bmatrix}$$

for each conjugate complex pair.

Restrictions on Attainable Roots

When m roots are specified, an exact solution can be found as long as the $m \times m$ matrix EW^T is nonsingular. If the system is controllable, $[b, F_1b, \cdots, F_{n-1}b]$ is of rank n, and, assuming that C has independent rows, W is of rank m. Also, if the specified roots are distinct, the rows of E, being similar to rows of a Vandermonde matrix [5], are independent, so that E is also of rank m. These conditions are not sufficient, however, to ensure that EW^T is nonsingular. Each element of the ith row of EW^T is a polynominal of degree $n-1$ in λ_{D_i} with coefficients from one of the columns of W^T. Assume that at least one of these polynomials is of degree $n-1$. Then, if the first $m-1$ roots λ_{D_i} are given, the vanishing of the determinant EW^T represents an equation of degree $n-1$, for λ_{D_m}. Each of the previously given λ_{D_i} is a root of this equation. The remaining $n-m$ roots represent unattainable choices of λ_{D_m}. When the maximum degree of the polynomials in EW^T is less than $n-1$, there are fewer restrictions. In fact, it is possible to prove the following Theorem.

Theorem 1

Assume that a) the system is completely controllable, b) C is of rank m, and c)

$$CA = 0, \quad CAb = 0, \cdots, CA^{n-m-1}b = 0.$$

Then real feedbacks from the output vector may be found which yield m arbitrarily chosen distinct roots, which are either real or appear in conjugate pairs.

Proof: Conditions a) and b) ensure that W is of rank m. From condition c), its first $n-m$ columns vanish. Thus the remaining m columns form a nonsingular matrix. The square matrix consisting of the last m columns of E is also nonsingular, so EW^T is nonsingular.

Condition c) implies complete observability. Thus $CA^{n-m}b$ cannot vanish, since W is of rank m, so at least one row of CA^{n-m} is independent of all the rows of $C, CA, \cdots, CA^{n-m-1}$. But if CA did not have at least one row independent of C, multiplying by A^{n-m-1}, CA^{n-m} could not have any rows independent of CA^{n-m-1}. Similarly, CA^2 has at least one row independent of C and CA, and so on. Thus $C, CA, \cdots, CA^{n-m}$ contains at least $m + n - m = n$ independent rows.

Assuming that the open-loop roots are distinct, the nature of the restrictions when the system is not completely controllable or not completely observable are most easily seen from the diagonal representation of A. Let $A = V\Lambda V^{-1}$, where the open-loop roots λ_i are the elements of the diagonal matrix Λ. Then $F(\lambda) = VU(\lambda)V^{-1}$, where $U(\lambda)$ is a diagonal matrix with elements $q(\lambda)/(\lambda - \lambda_i)$. Also, if the ith row of $V^{-1}b$ vanishes, the corresponding mode is uncontrollable, and if the ith column of CV vanishes, it is unobservable [3]. But then according to (8), $r(\lambda_i) = q(\lambda_i) = 0$. Thus the following theorem may be deduced.

Theorem 2

The root corresponding to either an uncontrollable or an unobservable mode of a system with distinct open-loop roots cannot be altered by feedbacks from the output vector.

Approximate Solutions

If it is desired to influence more roots than there are feedbacks, it is possible to look for a least-squares solution of (10). In case some of the feedbacks become undesirably large, a penalty may also be included on their magnitude. The solution is then determined by minimizing

$$J = \sum_{i=1}^{n} Q_i[r(\lambda_{D_i})]^2 + \sum_{j=1}^{m} R_jd_j^2 \tag{13}$$

where the coefficients Q_i determine the relative emphasis on the different roots, and the coefficients R_i determine the penalties, possibly zero, on the feedbacks. As long as the coefficients Q_i are equal for each member of a conjugate complex pair of desired roots, a real rectangular matrix Z and real vector f may be defined as

$$Z = \begin{bmatrix}TQEW^T\\ R\end{bmatrix}, \quad f = \begin{bmatrix}TQe\\ 0\end{bmatrix} \tag{14}$$

where Q and R are diagonal matrices formed from the coefficients Q_i and R_i. Then

$$J = \| Zd - f \|^2 \tag{15}$$

and the solution can be determined with the aid of the generalized inverse Z^I of Z [6] as

$$d = Z^I f \tag{16}$$

where, for an overdetermined system,

$$Z^I = (Z^T Z)^{-1} Z^T. \tag{17}$$

This method of obtaining an approximate solution, using a norm which measures the deviations from zero of the characteristic polynomial at the desired roots rather than a direct comparison between the closed-loop and desired roots, avoids the difficulty of deciding which root is to be compared with which.

