ON SENSITIVITY REDUCTION IN NONLINEAR FEEDBACK SYSTEMS

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ABSTRACT

The satisfaction of a well known sufficient condition for sensitivity reduction by use of feedback, is shown to imply severe restrictions on the feedback control law, and also on the weighting matrix in the integral-square criterion for closed-loop sensitivity reduction.

1. INTRODUCTION

Let the vector $\delta x(t)$ be a first-order variation of $x(t)$, the state trajectory of a system, due to first-order variations $\delta\mu(t)$ of a vector of parameters $\mu(t)$. Let δx_o and δx_c denote the respective variations of a pair of nominally equivalent open-loop and closed-loop systems. A criterion of sensitivity reduction in the closed-loop system is the inequality

$$\int_{t_0}^{t'} \delta x_o^T Z \delta x_o dt \geq \int_{t_0}^{t'} \delta x_c^T Z \delta x_c dt \quad , \qquad \text{all } t' > t_0 \quad , \tag{1.1}$$

where Z is a nonnegative weighting matrix. This criterion drew considerable attention since its introduction by Cruz and Perkins in 1964, because it is most suitable for small variations analysis, and because of its revealing relationship to the Bode sensitivity function (see [1] for details and references).†

We consider systems described by a vector differential equation

$$\dot{x} = f(t,x,u;\mu) \quad , \qquad x(t_0,\mu) = x_0(\mu) \quad , \tag{1.2}$$

where t is time, x the n-dimensional state, u is the m-dimensional control, and μ is a vector of continuous time-varying parameters. In an open-loop system the control vector u is a predetermined function of time assumed to be independent of μ, and in a closed-loop system u is a feedback control law

$$u = -k(t,x) \quad , \tag{1.3}$$

which now depends on μ via (1.2). Assuming f and k are continuously differentiable in all arguments, the first-order variation δx is given by the variational equation

$$\delta\dot{x} = f_x \delta x + f_u \delta u + f_\mu \delta\mu \quad , \qquad \delta x(t_0) = \xi \quad , \tag{1.4}$$

where f_x is the $n\times n$ matrix $(\partial f_i/\partial x_j)$ evaluated along the nominal $x(t)$, $u(t)$, and $\mu(t)$; f_u and f_μ are similarly defined.

In a pair of nominally equivalent open-loop and closed-loop systems, the control of the open-loop system is given by

$$u(t) = -k\big(t,x(t)\big) \quad , \tag{1.5}$$

where $x(t)$ is the solution of (1.2, 1.3). Let the subscripts o (to be distinguished from the subscript zero) and c denote quantities of a pair of nominally equivalent open-loop and closed-loop systems,

† Superior numerals refer to similarly-numbered references at the end of the paper.

respectively. Then from (1.4) and (1.3), and since $\delta u_0(t) \equiv 0$,

$$\begin{aligned} \delta\dot{x}_o &= f_x \delta x_o + f_\mu \delta\mu \quad , \\ \delta x_o(t_0) &= \xi \quad , \end{aligned} \tag{1.6}$$

$$\begin{aligned} \delta\dot{x}_c &= (f_x - f_u k_x)\delta x_c + f_\mu \delta\mu \quad , \\ \delta x_c(t_0) &= \xi \quad . \end{aligned} \tag{1.7}$$

The quantity $v(t)$

$$v(t) = \delta x_o(t) - \delta x_c(t) \quad , \tag{1.8}$$

is readily found to satisfy

$$\dot{v} = f_x v + f_u k_x \delta x_c \quad , \quad v(t_0) = 0 \quad . \tag{1.9}$$

