APPLICATIONS OF MULTI-GRID METHODS FOR
TRANSONIC FLOW CALCULATIONS

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Abstract

Multiple grid methods are discussed on the basis of the Poisson equation. In the second part, routine-type application of a multi-grid solver for the full potential equation in combination with a boundary layer integral method is demonstrated. Finally, the importance of multi-grid techniques for systems of partial differential equations (Euler, Navier Stokes) is discussed and some first results are presented for a scheme presently under consideration.

Introduction

Very fast and accurate methods for transonic flow computations are needed by design engineers to improve present day and to develop future transport as well as military aircraft. Such methods permit detailed transonic flow studies within a short time at low cost. Multiple grid methods have been demonstrated to be very powerful for non-elliptic two-dimensional, transonic flow problems by South and Brandt¹ as well as Jameson². Since practical flow analysis implies contour-conformal grid systems, which in general will be non-orthogonal and stretched, fast flow solvers must be insensitive against mesh spacing.

In order to understand the basic properties and the efficiency of multi-grid methods, a study of such methods has been performed on the basis of the Poisson equation. These numerical experiments indicated the combination of multi-grid plus an ADI-scheme to be the most robust procedure. Therefore the combination of such a solver for the full potential equation as proposed by Jameson² is presented in the second part of the present paper. Comparison with measurements is done after inclusion of viscous effects by means of integral boundary layer methods.

While such potential flow solutions prove to be useful for flows with moderate shock strength, the isentropic and irrotational flow assumption does lead to significant errors in the upper transonic speed range. Additional deficiencies occur in transonic lifting flow as pointed out in Ref. 3 due to the Kutta condition for potential flow theory. These problems can be overcome by solving either the time dependent exact inviscid equations (Euler equations) or the time dependent viscous equations (Navier Stokes equations). However, techniques being developed for the full potential equation (quasi-linear second order equation) cannot be simply applied for the Euler or Navier Stokes equations, since they represent systems of first order partial differential equations of hyperbolic nature in time. Since standard solution methods for these equations require very large computer time, efficient multi-grid solvers are highly desirable for such equations. In the last part of the present paper two different approaches for multi-grid schemes for the Euler equations are discussed. For the one similar for the Ni-scheme⁴ first results are presented.

Multi-Grid Study for the Poisson Equation

The model case which will be considered is a test function

$$f = c_1(x+y)^{d_1} \sin(a_1 x) \sin(b_1 y) + c_2(x+y)^{d_2} \sin(a_2 x) \sin(b_2 y) \quad (1)$$

which should be equal to the solution g of the two-dimensional Poisson equation

$$Lg = W \quad (2)$$

where the operator L on the solution is the central second order difference approximation in each point (i,j)

$$(Lg)_{i,j} = g_{i+1,j} - 2g_{i,j} + g_{i-1,j} + (\Delta x/\Delta y)^2 (g_{i,j+1} - 2g_{i,j} + g_{i,j-1}) \quad (3)$$

and the right hand side is the same approximation of the test function

$$\begin{aligned}
 W_{i,j} = & f_{i+1,j} - 2f_{i,j} + f_{i-1,j} \\
 & + (\Delta x/\Delta y)^2 (f_{i,j+1} - 2f_{i,j} + f_{i,j-1})
 \end{aligned}
 \tag{4}$$

such that in the converged state

$$g = f$$

The multi-grid equation is formulated as follows. Let

$$L_h g - W = 0 \tag{5}$$

be the equation with the grid width h . Then L_h approximates the linear differential operator L on the grid with a spacing proportional to the parameter h . Let U be the present estimate of g , and let V be the required correction to U such that $U+V$ satisfies (5). Then the basis of the multi-grid method is to replace (5) and determine V by

$$L_{2h} V + I_{2h}^h (L_h U - W) = 0 \tag{6}$$

where L_{2h} is the same approximation to L on a grid in which the spacing has been doubled, and I_{2h}^h is an operator which transfers to each grid point of the coarse grid the residual $L_h U - W$ of the coincident point of the fine mesh. After the solution of Eq. (6) the approximation on the fine grid is updated by interpolating the correction calculated on the coarse grid to the fine grid, so that U is replaced by

$$U^{\text{new}} = U + I_h^{2h} V \tag{7}$$

where I_h^{2h} is an interpolation operator.

