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A direct approach to the design of asymptotically optimal controllers†

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Many optimal control solutions require a complete set of measurements of current state variables, which may not be fully available. It is reasonable to ask whether compensators cannot be designed in such a way that the desirable qualities of the optimal control are reproduced.

One method of constructing a compensator that generates an asymptotically optimal control is to generate an estimate of the complete set of state variables by an auxiliary dynamic system, such as an observer or a Kalman filter. It can be shown, however, that a simpler design is often possible by employing the fact that usually a linear combination of the set of state variables is all that is required to reconstruct the optimal control. A simple direct method of determining such a controller is presented in this paper.

1. Introduction

Optimal control problems generally lead to solutions which require complete state feedback for their implementation. For situations where the entire state vector is not directly measurable, other methods must be found to synthesize the desired control. This paper describes a general method for designing low-order compensators which generate a control which approximates the optimal control, without requiring a complete set of measurements.

Consider the completely controllable (Kreindler and Sarachik 1964) time-invariant plant described by the nth-order differential equation:

$$\begin{aligned} \dot{\mathbf{x}}(t) &= A\mathbf{x}(t)+B\mathbf{u}(t), \\ \mathbf{y}(t) &= C\mathbf{x}(t), \end{aligned} \tag{1}$$

where A is an $n \times n$ system matrix, B is an $n \times p$ input matrix, C is an $m \times n$ output matrix of rank $m (m < n)$, $\mathbf{x}$ is an n-dimensional state vector, $\mathbf{u}$ is a p-dimensional input vector and $\mathbf{y}$ is an m-dimensional output vector. Consider the quadratic cost functional:

$$J(\mathbf{u}, \mathbf{x}) = \int_0^\infty [\mathbf{x}^{\mathrm{T}} Q \mathbf{x} + \mathbf{u}^{\mathrm{T}} R \mathbf{u}] \, d\tau, \tag{2}$$

where Q and R are, respectively, $n \times n$ and $p \times p$ symmetric positive definite matrices. It is well known (Kalman 1960 a) that the control law that forces the plant from some arbitrary initial state $\mathbf{x}(0)$ to the origin, while minimizing J, is given by:

$$\mathbf{u}^*(t) = R^{-1} B^{\mathrm{T}} P \mathbf{x}(t) = D\mathbf{x}, \tag{3}$$

where P is the solution of:

$$PA + A^{\mathrm{T}} P - PBR^{-1} B^{\mathrm{T}} P + Q = 0. \tag{4}$$

† Communicated by the Authors.

If not all the states are available for direct measurement, other means must be found to implement or at least to approximate $\mathbf{u}^*$.

One approach is to use an auxiliary dynamic system to generate an estimate $\hat{\mathbf{x}}$ of the state, and to construct the control as:

$$\mathbf{u} = D\hat{\mathbf{x}}.$$

Two well-known methods of doing this are to use a Kalman filter (Kalman 1960 b) or to use a compatible observer (Luenberger 1964, 1966). The Kalman filter incorporates a model of the plant, and is thus an auxiliary system of the same order as the plant. It is known that the optimal control in the presence of noisy measurements can be generated by using a Kalman filter in cascade with the deterministic optimal controller. A compatible observer is a system whose output equals the state of the plant to within an exponentially decaying error. Luenberger (1964) has shown that if the system is observable it is always possible to construct a compatible observer of order $n-m$, described by:

$$\dot{\mathbf{z}} = F\mathbf{z} + G\mathbf{y} + WB\mathbf{u}, \tag{5}$$

$$\mathbf{x} = \begin{bmatrix} C \\ \cdots \\ W \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{y} \\ \mathbf{z} \end{bmatrix} = M\mathbf{y} + N\mathbf{z}, \tag{6}$$

where F, G and W are $(n-m)\times(n-m)$, $(n-m)\times m$ and $(n-m)\times n$-dimensional matrices, respectively, which satisfy:

$$WA - FW = GC. \tag{7}$$

Bongiorno and Youla (1968) extend the results of Luenberger (1964) by establishing necessary and sufficient conditions for which a compatible observer can be employed. Dellon and Sarachik (1968) present a technique that is applicable to time-varying linear systems in which the resulting controller dynamics is also of order $n-m$.

