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Bodies having Minimum Pressure Drag in Supersonic Flow - Investigating Nonlinear Effects

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One of the most interesting problems in supersonic aerodynamics has been to find profile shapes that have minimum pressure drag, subject to a set of geometric constraints. Considerable work was done in this area in the fifties, all of which relied on a linearized flow model. We revisit this problem using a more sophisticated flow model. It is logical to expect the optimum profile shape to look different. We confirm this, but also note that the differences are very small. We then examine the equations of fluid flow and try to see why the linearized flow model works well for this problem and where the differences come from.

I. Introduction

The life-cycle of Engineering Design involves innovating to introduce a product, and when that is done, looking for ways to make it work better. Supersonic Aerodynamics can be looked at with a similar perspective. Man has always wanted to fly faster. While the basic mechanics of supersonic flight was laid out in the fifties, people are yet to find ways in which to make Supersonic flight more efficient.

One of the classic research problems in supersonic flight has been that of finding two-dimensional and axisymmetric profiles that have minimum pressure drag in supersonic flow. The two-dimensional sections are used as wing-profile sections, and the axisymmetric profiles are useful in that the distribution of Cross-sectional area is made to follow the optimum distribution (The Area Rule). This problem becomes redundant without suitable constraints. We know that the minimum drag shape is a flat plate in two-dimensional flow and a needle-like profile in axisymmetric flow. But this is not the answer we are looking for. Hence to make the problem more meaningful, we ensure that the enclosed area/volume is constant. We also ensure that the ends are pointed. This is to anchor the shocks firmly to the leading and trailind edges.

We then note that this problem has been solved in the fities using a linear flow model. However, recent advances in Computational Fluid Dynamics and Aerodynamic Shape Optimization have made it possible for this problem to be analyzed using a nonlinear flow model. We do this, and discuss the results.

II. Results from Classical Theory

Analytical solutions for the problem being studied have been obtained, assuming a linearized flow model. For the $2-d$ case the optimum profile is parabolic.

$$y(x) = 3Ax(1-x) \ , \quad \tau = \frac{3A}{2} \ , \tag{1}$$

where A is the area enclosed and τ is the thickness-chord ratio. The drag coefficient is given by

$$C_d = \frac{12A^2}{\sqrt{M^2-1}} \ . \tag{2}$$

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For the axisymmetric case, the profile shapes that solve this problem are the well known Sears-Haack profiles, discovered independently by Sears(1947) and Haack(1947). The derivation of the Sears-Haack profiles is outlined in the book by Ashley and Landahl[1] and also in an article by Carlo Ferrari.[2] The Sears-Haack profile is given by

$$y(x) = \sqrt{\frac{16V}{3\pi^2}}\,[4x(1-x)]^{\frac{3}{4}} \ \ , \tau = \sqrt{\frac{64V}{3\pi^2}} \ , \tag{3}$$

where V is the enclosed volume and τ is the fineness ratio. The drag coefficient is given by

$$C_D = 24V \ . \tag{4}$$

As can be observed, these profile shapes have some interesting properties. Firstly, they are unique solutions to the optimization problem. Moreover, they are just a function of the enclosed area/volume and not the Mach Number.

III. Nonlinear Optimization via Control Theory

In this work we apply the adjoint method developed by Jameson and his associates during the last 15 years. ([3–6]) The aerodynamic shape optimization problem involves minimizing(or maximizing) a given cost function, with parameters that define the shape of the body as the design variables, usually of the form

$$I = \int_{\mathcal{B}_\xi} \mathcal{M}(w,S)\, d\mathcal{B}_\xi \ , \tag{5}$$

where w is the vector of flow state variables and S_{ij} are the coefficients of the Jacobian matrix of the transformation from physical space to computational space. $\mathcal{M}(w,S)$ in our case is just C_p, the pressure coefficient. We also have the constraint that the state variables at the computational points have to satisfy the flow equations, irrespective of the shape of the boundary.

$$\int_{\mathcal{B}} n_i \phi^T f_i(w) d\mathcal{B} = \int_{\mathcal{D}} \frac{\partial \phi^T}{\partial x_i} f_i(w) d\mathcal{D} \ , \tag{6}$$

or, when transformed to computational space

$$\int_{\mathcal{B}_\xi} n_i \phi^T S_{ij} f_j(w) d\mathcal{B}_\xi = \int_{\mathcal{D}_\xi} \frac{\partial \phi^T}{\partial \xi_i} S_{ij} f_j(w) d\mathcal{D}_\xi \ , \tag{7}$$

where ϕ is any arbitrary test function.