Equation (6) can in turn be solved by introducing an approximation on a yet coarser grid, so that a multiple sequence of grids may be used, leading to a rapid solution procedure for two reasons. First, the number of operations required for a relaxation sweep on one of the coarse grids is much smaller than the number required on the fine grid. Second, the rate of convergence is faster on a coarse grid, reflecting the fact that corrections can be propagated from one end of the grid to the other in a smaller number of steps.

To extend this idea to nonlinear equations and to avoid a perturbation form of the equation, Eq.(6) is reorganised by adding and subtracting

the current solution U to give

$$L_{2h} (U+V) - L_{2h} U + I_{2h}^h (L_h U - W) = 0$$

$$\text{or } L_{2h} \bar{U} - \bar{W} = 0 \quad (8)$$

where $\bar{U}$ is the improved estimate of the solution to be determined on the coarse grid, and $\bar{W}$ is an appropriately modified right-hand side

$$\bar{W} = L_{2h} U - I_{2h}^h (L_h U - W) \quad (9)$$

The updating formula (7) now becomes

$$U^{\text{new}} = U + I_h^{2h} (\bar{U} - U) \quad (10)$$

As updating operator I_h^{2h} fourth order interpolation formulas are used (third order at boundaries).

The success of the multiple grid method generally depends on the use of a relaxation algorithm which rapidly reduces the high frequency components of error on any given grid. To analyse the influence of this smoothing algorithm, three different relaxation schemes were tested:

- horizontal line relaxation (SLOR)
- alternating direction scheme AF1⁵

$$(\alpha - \delta_x^2) \left(\frac{\alpha}{r} - \delta_y^2 \right) \delta g = \omega \frac{\alpha}{r} Lg \quad (11)$$

where

$r = (\Delta x / \Delta y)^2$, α is a parameter to be chosen, ω is an over-relaxation factor, and the residual Lg is calculated using the result of the previous iteration

- alternating direction scheme AF2⁶

$$(\alpha r \delta_y^- - \delta_x^2) (\alpha - \delta_y^+) \delta g = \omega \alpha Lg \quad (12)$$

using one-sided difference operators δ_y^- and δ_y^+ .

On an orthogonal mesh with equidistant spacing in Δx and Δy the computational results indicate that horizontal line relaxation and AF2 work well in the multi-grid mode when

$$\Delta x < \Delta y \quad (128 \times 32 \text{ grid})$$

but not so well when

$$\Delta x > \Delta y \quad (32 \times 128 \text{ grid})$$

On an equally spaced grid ($\Delta x = \Delta y$) all three schemes worked, but AF1 was best. The average rate of reduction of residual per work unit (measured as fine grid iteration) that could be achieved with AF1 in multi-grid mode was

- 0.230 on a 128 x 128 grid
- 0.259 on a 64 x 64 grid
- 0.272 on a 32 x 32 grid

It is interesting that the observed convergence rate gets faster as the grid gets finer. The result on the 128x128 grid represents a reduction from an initial average residual of $0.108 \cdot 10^{-1}$ to a final average residual of $0.576 \cdot 10^{-13}$ in eleven multi-grid cycles (17.66 work units).

The average error $|f-g|$ was reduced from $0.573 \cdot 10^{-1}$ to $0.554 \cdot 10^{-13}$ for a mean rate of reduction of 0.162 per work unit. For comparison the rate of convergence with line relaxation on a 64x64 grid (using 1 grid) was 0.982 for the residual and 0.993 for the error. In multi-grid mode line relaxation gave rates of 0.576 for the residual and 0.511 for the error.

An extensive study was made for the rate of convergence of the AF1 scheme in multigrid mode with different choices of the parameters α and ω . Fig. 1 shows curves of the rate of convergence for the 128x128 grid plotted against ω for different values of α . Optimal convergence rates can be obtained for certain combinations of α , ω only, however these combinations do depend on grid spacing and the optimum is very narrow. In production-type codes automatic adjustment for optimal α - ω combination would be highly desirable, however the logics for that are not understood yet.