2. Asymptotically optimal controllers

It is desirable that the compensator should generate a control which differs from the optimal control by an exponentially decaying error. Compensators using Kalman filters or observers have this property of being asymptotically optimal. Motivated by the desire to reduce the complexity of the design, we may, however, ask if it is necessary to reconstruct an estimate of the complete state vector, particularly if $p \ll n$, when all that is really required are the p independent linear combinations of $x_1, x_2, \ldots, x_n$ which constitute the optimal control. Luenberger (1966) noted that if it is only required to estimate a single linear combination of the state variables, the order of the observer can be reduced, but did not provide an easy method of constructing a compensator which makes use of this idea. A direct method of designing asymptotically optimal controllersof low order is described in the following paragraphs. It requires the solution of a set of linear equations, and is simple to execute.

Assume that the control is:

$$\mathbf{u} = D\mathbf{x} + \mathbf{e}, \tag{8}$$

where $\mathbf{e}$ is the error due to the constraint that only $\mathbf{y}$ and not $\mathbf{x}$ is available. Consider the possibility of designing the controller so that

$$\mathbf{e} = K\mathbf{v}, \tag{9}$$

where $\mathbf{v}$ has dimension r, and satisfies the dynamic equation:

$$\dot{\mathbf{v}} = F\mathbf{v}. \tag{10}$$

If the eigenvalues of F have negative real parts, then $\lim_{t\to\infty} \mathbf{v}(t) = 0$, and the control will asymptotically approach the optimal control.

Consider the transformation:

$$\mathbf{z} = \mathbf{v} + W\mathbf{x}. \tag{11}$$

Differentiating (11), and using (1) and (10), we have:

$$\begin{aligned}\dot{\mathbf{z}} &= F\mathbf{v} + WA\mathbf{x} + WB\mathbf{u} \\ &= F\mathbf{z} + (WA - FW)\mathbf{x} + WB\mathbf{u},\end{aligned} \tag{12}$$

also:

$$\begin{aligned}\mathbf{u} &= D\mathbf{x} + K\mathbf{v} \\ &= (D - KW)\mathbf{x} + K\mathbf{z}.\end{aligned} \tag{13}$$

Suppose that there exist matrices W, G, H, K satisfying:

$$WA - FW = GC, \tag{14}$$

$$D - KW = HC. \tag{15}$$

Then (12) and (13) reduce to:

$$\dot{\mathbf{z}} = F\mathbf{z} + G\mathbf{y} + WB\mathbf{u}, \tag{16}$$

$$\mathbf{u} = H\mathbf{y} + K\mathbf{z}. \tag{17}$$

These would represent the equations of an auxiliary dynamic controller, with $\mathbf{y}$ as its input, that would generate the desired control. Our problem is thus to find a simultaneous solution to (14) and (15).

3. Minimum order controller for single-input plant

Consider a single-input plant. The problem is to find F of minimum order r such that (14) and (15) can be solved. The fact that u is a scalar reduces (14) and (15) to

$$WA - FW = GC, \tag{18}$$

$$\mathbf{d}^{\mathrm{T}} - \mathbf{k}^{\mathrm{T}} W = \mathbf{h}^{\mathrm{T}} C, \tag{19}$$

where W is $r \times n$, F is $r \times r$, G is $r \times m$, $\mathbf{k}$ is an r vector and $\mathbf{h}$ is an m vector.