It then follows that (1.1) holds if and only if the inequality

$$\int_{t_0}^{t'} (2\delta x_c^T Z v + v^T Z v)dt \geq 0 \quad , \qquad \text{all} \quad t' > t_0 \quad , \tag{1.10}$$

is satisfied. It is customary to replace the continuously differentiable δx_c, which is given by (1.7) and depends on $\delta\mu(t)$, by an arbitrary continuously differentiable m-vector $z(t)$. Then (1.10) and (1.9) become

$$\int_{t_0}^{t'} (2z^T Z\nu + \nu^T Z\nu)dt \geq 0 \quad , \qquad \text{all} \quad t' > t_0 \quad , \tag{1.11}$$

where ν replaces v and satisfies

$$\dot{\nu} = f_x \nu + f_u k_x z \quad , \quad \nu(t_0) = 0 \quad . \tag{1.12}$$

Condition (1.11), called a <u>sensitivity inequality</u>, is sufficient, but is generally no longer necessary for (1.1) to hold; however, (1.11) is independent of δx_c and is thus more amenable to theoretical analysis. In particular, for linear time-invariant systems,

$$\dot{x} = A(\mu)x + B(\mu)u \quad , \tag{1.13}$$

$$u = -Kx \quad , \tag{1.14}$$

the condition (1.11) can (by completing the square and using Parseval's theorem) be put in the form[1]

$$[I + (-j\omega I - A)^{-1}BK]^T Z[I + (j\omega I - A)^{-1}BK] \geq Z \quad , \quad \text{all real } \omega \quad , \tag{1.15}$$

first derived by Cruz and Perkins.[2]

It first appeared that sufficient conditions like (1.11) and (1.15) could be used to design feedback systems for sensitivity reduction according to (1.1) with some prescribed weighting matrix Z. However, we have shown[3] that under the assumption that on every interval $[t_0, t']$ the pair of matrices $[f_x, f_\mu]$ is completely controllable and the rows of k_x are linearly independent, Z must be of the form

$$Z = k_x^T M k_x \tag{1.16}$$

for some nonnegative definite matrix M. This simply means that the sensitivity reduction should, in general, be measured in terms of the quantity η,

$$\eta = k_x \delta x \quad , \tag{1.17}$$

that is being fed back, rather then by δx.

By use of (1.16) and (1.17) in (1.1), (1.11) and (1.12), we find that for closed-loop sensitivity reduction according to

$$\int_{t_0}^{t'} \eta_o^T M \eta_o \delta t \geq \int_{t_0}^{t'} \eta_c^T M \eta_c dt \quad , \qquad \text{all} \quad t' > t_0 \quad , \tag{1.18}$$

the following condition is sufficient: the inequality

$$\int_{t_0}^{t'} (2\upsilon^T M k_x \nu + \nu^T k_x^T M k_x \nu)\,dt \geq 0 \quad , \tag{1.19}$$

$$\text{all} \quad t' > t_0 \quad ,$$

must hold for every solution $\nu(t)$ of

$$\dot{\nu} = f_x \nu + f_u \upsilon \quad , \quad \nu(t_0) = 0 \quad , \tag{1.20}$$

where now υ replaces $k_x z$, and is an arbitrary continuously differentiable function of t of dimension m.

By completing the square, (1.19) can be written in the symmetric form

$$\int_{t_0}^{t'} (\upsilon + k_x \nu)^T M (\upsilon + k_x \nu)\,dt \geq \int_{t_0}^{t'} \upsilon^T M \upsilon\,dt \quad , \tag{1.21}$$

$$\text{all} \quad t' > t_0 \quad .$$

For linear time-invariant systems, (1.21) becomes[1]

$$[I + K(-j\omega I - A)^{-1}B]^T M [I + K(j\omega I - A)^{-1}B] \tag{1.22}$$

$$\geq M \quad , \quad \text{all real} \quad \omega \quad ,$$

We observe that (1.22) is in terms of an $m \times m$ Hermitian matrix, while (1.15) is in terms of an $n \times n$ matrix; the matrix $I + K(j\omega I - A)^{-1}B$ is seen as a generalized return difference for the loop opened at the control input, while $I + (j\omega I - A)^{-1}BK$ is the return difference for the loop opened at the state output.