MAD-Full Potential Solver

In the case of transonic flow we have to allow for a change from elliptic to hyperbolic type as the flow becomes locally supersonic. In the model problem

$$a \phi_{xx} + b \phi_{yy} = 0 \quad (13)$$

this corresponds to one of the coefficients, say a , to become negative. The classical alternating direction scheme AF1

$$(\alpha - A\delta_x^2) (\alpha - B\delta_y^2) \delta\phi = \omega\alpha L\phi \quad (14)$$

then has the disadvantage⁷ that if one regards the iterations to represent time steps Δt in an artificial time direction t , it simulates the time dependent equation

$$\alpha \Delta t \phi_t = \alpha \phi_{xx} + b \phi_{yy} \quad (15)$$

When $\alpha < 0$ and Cauchy data is given at $x = 0$ - corresponding to supersonic inflow - this leads to an ill posed problem which admits oscillatory solutions which are undamped in time and grow in the x -direction.

Therefore Jameson² proposed the following generalised alternating direction scheme. Let the scalar parameter α in Eq. (14) be replaced by a difference operator

$$S \equiv \alpha_0 + \alpha_1 \delta_x^- + \alpha_2 \delta_y^- \quad (16)$$

where δ_x^- and δ_y^- denote one sided difference operators in the x and y directions. This yields the scheme

$$(S - A\delta_x^2) (S - B\delta_y^2) \delta\phi = \omega SL\phi \quad (17)$$

in which the residual $L\phi$ is differenced by the operator S . The corresponding time dependent equation is now a hyperbolic one of the form

$$\beta_0 \phi_t + \beta_1 \phi_{xt} + \beta_2 \phi_{yt} = a \phi_{xx} + b \phi_{yy} \quad (18)$$

where the coefficients $\beta_0, \beta_1, \beta_2$ depend on the parameters $\alpha_0, \alpha_1, \alpha_2$.

Two remarks should be made to this scheme:

- There are additional error terms because the operator S does not commute with $a\delta_x^2$, $b\delta_y^2$, and the order of the factors may matter.
- Applied on a single grid it is necessary to use a sequence of the parameters α_0 , α_1 , α_2 to reduce all frequency bands of the error.

From the different possible strategies a very simple one has been chosen. Each cycle begins on the fine grid. The alternating direction iteration is performed once on each grid until the coarsest grid is reached. Then it is performed once on each grid going back up to the second finest grid, and the cycle terminates with the interpolation of the correction from the second finest to the fine grid. For viscous flow computations, this result is being used for a boundary layer computation, the displacement thickness of which is entering the next multi-grid cycle to provide finally a converged viscid solution. The additional effort for the viscid analysis is less than the time of one MAD-cycle per cycle, without increasing the total amount of MAD-cycles needed for convergence.

Some typical results are presented here. Further results are given in Ref. 8. All the examples were calculated on a circular domain generated by conformal mapping of the airfoil to a unit circle, with 160×32 in the θ and r direction on the fine grid. Five grids were used in the multi-grid scheme, giving a coarse grid of 10×2 cells; Fig. 2 shows a typical result for the RAE 2822 airfoil including viscous effects. The result was obtained after 12 MAD-cycles reducing the average residual from $0.1194 \cdot 10^{-3}$ to $0.7355 \cdot 10^{-5}$. Total CPU-time on an IBM 3031 computer for this case is 120 sec.

Fig. 3 shows similar results for the DO-A1 airfoil (CAST 7). Again there is very good agreement with the experimental data. Finally, Fig. 4 portrays inviscid results for the cylinder flow at $M = 0.50$ after 29 cycles and a reduction in average residual from $0.7876 \cdot 10^{-2}$ to $0.71 \cdot 10^{-6}$.

Multiple Grid Schemes for Systems of First Order Partial Differential Equations

The example which will be considered is two-dimensional unsteady inviscid flow which is described by the unsteady two-dimensional Euler equations

$$\frac{\partial \vec{U}}{\partial t} + \frac{\partial \vec{F}}{\partial x} + \frac{\partial \vec{G}}{\partial y} = 0 \quad (19)$$

$$\text{where} \quad \vec{U} = \begin{Bmatrix} \rho \\ \rho u \\ \rho v \\ \rho E \end{Bmatrix}, \quad \vec{F} = \begin{Bmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ \rho uH \end{Bmatrix}, \quad \vec{G} = \begin{Bmatrix} \rho v \\ \rho vu \\ \rho v^2 + p \\ \rho vH \end{Bmatrix} \quad (20)$$

and p , ρ , u , v , E and H denote the static pressure, density, Cartesian velocity components, total energy and total enthalpy. For a perfect gas is

$$E = \frac{p}{(\gamma-1)\rho} + \frac{1}{2} (u^2 + v^2), \quad H = E + p/\rho \quad (21)$$

If only homoenergetic steady flow ($H = H_0 = \text{const}$) is of interest, the transient phase does not have to be time-accurate, thus allowing for accelerating techniques. Two possible ones are described in Ref. 9, the local time stepping and a forcing term proportional to $H - H_0$. Both acceleration techniques basically permit the additional use of multiple grid techniques.