Our objective is to obtain an expression for W from (18) and eliminate it from (19). Let the characteristic equation of F be:

$$\det(\lambda I - F) = \lambda^r + f_1 \lambda^{r-1} + \ldots + f_r = 0.$$

Utilizing a method presented in Jameson (1968) and Rothschild and Jameson (1970) for solving equations of the form (18) given G, C, F and A, we proceed by defining the following sequence:

$$C_i = GCA^{i-1} + FC_{i-1} = WA^i - F^i W, \tag{20}$$

where

$$C_0 = 0.$$

Multiplying each C_i by f_{r-1} and summing we obtain:

$$\sum_{i=0}^{r} W f_i A^{r-i} = \sum_{i=0}^{r} f_i C_{r-i} + \sum_{i=0}^{r} f_i F^{r-i} W, \tag{21}$$

where $f_0 = 1$. From the Cayley–Hamilton theorem, F satisfies:

$$\sum_{i=0}^{r} f_i F^{r-i} = 0, \tag{22}$$

so that (21) reduces to:

$$W(A^r + f_1 A^{r-1} + \dots f_r I) = C_r + f_1 C_{r-1} + \dots + f_{r-1} C_1. \tag{23}$$

Expanding the right-hand side in (23) gives:

$$\begin{aligned} W(A^r + f_1 A^{r-1} + \dots f_r I) = GCA^{r-1} + (F + f_1 I) GCA^{r-2} \\ + (F^2 + f_1 F + f_2 I) GCA^{r-3} + \dots \\ + (F^{r-1} + f_1 F^{r-2} + \dots f_{r-1} I) GC. \end{aligned} \tag{24}$$

Multiplying (24) on the left by $\mathbf{k}^{\mathrm{T}}$, and using (19), W is eliminated and we obtain:

$$\begin{aligned} \mathbf{h}^{\mathrm{T}} C(A^r + f_1 A^{r-1} + \dots f_r I) + \mathbf{s}_1{}^{\mathrm{T}} GCA^{r-1} + \mathbf{s}_2{}^{\mathrm{T}} GCA^{r-2} + \dots \mathbf{s}_r{}^{\mathrm{T}} GC \\ = \mathbf{d}^{\mathrm{T}}(A^r + f_1 A^{r-1} + \dots f_r I), \end{aligned} \tag{25}$$

where $\mathbf{s}_i$ represents the ith column of the S matrix defined by:

$$S = [\mathbf{k} \,\vdots\, (F^{\mathrm{T}} + f_1 I) k \,\vdots\, \dots \,\vdots\, (F^{\mathrm{T}(r-1)} + f_1 F^{\mathrm{T}(r-2)} + \dots f_r I)\,\mathbf{k}]. \tag{26}$$

Also, multiplying (24) on the right by $(A^r + f_1 A^{-1} + \dots f_r I)^{-1}$ gives:

$$\begin{aligned} W = [GCA^{r-1} + (F + f_1 I) GCA^{r-2} + \dots (F^{r-1} + f_1 F^{r-2} + \dots) GC] \\ \times [A^r + f_1 A^{r-1} + \dots f_r I]^{-1}. \end{aligned} \tag{27}$$

Under the assumption that A and F do not have any common eigenvalues, Gantmacher (1960) shows that (18) has a unique solution for W. The solution for (18) must also satisfy (24), whose solution is unique and given by (27), since $(A^r + f_1 A^{r-1} + \dots f_r I)^{-1}$ is unique. If (25) is satisfied along with (27), we see that the value of $(\mathbf{d}^{\mathrm{T}} - \mathbf{h}^{\mathrm{T}} C)$ obtained from (25) must equal $\mathbf{k}^{\mathrm{T}}$ multiplied by the expression for W in (27). Thus satisfying (25) and (27) implies satisfaction of (18) and (19).