Since equation (7) is true for any test function ϕ, we can choose ϕ to be the adjoint variable ψ. We can then add equation (7) to the cost function defined in (5) to form the following augmented cost function.

$$I = \int_{\mathcal{B}_\xi} \mathcal{M}(w,S)\, d\mathcal{B}_\xi + \int_{\mathcal{B}_\xi} n_i \psi^T S_{ij} f_j(w) d\mathcal{B}_\xi - \int_{\mathcal{D}_\xi} \frac{\partial \psi^T}{\partial \xi_i} S_{ij} f_j(w) d\mathcal{D}_\xi \ . \tag{8}$$

We then take a variation of the cost function described in (8)

$$\begin{aligned} \delta I \ &= \ \int_{\mathcal{B}_\xi} \left(\frac{\partial \mathcal{M}}{\partial w} \delta w + \delta \mathcal{M}_{II} \right) d\mathcal{B}_\xi \\ &+ \ \int_{\mathcal{B}_\xi} n_i \psi^T \left(S_{ij} \frac{\partial f_j(w)}{\partial w} \delta w + \delta S_{ij} f_j(w) \right) d\mathcal{B}_\xi \\ &- \ \int_{\mathcal{D}_\xi} \frac{\partial \psi^T}{\partial \xi_i} \left(S_{ij} \frac{\partial f_j(w)}{\partial w} \delta w + \delta S_{ij} f_j(w) \right) d\mathcal{D}_\xi \ . \end{aligned} \tag{9}$$

We choose ψ such that the variation in the cost function δI does not depend on the variation of the solution δw. ψ is then a solution of the adjoint equations

$$\begin{aligned} \frac{\partial \mathcal{M}}{\partial w} &= -n_i \psi^T S_{ij} \frac{\partial f_j(w)}{\partial w}, \text{on } \mathcal{B}_\xi , \\ \left(S_{ij} \frac{\partial f_j(w)}{\partial w} \right)^T \frac{\partial \psi}{\partial \xi_i} &= 0, \quad \text{on } \mathcal{D}_\xi . \end{aligned} \tag{10}$$

One thus obtains an expression for the change in the cost function of the form

$$\delta I = \int_{\mathcal{B}_\xi} \mathcal{G} \delta \mathcal{F} d\mathcal{B}_\xi , \tag{11}$$

where $\mathcal{F}(\xi)$ is a function defining the shape and $\mathcal{G}$ is the required gradient.

The gradient with respect to the design variables is obtained from the solutions to the adjoint equations by a reduced gradient formulation ([5]). This is modified to account for the area/volume constraints. In order to preserve the smoothness of the profile the gradient is smoothed by an implicit smoothing formula. This corresponds to redefining the gradient with respect to a weighted Sobolev inner product ([4]). The optimum is then found by a sequential procedure in which the shape is modified in a descent direction defined by the smoothed gradient at each step, and the flow solution and the gradient are recalculated after each shape change.

IV. Results and Discussions

CONVERGENCE FROM DIFFERENT INITIAL CONDITIONS The main test of the correctness of the optimization algorithm is to see if it converges to the same optimum profile regardless of what the initial profile is. Figures 1 and 2 show the optimization history from two different initial profile shapes for two-dimensional flow. They both enclose the same area. It can be seen that they converge to the same optimum profile. This gives us confidence in the correctness of our optimization setup.

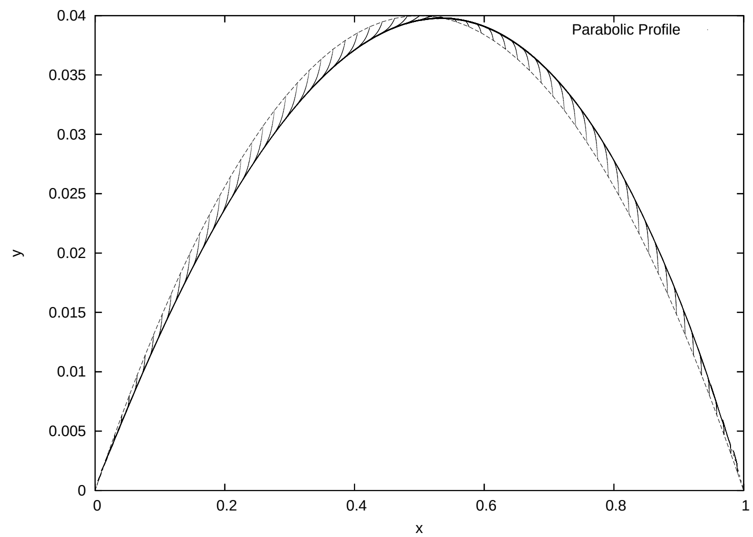

Figure 1. Convergence of the optimization algorithm from a Parabolic Initial Profile