In this paper we show, in the next Section 2, that the sensitivity inequalities (1.21) or (1.22) can be satisfied only for M of a particular structure, and that k_x must have certain properties. The implications of these restrictions are discussed in Section 3.

2. RESULTS

THEOREM 2.1. <u>For the sensitivity inequality (1.19) to hold, the $m \times m$ matrix $Mk_x f_u$ must for all $t \geq t_0$ be symmetric and nonnegative definite</u>.

<u>Proof</u>. The symmetry of $Mk_x f_u$ is proved in the Appendix. The symmetry of MKB in the linear time-invariant case follows as a corollary; however, it is of interest to present an independent proof in a manner essentially pointed out to us by B. D. O. Anderson. An expansion of (1.22) in powers of $1/j\omega$ yields

$$1/j\omega[MKB - (MKB)^T] + \ldots \geq 0 \quad .$$

For large $|\omega|$ this term dominates and it must therefore be a nonnegative definite Hermitian matrix for both positive and negative ω; this implies it must be zero, i.e.,

$$MKB = (MKB)^T \quad . \tag{2.1}$$

To prove that $Mk_x f_u \geq 0$, let

$$\upsilon(t) = g(t - t^o; \rho) y \quad , \tag{2.2}$$

where $g(t - t^o; \rho)$ is a continuously differentiable scalar function that as $\rho \to 0$ approaches an impulse at $t = t^o$, and is zero outside the interval $[t^o - \epsilon, t^o + \epsilon]$; y is a m-vector. Letting $t' = t^o + \epsilon$, (1.19) yields

$$y^T M k_x f_u y + h(\rho, \epsilon) \geq 0 \quad , \tag{2.3}$$

where $h(\rho, \epsilon) \to 0$ as $\rho \to 0$, $\epsilon \to 0$. Thus $Mk_x f_u$ must be nonnegative definite. <u>Q.E.D.</u>

In most cases, the weighting matrix M in (1.18) is taken to be positive definite. For $M > 0$, Theorem 2.1 implies

THEOREM 2.2. <u>For the sensitivity inequality (1.19) to hold with $M > 0$, the $m \times m$ matrix $(k_x f_u)^T$, or equivalently $k_x f_u$, must for all $t \geq t_0$ have m linearly independent real eigenvectors, and nonnegative real eigenvalues. Furthermore, M must be of the form</u>

$$M = V \Gamma V^T \quad , \tag{2.4}$$

<u>where the columns of V are eigenvectors of $(k_x f_u)^T$ and Γ is a real, symmetric, positive definite matrix that commutes with the diagonal matrix Λ of eigenvalues of $(k_x f_u)^T$ or $k_x f_u$.</u>

<u>Proof</u>. Let $L^T L = M > 0$ be such that $Mk_x f_u$ is symmetric and nonnegative definite. Then

$$(L^{-1})^T M k_x f_u L^{-1} = L k_x f_u L^{-1}$$

is symmetric and nonnegative definite, and it is similar to $k_x f_u$. Thus, the eigenvectors and eigenvalues of $k_x f_u$ have the same properties as a symmetric nonnegative definite matrix, namely, those claimed. The same holds for $(k_x f_u)^T$, as can be seen by applying the preceeding argument to the symmetric matrix $(k_x f_u)^T M$. To prove (2.4), we note that since $(k_x f_u)^T$ has linearly independent eigenvectors, the relation $(k_x f_u)^T V = V\Lambda$ yields

$$(k_x f_u)^T = V\Lambda V^{-1} \quad , \tag{2.5}$$

which we substitute into the symmetry condition

$$M k_x f_u = (k_x f_u)^T M \quad , \tag{2.6}$$

to obtain

$$M V^{-1^T} \Lambda V^T = V\Lambda V^{-1} M \quad . \tag{2.7}$$

Multiplying on the right by $(V^T)^{-1}$ and on the left by V^{-1}, (2.7) becomes