Our problem has been reduced to finding a solution to (25) for stable F. Let F and $\mathbf{k}^{\mathrm{T}}$ be specified. If we denote the r rows of G as $\mathbf{g}_1{}^{\mathrm{T}}, \mathbf{g}_2{}^{\mathrm{T}}, \dots, \mathbf{g}_r{}^{\mathrm{T}}$, upon transposing (25) with the aid of Kronecker products (Gantmacher 1960) we obtain:

$$\begin{aligned} (A^{\mathrm{T}r} + f_1 A^{\mathrm{T}r-1} + \dots f_r I) C^{\mathrm{T}} \mathbf{h} + (s_{11} A^{\mathrm{T}r-1} + s_{12} A^{\mathrm{T}r-2} + \dots s_{1r} I) C^{\mathrm{T}} \mathbf{g}_1 \\ + (s_{21} A^{\mathrm{T}r-1} + s_{22} A^{\mathrm{T}r-2} + \dots s_{2r} I) C^{\mathrm{T}} \mathbf{g}_2 + \dots \\ + (s_{r1} A^{\mathrm{T}r-1} + s_{r2} A^{\mathrm{T}r-2} + \dots s_{rr} I) C^{\mathrm{T}} \mathbf{g}_r = (A^{\mathrm{T}r} + f_1 A^{\mathrm{T}r-1} + \dots f_r I)\,\mathbf{d}. \end{aligned} \tag{28}$$

Regarding the unknowns as an $(\mathrm{r}+1) m$ vector, $[\mathbf{h}^{\mathrm{T}} \mathbf{g}_1{}^{\mathrm{T}} \mathbf{g}_2{}^{\mathrm{T}} \dots \mathbf{g}_r{}^{\mathrm{T}}]^{\mathrm{T}}$, the above equations have an $n \times (r-1) m$ matrix. From (26) we note that if F and $\mathbf{k}^{\mathrm{T}}$ are chosen to be an observable pair, it would follow that the columns of the $mr \times mr$ matrix:

$$P = \begin{bmatrix} s_{11} I_m & s_{21} I_m & \dots & s_{r1} I_m \\ s_{12} I_m & s_{22} I_m & \dots & s_{r2} I_m \\ \vdots & \vdots & \vdots & \vdots \\ s_{1r} I_m & \dots & \dots & s_{rr} I_m \end{bmatrix},$$

are linearly independent, where I_m represents the m-dimensional identity matrix. The last mr columns of the $n \times (r+1)m$ matrix (28) can be represented by Ω, where

$$\Omega \triangleq [(s_{11} A^{\mathrm{T}^{r-1}} + s_{12} A^{\mathrm{T}^{r-2}} + \dots s_{1r} I) C^{\mathrm{T}} \mid \dots \mid (s_{r1} A^{\mathrm{T}^{r-1}} + s_{r2} A^{\mathrm{T}^{r-2}} + \dots s_{rr} I) C^{\mathrm{T}}].$$

We note that Ω can be written as:

$$\Omega = [A^{\mathrm{T}^{r-1}} C^{\mathrm{T}} \mid A^{\mathrm{T}^{r-2}} C^{\mathrm{T}} \mid \dots \mid C^{\mathrm{T}}] P. \tag{29}$$

Since P is invertible, we have rank $\Omega = \text{rank}\,[A^{\mathrm{T}^{r-1}} C^{\mathrm{T}} \mid A^{\mathrm{T}^{r-2}} C^{\mathrm{T}} \mid \dots \mid C^{\mathrm{T}}]$. If we now augment Ω with the first m columns of (28), namely

$$(A^{\mathrm{T}^r} + f_1 A^{\mathrm{T}^{r-1}} + \dots f_r I) C^{\mathrm{T}},$$

we find that the $m(r+1)$ columns of (28) span the same space and contain the same number of columns as $A^{\mathrm{T}^r} C^{\mathrm{T}}, A^{\mathrm{T}^{r-1}} C^{\mathrm{T}}, \dots, C^{\mathrm{T}}$. It is clear that a sufficient condition for the existence of a solution to (28) is that $r \geqslant \alpha - 1$, where α is the observability index of A, C, or the least integer such the matrix

$$[C^{\mathrm{T}} \mid A^{\mathrm{T}} C^{\mathrm{T}} \mid \dots \mid A^{\mathrm{T}^{\alpha-1}} C^{\mathrm{T}}]$$

has rank n.