OPTIMUM PROFILE SHAPES The results of the 2D optimization can be seen in Fig. 3 and the results of the axisymmetric optimization can be seen in Fig. 4. As can be observed, the nonlinear optimum profiles are slightly different from the classical optimum profiles. They have a more rearward point of maximum thickness. The primary difference between a linearized flow model and a nonlinear model is the appearance of shocks at the leading edge in the case of the nonlinear flow model. Reducing the included angle at the leading edge and moving the point of maximum thickness backward is consistent with reducing the magnitude of

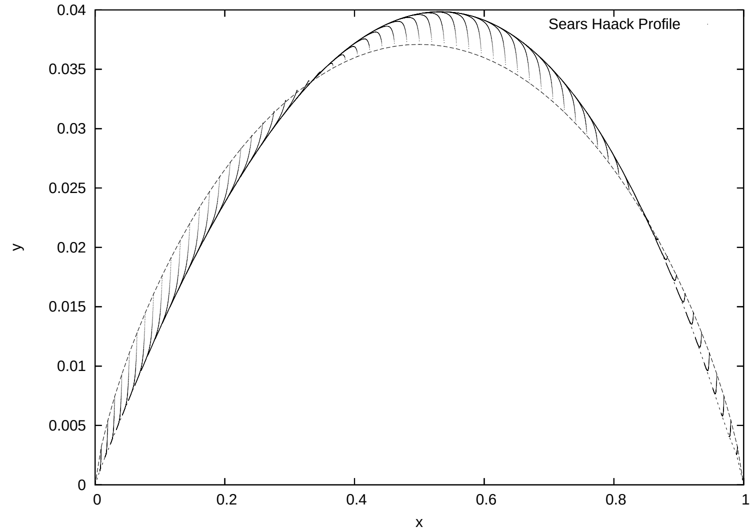

Figure 2. Convergence of the optimization algorithm from a Sears-Haack Initial Profile

the leading edge shock. This results in a lower drag and at the same time brings the flow closer to the linear regime.

The difference is hardly noticable for small thickness-chord/fineness ratios. This is just an indicator to the fact that linear theory is a very good approximation for small fineness ratios. Moreover, the nonlinear optimum profiles for axisymmetric flow are a lot closer to their corresponding classical profiles than for 2D flow. This is because of the three-dimensional relieving effect experienced in axisymmetric flow.

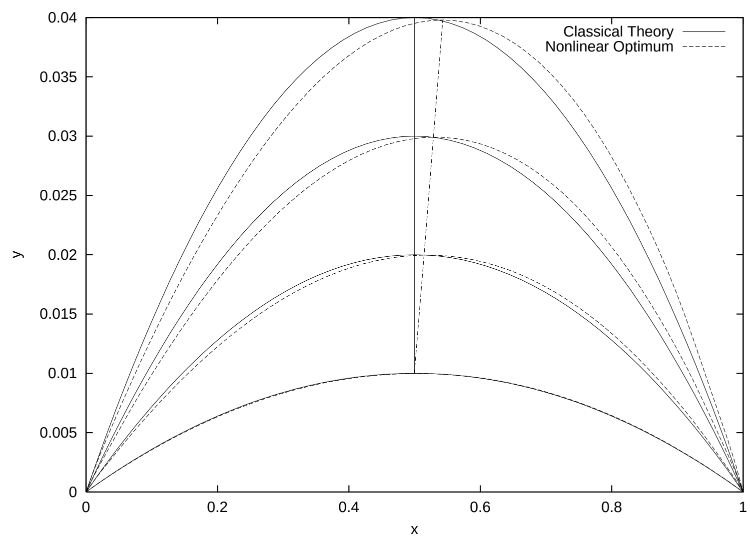

Figure 3. Classical and Nonlinear Optimum Profiles for 2-D flow

Variation with Mach Number The optimum profile for 2D flow changes with Mach number. The optimum shape for two Mach numbers is shown in Fig. 5. It is seen that the point of mazimum thickness is more backward for the higher Mach number. This again is consistent with our earlier argument that the main goal of the nonlinear optimization is to reduce the magnitude of the leading edge shock. Such a variation is not observed for axisymmetric optimum profiles. This is due to the fact that the drag coefficient is not sensitive to changes in Mach number in this case. This can also be seen from Eqn. 4.