All three schemes presently under multi-grid evaluation are based on the integral conservation formulation of Eq. (19)

$$\iint \frac{\partial}{\partial t} \vec{U} \, d\text{vol} + \oint \vec{H} \cdot \hat{n} \, ds = 0 \quad (22)$$

describing the change of the flow vector $\vec{U}$ in time in one volume to be equal to fluxes of the flow in mass, momentum, energy through the surfaces of this volume.

The three schemes differ in the approximation of the flow in each cell

- central point Runge Kutta Stepping (Ref. 9)
where the flow is discretised by values in the volume center

- nodal point Runge Kutta Stepping
where the flow is discretised by values in the volume corners
- one step distribution formula scheme (Ref.4)

The nodal point and the distribution formula schemes are basically better suited to multi-grid techniques since after mesh halving points will remain mesh points.

Best results in multi-grid so far have been obtained using the one step distribution formula scheme. Therefore this procedure is described briefly.

The basic idea again is to use the coarser grids to propagate the fine grid connections properly and rapidly throughout the field, thus improving convergence rate to steady state while maintaining low truncation errors by using the fine grid discretisation.

The changes ΔU_{2h} in the coarse grid, obtained by removing every other line from the fine grid, are determined by

$$\Delta U_{2h} = I_{2h}^h \delta U_h \quad (23)$$

Where I_{2h}^h is an operator which transfers to each control volume of the coarse grid the correction δU_h of the fine grid using specific distribution formulas.

After computing the corrections δU_{2h} on all coarse grid points, the flow properties at the finest grid are updated by

$$U^{\text{new}} = U + I_h^{2h} \delta U_{2h} \quad (24)$$

where I_h^{2h} is a linear interpolation operator which interpolates the coarse grid corrections to give the corrections at each fine grid point of the finest mesh.

The scheme is quite sensitive to the choice of distribution formulas and the boundary condition treatment. Also, compared with the Runge-Kutta stepping schemes in Ref. 9, a fairly large dissipation has to be incorporated to make multi-grid converging efficiently. Fig. 5 portrays the converged pressure distribution C_p for the NACA 0012 airfoil at $M_\infty = 0,80$ and $\alpha = 0^\circ$. The added convergence history with an average

reduction rate per cycle for 500 cycles of 0.9686 is quite good. Fig. 6 shows similar results for the cylinder at $M_\infty = 0.35$. Here, the reduction in $\partial\rho/\partial t$ is equal to 0.9785. However, both results show effects of numerical viscosity. The cylinder-case gives no fully symmetric solution, the NACA airfoil case shows errors in the shock region compared with fully converged solutions of the very accurate Runge Kutta stepping scheme in Ref. 9.

Conclusions

Engineering application of computational methods requires fast and reliable numerical schemes. Multi-grid techniques are very well suited to meet those requirements, and have proved to be very useful for problems governed by quasi-linear equations. However, more has to be done to develop fast and accurate schemes for systems of first order partial differential equations since these equations are the most relevant ones for computational fluid mechanics.

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FIG. 1: INFLUENCE OF PARAMETERS α, ω ON CONVERGENCE

FIG. 2: RAE AIRFOIL 2822

FIG. 3: DO-A1 (CAST 7) AIRFOIL

● ○ EXPERIMENT CL = .110 CD = .0102 CM = -.118
M_∞ = .760 ± .001 α = -2.0 Re = 6·10⁶ (Ref. AGARD AR-138)
+ x DOFOIL COMPUTATION

CL= 0.134053 CD= 0.010449 CM= -0.123160

FIG. 4: FULL POTENTIAL MAD – SOLUTION

FIG. 5: NACA 0012

FIG. 6: CIRCLE