Thus for a single-input plant, a dynamic controller of order $\alpha - 1$ can always be found which will produce a u that asymptotically approaches $\mathbf{d}^{\mathrm{T}} \mathbf{x}$. It can be shown that in general for an observable system:

$$n/m \leqslant \alpha \leqslant n - m + 1.$$

Thus the lowest allowed range for r becomes:

$$(n-m)/m \leqslant r \leqslant n - m.$$

For a multi-output plant ($m > 1$), the order r of F can be lower than $n - m$, the order required by the observer solution. For example if $n = 10$, $m = 5$, and if the measurement matrix were chosen properly, a first-order filter is possible, whereas the observer solution would require an auxiliary system of fifth order. Note that when $m(r+1) > n$ the solution is not unique.

In the case where the observability index is 2, which can occur if m is at least $n/2$, the equations reduce to a very simple form since a first-order auxiliary system is possible. Equations (18) and (19) reduce to:

$$\mathbf{w}^{\mathrm{T}} A - f\mathbf{w} = \mathbf{g}^{\mathrm{T}} C, \tag{30}$$

$$\mathbf{d}^{\mathrm{T}} - k\mathbf{w}^{\mathrm{T}} = \mathbf{h}^{\mathrm{T}} C, \tag{31}$$

where W and G have been reduced to row vectors $\mathbf{w}^{\mathrm{T}}, \mathbf{g}^{\mathrm{T}}$, and F and K have become scalars f and k. Upon transposing the above, and eliminating $\mathbf{w}$, we obtain:

$$[kC^{\mathrm{T}} \mid (A-fI)^{\mathrm{T}} C^{\mathrm{T}}] \begin{bmatrix} \mathbf{g} \\ \mathbf{h} \end{bmatrix} = (A-fI)^{\mathrm{T}} \mathbf{d}, \tag{32}$$

where $\begin{bmatrix} \mathbf{g} \\ \mathbf{h} \end{bmatrix}$ is a composite vector of dimension $2m$.

Provided that $\alpha = 2$, we are assured of the $n \times 2m$ matrix $[kC^{\mathrm{T}} \mid (A-fI)^{\mathrm{T}} C^{\mathrm{T}}]$ having rank n, with $2m \geqslant n$. We thus guarantee the existence of a solution for $\mathbf{g}$ and $\mathbf{h}$ which of course is not unique if $2m > n$. The resulting compensator can be constructed as:

$$u = \mathbf{h}^{\mathrm{T}} \mathbf{y} + kz, \tag{33}$$

$$\dot{z} = fz + \mathbf{g}^{\mathrm{T}} \mathbf{y} + \mathbf{w}^{\mathrm{T}} \mathbf{b} u. \tag{34}$$

4. Multi-input plants

For a system with p inputs the procedure of the last section may be applied separately for each input. This amounts to choosing F and K in a particular way and solving for G and H. The resulting compensator is of order $r = p(\alpha - 1)$. Alternatively one may specify F and G, solve (14) for W, and then solve (15) for H and K. Since (15) represents pn conditions, with pm unknown elements for H and pr unknown elements for K, a solution can be found only if $r \geqslant n - m$. If $r = n - m$ this method leads to the usual solution for an observer, since (15) can be written as:

$$[H \mid K] \begin{bmatrix} C \\ \hdashline W \end{bmatrix} = D, \tag{35}$$

where $\begin{bmatrix} C \\ \hdashline W \end{bmatrix}$ is a square matrix with an inverse which can be partitioned as $[M \mid N]$. Then:

$$[H \mid K] = D[M \mid N] \tag{36}$$

and

$$u = D[M\mathbf{y} + K\mathbf{z}]. \tag{37}$$

Also:

$$\begin{aligned} M\mathbf{y} + N\mathbf{z} &= MC\mathbf{x} + NW\mathbf{x} + N\mathbf{v} \\ &= [M \mid N] \begin{bmatrix} C \\ \hdashline W \end{bmatrix} \mathbf{x} + N\mathbf{v} \\ &= \mathbf{x} + N\mathbf{v} \end{aligned} \tag{38}$$

and since $\mathbf{v}$ decays to zero, $(M\mathbf{y} + N\mathbf{z})$ is an estimate of the state $\mathbf{x}$.