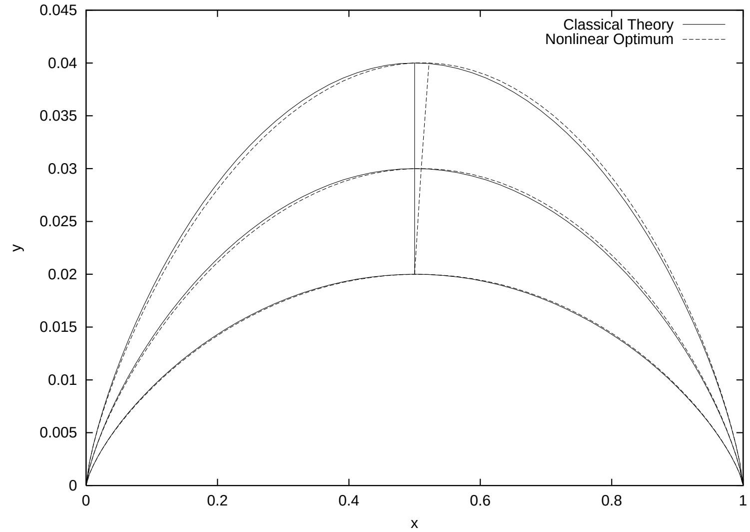

Figure 4. Classical and Nonlinear Optimum Profiles for Axisymmetric flow

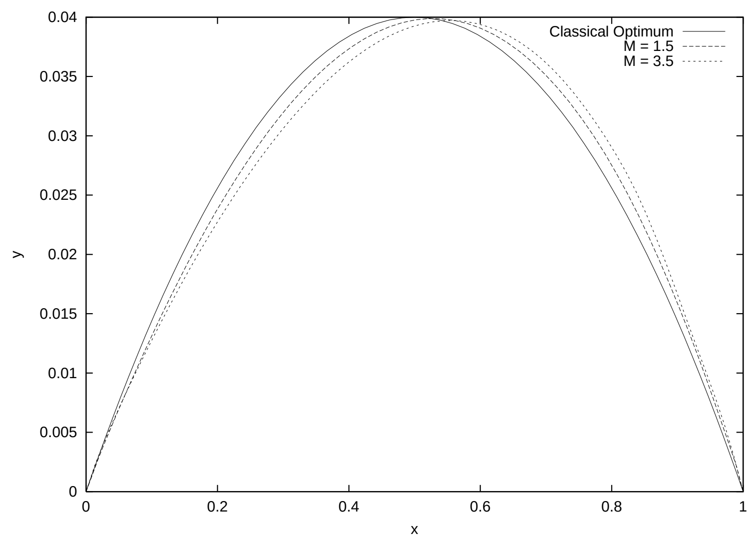

Figure 5. Variation of 2D Optimum Profiles with Mach Number

Discussion of Results The main assumptions of linear theory are that the flow is isentropic, irrotational, and the perturbation velocities in the axial and normal directions are very small compared to the free-stream velocity. We observe that these are valid assumptions except in the vicinity of the leading and the trailing edges, where we have stagnation points. Here we have shocks that cause entropy jumps. Moreover, the perturbation velocities are no longer small enough. Thus we expect to see the biggest change at these points. This is found to be true.

V. Conclusions

The minimum pressure drag problem was set-up and solved using a nonlinear flow model. Optimization was carried out using the Adjoint Method. In both the two-dimensional case and the axisymmetric case, the resulting optimum profiles were compared to the optimum profiles obtained from linear theory. It was seen that they were different, though not vastly so. The optimum profiles are different for different Mach Numbers in the case of two-dimensional flow. There is no Mach number dependence for axisymmetric flow. The optimal shapes become closer to their linear theory counterparts at low mach numbers and small fineness ratios. Moreover, the differences are more obvious for two-dimensional sections than for axisymmetric profiles.

The main conclusion that is made is that the results from linear theory are very good as far as the optimum drag shapes are concerned. In this respect, it makes very good engineering sense to use results from linear theory for any preliminary design.

VI. Acknowledgements

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References

[1] Ashley and Landahl: *Aerodynamics of Wings and Bodies*, (Dover Publications Inc New York 1985)

[2] Carlo Ferrari: Bodies of Revolution having Minimum Pressure Drag. In: *Theory of Optimum Aerodynamic Shapes*, ed by Angelo Miele (Academic Press Inc, New York 1965) pp 103–124

[3] Antony Jameson: Aerodynamic Design via Control Theory, *Journal of Scientific Computing*, pp 3:233–260 (1988)

[4] Antony Jameson: A Perspective on Computational Algorithms for Aerodynamic Analysis and Design. In: *Progress in Aerospace Sciences*, (2001) pp 37:197–243

[5] Antony Jameson and Sangho Kim: Reduction of the Adjoint Gradient Formula in the Continuous Limit, *AIAA Paper 2003-0040*. (41st AIAA Aerospace Sciences Meeting and Exhibit, Reno, NV, January 2003)

[6] Siva Nadarajah: The Discrete Adjoint Approach to Aerodynamic Shape Optimization: PhD Dissertation, Stanford University, Stanford (2003)