$$V^{-1} M V^{-1^T} \Lambda = \Lambda V^{-1} M V^{-1^T} \quad . \tag{2.8}$$

Letting

$$V^{-1} M V^{-1^T} \triangleq \Gamma \quad , \tag{2.9}$$

we have that

$$\Gamma\Lambda = \Lambda\Gamma \tag{2.10}$$

and M is given by (2.4). <u>Q.E.D.</u>

Note that the proof of (2.4) used only the nonsingularity of V; thus if V is nonsingular, $M \geq 0$ must be of the form (2.4). By a lengthy proof (to be published in a different context) it can be shown that Theorem 2.2 is valid for $M \geq 0$, except that the eigenvectors of $(k_x f_u)^T$ may be linearly dependent, Γ need only be nonnegative definite, and only those eigenvalues which correspond to the eigenvectors that comprise the columns of V in (2.4) must be real and nonnegative.

For single-input systems, where the feedback control law $k(t,x)$ is a scalar and $\mathbf{k}_x$ an n-vector, the requirements of Theorem 2.2 reduce to a simple scalar inequality, i.e., <u>for the sensitivity inequality</u> (1.21), now reduced to

$$\int_{t_0}^{t'} (\upsilon + k_x^T \nu)^2 dt \geq \int_{t_0}^{t'} \upsilon^2 dt \quad , \tag{2.11}$$

$$\underline{\text{all}} \quad t' > t_0 \quad ,$$

<u>to hold</u>, $k(t,x)$ <u>must be such that</u>

$$k_x^T f_u \geq 0 \quad , \quad \underline{\text{for all}} \quad t \geq t_0 . \tag{2.12}$$

The properties of the matrix $k_x f_u$ apply, of course, to KB in the linear case.

3. DISCUSSION

The restrictions the form (2.4) imposes on the weighting matrix M are quite severe. If the eigenvalues of $k_x f_u$ [KB in the linear case] are distinct, then Γ must be diagonal, amounting to no more than a scaling of the eigenvectors of $(k_x f_u)^T$ (of course, for a given V, the sensitivity inequality might be satisfied only for some such Γ). Thus, the freedom of choosing an M is very limited. For $M > 0$, the necessary conditions on k_x given in Theorem 2.2 are also restrictive, in particular that all eigenvalues of $k_x f_u$ must be real and nonnegative; for $M \geq 0$, $k_x f_u$ must have at least one such eigenvalue.

There is nonetheless a large class of feedback systems (1.2, 1.3) that satisfies the sensitivity inequality, a certain class of optimal systems.(1) In fact, the most effective way to insure closed-loop sensitivity reduction according to (1.1) appears to be the synthesis of an optimal system.

The restrictions on the weighting matrices Z and M do not necessarily apply to (1.1) and (1.18), because the sensitivity inequalities (1.11) and (1.19) are only sufficient conditions for (1.1) and (1.18), respectively. It is only in systems where f_μ is of full rank, and where therefore every continuously differentiable $\delta x_c(t)$ can be produced by some $\delta\mu(t)$, that the sensitivity inequalities are also necessary.(3) We have therefore an interesting reciprocal relationship: the more restricted the class of $\delta x_c(t)$, the less restricted is the class of possible weighting matrices Z in (1.1). For example, for linear optimal systems where the plant is in phase variable form and remains so under parameter variations, Z in (1.1) can be an almost arbitrary nonnegative definite matrix.[4,5]

APPENDIX: PROOF OF SYMMETRY OF $Mk_x f_u$

For convenience, we repeat equations (1.19) and (1.20)

$$\int_{t_0}^{t'} (2\upsilon^T Mk_x \nu + \nu^T k_x^T Mk_x \nu)dt \geq 0 \quad , \tag{A.1}$$

$$\text{all} \quad t' > t_0 \quad ,$$

$$\dot{\nu} = f_x \nu + f_u \upsilon \quad , \quad \nu(t_0) = 0 \quad . \tag{A.2}$$