Evidently, it is always possible to design a controller of order not greater than the smaller of $p(\alpha - 1)$ and $(n - m)$ by choosing either F and K or F and G. It remains an open question whether a compensator of still lower order could be produced by specifying F alone, and treating equation (28), generalized for the multi-input case, as a non-linear relation to be satisfied by G, H and K.

5. Closed-loop dynamics

The dynamics of the closed-loop system can be explored by transforming from plant-controller state space coordinates to plant-error state space co-ordinates:

$$\begin{bmatrix} \mathbf{x} \\ \mathbf{v} \end{bmatrix} = \begin{bmatrix} I & 0 \\ \hdashline -W & I \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{z} \end{bmatrix}. \tag{39}$$

Applying (25), eqns. (1), (13) and (14) reduce to:

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{v}} \end{bmatrix} = \left[\begin{array}{c|c} A+BD & BK \\ \hline 0 & F \end{array}\right] \begin{bmatrix} \mathbf{x} \\ \mathbf{v} \end{bmatrix}. \tag{40}$$

Thus the eigenvalues of the complete closed-loop system are those of $(A+BD)$ together with those of F. Note that the eigenvalues of $(A+BD)$ are those of the closed-loop system that would be obtained if a perfect estimate of the plant state vector were available. Assuming complete controllability and observability for the plant, we are then assured that $(A+BD)$ is stable.

Even if we disregard the desire for optimality, we can employ the preceding methods to place arbitrarily all the eigenvalues of the closed-loop system. Wonham (1967) shows that if the plant is completely controllable it is possible to establish that for any set $\Omega = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, with complex λ_i occurring in conjugate pairs, there exists a real D for which $(A+BD)$ has as its eigenvalues the set Ω. Since the eigenvalues of F are also arbitrarily chosen this completes the proof that all roots may be arbitrarily chosen.

6. Comparison of performance between the asymptotically optimal and optimal controllers

It is of interest to determine the loss of performance which may be expected when an optimal controller is replaced by an asymptotically optimal controller. Now the optimal cost is:

$$J^* = \mathbf{x}(0)^{\mathrm{T}} P \mathbf{x}(0), \tag{41}$$

where P is the solution to (4). Also, premultiplying (4) by $\mathbf{x}^{\mathrm{T}}$ and post multiplying by $\mathbf{x}$, and substituting from (1):

$$\dot{\mathbf{x}}^{\mathrm{T}} P \mathbf{x} + \mathbf{x}^{\mathrm{T}} P \dot{\mathbf{x}} = \mathbf{x}^{\mathrm{T}} Q \mathbf{x} - \mathbf{x}^{\mathrm{T}} PB\mathbf{u} - \mathbf{u}^{\mathrm{T}} B^{\mathrm{T}} P \mathbf{x} - \mathbf{x}^{\mathrm{T}} PBR^{-1} B^{\mathrm{T}} P \mathbf{x}. \tag{42}$$

Thus, substituting from (3):

$$\frac{d}{dt}(\mathbf{x}^{\mathrm{T}} P \mathbf{x}) = \mathbf{x}^{\mathrm{T}} Q \mathbf{x} + \mathbf{u}^{\mathrm{T}} R \mathbf{u} + (\mathbf{u} - D\mathbf{x})^{\mathrm{T}} R (\mathbf{u} - D\mathbf{x}), \tag{43}$$

where D is the optimal feedback matrix, and this may be integrated to give:

$$J = J^* + \int_0^\infty (\mathbf{u} - D\mathbf{x})^{\mathrm{T}} R (\mathbf{u} - D\mathbf{x})\, dt. \tag{44}$$