Example 1

Consider the system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & -10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix} u$$

which might represent a truck under the action of a spring and a motor. Then x_1 and x_2 would be the displacement and velocity of the truck, and x_3 the force exerted by the motor. The open-loop characteristic polynomial is

$$q(\lambda) = (\lambda + i)(\lambda - i)(\lambda + 10)$$

with roots $\pm i$ representing undamped oscillations, and -10 the time lag of the motor. Let x_1 and x_2 be measured, and two distinct roots λ_{D_1} and λ_{D_2} be specified. Equation (10) reduces to

$$\begin{bmatrix} 10 & 10\lambda_{D_1} \\ 10 & 10\lambda_{D_2} \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} q(\lambda_{D_1}) \\ q(\lambda_{D_2}) \end{bmatrix}.$$

The conditions of Theorem 1 are met, so there are no restrictions on λ_{D_1} and λ_{D_2}. The third root becomes $10 + \lambda_{D_1} + \lambda_{D_2}$. Any desired amount of damping can be introduced at the expense of a reduction in the effective speed of the motor. For example, if λ_{D_1} and λ_{D_2} are $-1 \pm i$, the third root is -8.

If only the velocity x_2 is measured, and a single root λ_D specified, (10) reduces to

$$10\lambda_D d = q(\lambda_D).$$

Any finite value of λ_D except zero is attainable. It is more likely that it would be desired to increase the damping of the principal roots. If two desired roots are specified as $-1 \pm i$, the least-squares solution is $d = -1.4$, yielding roots $-0.827 \pm 0.717i$, -0.835. The introduction of a single feedback would thus suffice to produce a well-damped stable system.

Example 2

The lateral equations of motion of a typical medium-sized aircraft under the influence of its rudder are

$$\begin{bmatrix} \dot{\beta} \\ \dot{p} \\ \dot{\phi} \\ \dot{r} \end{bmatrix} = \begin{bmatrix} -0.154 & 0.004 & 0.178 & -0.990 \\ -1.25 & -2.85 & 0 & 1.43 \\ 0 & 1.0 & 0 & 0 \\ 0.568 & -0.277 & 0 & -0.284 \end{bmatrix} \begin{bmatrix} \beta \\ p \\ \phi \\ r \end{bmatrix} + \begin{bmatrix} 0.075 \\ -0.727 \\ 0 \\ -2.05 \end{bmatrix} \delta_r$$

$$\dot{\delta}_r = -10\delta_r + 10u$$

where β, p, ϕ, and r are the sideslip, roll rate, roll angle, and yaw rate, δ_r is the rudder angle, and the last equation represents the rudder actuator. The roots of the open-loop system are

$-0.268 \pm 0.895_i$	Dutch roll
-2.78	roll subsidence
0.033	spiral
-10.0	actuator.

If no feedback is allowed from δ_r (around the actuator), the first four roots can be placed exactly. For example, let the desired roots be

$-1.2 \pm 1.5_r$	Dutch roll
-2.8	roll subsidence
-0.05	spiral.

These are obtained with the control $u = -0.976\beta - 0.054p + 0.175\phi + 0.848r$. The actuator root then becomes -8.04. To prevent difficulties when the roll angle ϕ becomes large, it may be desired to eliminate the feedback from ϕ. Then, if the same four roots are specified with equal weights on each, the least-squares solution is $u = -0.935\beta - 0.001p + 0.797r$. The corresponding closed-loop roots are

$-1.20 \pm 1.49_r$	Dutch roll
-2.78	roll subsidence
0.022	spiral
-8.13	actuator.

The desired well-damped Dutch roll is obtained. The spiral instability is reduced but not eliminated. It is known, however, that pilots experience no difficulty in controlling spiral instability as long as the doubling time is of the order of 20 seconds or more. These feedbacks would constitute quite an acceptable stability augmentation system for this airframe. The least-squares equations were set up and solved in about 0.02 second on an IBM 360.

Conclusions

It is generally possible to force m roots of an nth-order single-input system to assume prescribed values by introducing feedbacks from m measurements. The feedbacks may be determined by solving a system of m linear equations. The restrictions on the prescription of the values of the m roots are slight, but the remaining $n - m$ roots assume arbitrary values. When a few roots dominate the behavior of the system, this method is a convenient way of designing a simple controller, which uses a few measurements and avoids the use of dynamic elements. If more than m roots are important, one may try to bring them close to desired values. If this is possible, suitable feedback gains may be found by computing the least-squares solution of the overdetermined set of equations which would be satisfied if the desired values were actually attainable. In comparison with time-domain methods for designing controllers with a restricted set of feedbacks [7], [8], the computations are extremely easy. A time-domain method has the advantage, however, that the desired qualities of the response can be directly represented in the definition of the performance index. The determination of the minimum number of measurements which will permit complete stabilization of a given plant without the introduction of dynamic elements remains an open question.

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