If $\upsilon(t)$ is a harmonic function of sufficiently high frequency ω, then the magnitude of $\nu(t)$ is, approximately, of order $1/\omega$ smaller then that of $\upsilon(t)$; thus, for large $|\omega|$, the first term in (A.1) dominates the second, and we must have

$$\int_{t_0}^{t'} \upsilon^T Mk_x \nu dt \geq 0 \quad . \tag{A.3}$$

Let $\upsilon(t)$ satisfy

$$\ddot{\upsilon} + \omega^2 \upsilon = 0 \quad . \tag{A.4}$$

Substituting $\upsilon = -1/\omega^2 \ddot{\upsilon}$ in (A.3), we have

$$-\int_{t_0}^{t'} \ddot{\upsilon}^T Mk_x \nu dt \geq 0 \quad , \tag{A.5}$$

and by integration by parts

$$-\dot{\upsilon}^T Mk_x \nu \Big|_{t_0}^{t'} + \int_{t_0}^{t'} \dot{\upsilon}^T \left[\frac{d}{dt}(Mk_x) + Mk_x f_x\right] \nu dt \tag{A.6}$$

$$+ \int_{t_0}^{t'} \dot{\upsilon}^T Mk_x f_u \upsilon \geq 0 \quad .$$

The second term is, approximately, of magnitude $1/\omega$ smaller then the last term, and thus we must have

$$-\dot{\upsilon}^T Mk_x \nu \Big|_{t_0}^{t'} + \int_{t_0}^{t'} \dot{\upsilon}^T Mk_x f_u \upsilon \geq 0 \quad . \tag{A.7}$$

We want to show that $Mk_x f_u$ is symmetric at an arbitrary time $t^o \geq t_0$ where $M(t^o)k_x(t^o)f_u(t^o) \triangleq (Mk_x f_u)^o \neq 0$. Letting $\upsilon(t) = 0$ on $[t_0, t^o - \epsilon]$, and setting $t' = t^o + \epsilon$, (A.7) becomes

$$-\dot{\upsilon}^T Mk_x \nu \Big|_{t^o-\epsilon}^{t^o+\epsilon} + \int_{t^o-\epsilon}^{t^o+\epsilon} \dot{\upsilon}^T (Mk_x f_u)^o \upsilon dt \tag{A.8}$$

$$+ g(\epsilon) \geq 0 \quad ,$$

where $g(\epsilon)$ represents the contribution from the time-varying part of $Mk_x f_u$ on $[t^o-\epsilon, t^o+\epsilon]$. Since the $g(\epsilon)$ goes to zero with ϵ faster than the middle term, it can be neglected. Recalling that $\nu(t^o-\epsilon) = 0$, and choosing ϵ such that $\upsilon(t^o+\epsilon) = 0$, we have

$$\int_{t^o-\epsilon}^{t^o+\epsilon} \dot{\upsilon}^T (Mk_x f_u)^o \upsilon dt \geq 0 \quad . \tag{A.9}$$

Let

$$\upsilon(t) = c\, e^{j\omega t} + \bar{c}\, e^{-j\omega t} \quad , \tag{A.10}$$

where c is a complex vector and $\bar{c}$ its complex conjugate. Then (A.9) becomes

$$2\epsilon j\omega c^T \left[(Mk_x f_u)^o - (Mk_x f_u)^{o^T}\right] \bar{c} +$$

$$\frac{1}{2}\left[c^T (Mk_x f_u)^o c\, e^{2j\omega t} \right. \tag{A.11}$$

$$\left. - \bar{c}^T (Mk_x f_u)^o \bar{c}\, e^{-2j\omega t}\right]_{t^o-\epsilon}^{t^o+\epsilon} \geq 0 \quad .$$

For sufficiently large $|\omega|$ the first term dominates, and since by a choice of c its sign can be reversed, we must have

$$Mk_x f_u = (Mk_x f_u)^T \quad , \tag{A.12}$$

for all $t^o \geq t_0$ for which $Mk_x f_u \neq 0$, proving the symmetry of $Mk_x f_u$ for all $t \geq t_0$.

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