For the asymptotically optimal control it then follows from (8) and (9) that

$$J = J^* + \int_0^\infty \mathbf{v}^{\mathrm{T}} K^{\mathrm{T}} RK \mathbf{v}\, dt. \tag{45}$$

Substituting the solution of (10) for $v(t)$ this becomes:

$$J - J^* = \mathbf{v}(0)^{\mathrm{T}} S \mathbf{v}(0), \tag{46}$$

where

$$S = \int_0^\infty \exp(F^{\mathrm{T}} t) K^{\mathrm{T}} RK \exp(Ft)\, dt. \tag{47}$$

and

$$F^{\mathrm{T}} S + SF = \int_0^\infty \frac{d}{dt}[\exp(F^{\mathrm{T}} t) K^{\mathrm{T}} RK \exp(Ft)]\, dt = -K^{\mathrm{T}} RK. \tag{48}$$

It is realistic to assume that initially the auxiliary system is at rest:

$$\mathbf{z}(0) = 0,$$

since once the plant is in operation there is no means of updating z, and the affect of a sudden disturbance would be to change x but not z. In this case:

$$\mathbf{v}(0) \doteq W\mathbf{x}(0)$$

and

$$J - J^* = \mathbf{x}(0^{\mathrm{T}})\, W^{\mathrm{T}} S W \mathbf{x}(0). \tag{49}$$

The designer is free to choose F, and he may also have some latitude in choosing G and H in cases when the solution of equation (29) is not unique. It turns out, as has been noted by Bongiorno and Youla (1968) for the case of an observer, that it is not generally possible to make $J - J^*$ arbitrarily small by choosing F with large negative eigenvalues, because the solution for W depends on the choice of F. A simple example is instructive.

Consider the following time-invariant plant whose state equations are:

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u, \tag{50}$$

$$\mathbf{y} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \mathbf{x}. \tag{51}$$

If we set $R = 1$, and Q arbitrary but positive definite in (2), the resulting optimal control law from (3) and (4) is:

$$u^*(t) = [d_1 \quad d_2 \quad d_3 \quad d_4]\, \mathbf{x}. \tag{52}$$

The plant is controllable, with observability index equal to two, so a first-order auxiliary system is realizable. Setting $F = f$, and $k = 1$ arbitrarily, we may solve (30) and (31) for $\mathbf{g}$, $\mathbf{h}$ and $\mathbf{w}$:

$$\mathbf{w}^{\mathrm{T}} = [f d_2 \quad d_2 \quad f d_4 \quad d_4], \tag{53}$$

where f must be negative for stability. Then applying (48):

$$S = -\frac{1}{2f}. \tag{54}$$

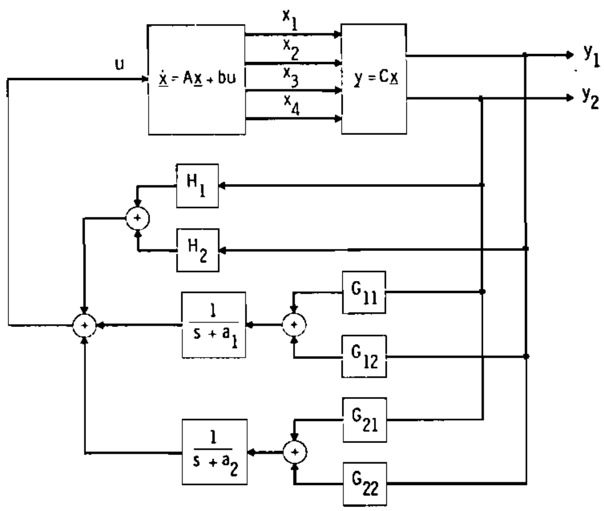
Substituting (53), (54) in (49):

$$\Delta J = J - J^* = -\tfrac{1}{2}\mathbf{x}_0^{\mathrm{T}} \begin{bmatrix} f d_2^2 & d_2^2 & f d_2 d_4 & d_2 d_4 \\ d_2^2 & \dfrac{d_2^2}{f} & d_2 d_4 & \dfrac{d_2 d_4}{f} \\ f d_2 d_4 & d_2 d_4 & f d_4^2 & d_4^2 \\ d_2 d_4 & \dfrac{d_2 d_4}{f} & d_4^2 & \dfrac{d_4^2}{f} \end{bmatrix} \mathbf{x}_0. \tag{55}$$

It is interesting to note that $\Delta J \to \infty$ as $f \to -\infty$ if $x_1(0) \neq 0$ or $x_3(0) \neq 0$. On the other hand, $\Delta J \to 0$ as $f \to -\infty$ if $x_1(0) = x_3(0) = 0$.

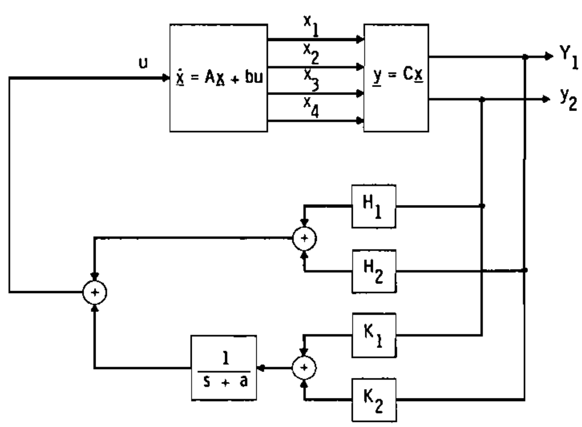
The corresponding observer solution to the above example will require second-order dynamics in the auxiliary system. The resulting typical configurations for the controller using an observer and the asymptotically optimal controller can be compared in figs. 1 and 2.

Fig. 1



Controller configuration for fourth-order plant using observer solution.

Fig. 2



Asymptotically optimal control.

As exemplified in the problem here, the optimal choice of F and any free parameters in G and H will in general depend on the initial conditions since these appear in (49). In order to optimize the compensator it may then be desirable to modify the performance index to eliminate this dependence (Kleinman *et al.* 1968, Levine and Athans 1969, Jameson 1970, Kosut 1970, Jameson and Rossi 1968).

Some possible choices are:

$$J_1 = \operatorname*{ave}_{x(0)} \frac{J}{\|x(0)\|^2}, \tag{56}$$

$$J_2 = \max_{x(0)} \frac{J}{\|x(0)\|^2}, \tag{57}$$

$$J_3 = \max_{x(0)} \frac{J}{J^*}. \tag{58}$$

It is easily shown (Jameson and Rossi 1968, Jameson 1970) that

$$J_1 = \operatorname{tr}(P + W^{\mathrm{T}} S W), \tag{59}$$

$$J_2 = \lambda_{\max}(P + W^{\mathrm{T}} S W), \tag{60}$$

$$J_3 = \lambda_{\max}[P^{*-1}(P + W^{\mathrm{T}} S W)]. \tag{61}$$

7. Conclusions

In areas where compensator design has been previously made chiefly from intuition and experience, it is expected that the theory of asymptotically optimal control will help lead to a more complete understanding of how a system's behaviour may be modified through the addition of dynamic elements.

It is important to observe that the method does not dispense with the judgement of the designer, who must still decide on the appropriate measurements to be used, but it will help indicate a desirable configuration of the compensator for a particular measurement set. The roots of the closed-loop system are the roots of the optimal system plus the additional roots of the compensator, so the designer has freedom to prescribe the additional roots as any stable set.

For plants which exhibit parameter variations during the normal course of operation, such as an aircraft at different points of its flight envelope, the present method could be used to indicate a suitable configuration for the compensator. It would then be necessary to pick best compromise values of the filter constants over the expected range of plant parameter variations.

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