# Roots of Polynomials

Antony Jameson

*Department of Aeronautics and Astronautics, Stanford University, Stanford, California, 94305*

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## Roots of Polynomials

1. Evaluation of polynomials and derivatives by nested multiplication
2. Approximate location of roots
3. Bernoulli's method
4. Newton's method
5. Bairstow's method

# 1 Evaluation of polynomials

Let $P_n(x) = a_0x^n + a_1x^{n-1} \ldots + a_n$. To calculate $P_n(\xi)$ use nesting.

$$b_0 = a_0$$

$$b_1 = b_0\xi + a_1 = a_0\xi + a_1$$

$$b_2 = b_1\xi + a_2 = a_0\xi^2 + a_1\xi + a_2$$

$$\ldots$$

$$b_n = P_n(\xi)$$

If we set $P_n(x) = (x - \xi)Q_{n-1}(x) + R_0$ where $R_0 = P_n(\xi)$, then on multiplying out and equating coefficients we find

$$Q_{n-1}(x) = b_0x^{n-1} + b_1x^{n-2} \ldots + b_{n-1}, \quad P_0 = b_n$$

Repeating the division we have

$$Q_{n-1}(x) = (x - \xi)Q_{n-2}(x) + R_1$$

where $R_1 = Q_{n-1}(\xi)$, and thus

$$P_n(x) = (x - \xi)^2Q_{n-2}(x) + (x - \xi)R_1 + R_0.$$

Differentiating with respect to $x$ and setting $x = \xi$

$$P_n'(\xi) = R_1.$$

The procedure can be continued to yield

$$P_n(x) = R_n(x-\xi)^n \ldots + R_1(x-\xi) + R_0$$

where

$$R_k = \frac{1}{k!}\frac{d}{dx^k}P_n(x)\bigg|_{x=\xi}$$

The evaluation of the coefficients is indicated by the array

$$\begin{array}{ccccc}
a_0 & b_0 & c_0 & \cdots & & R_n \\
a_1 & b_1 & c_1 & \cdots & R_{n-1} & \\
\vdots & & & & & \\
a_{n-2} & b_{n-2} & c_{n-2} & & & \\
a_{n-1} & b_{n-1} & R_1 & & & \\
a_n & R_0 & & & &
\end{array}$$

where any entry outside the $1_{st}$ row and column is found by multiplying the entry above by $\xi$ and adding the entry to the left

$$c_k = c_{k-1}\xi + \beta_k$$

etc.

Nested multiplication (Horner's rule) for polynomial

Let

$$
\begin{aligned}
P_3(z) &= a_0 z^3 + a_1 z^2 + a_2 z + a_3 \\
&= \left((a_0 z + a_1) z + a_2\right) z + a_3
\end{aligned}
$$

To sum $p_n(z)$ let

$$b_0 = a_0$$

$$b_1 = a_1 + b_0 z$$

$$b_i = a_i + b_{i-1} z$$

$$\ldots$$

Then

$$p_n(z) = b_n$$

Also we have

Division theorem

$$\frac{p(x) - p(z)}{x - z} = \sum_{i=0}^{n-1} b_i x^{n-1-i}$$

Denote right side by $q_{n-1}(x)$

$$\begin{aligned}(x-z)q_{n-1}(x) &= \sum_{i=0}^{n-1} b_i x^{n-i} - \sum_{i=0}^{n-1} b_i z x^{n-i-1} \\ &= \sum_{i=1}^{n}(b_i - b_{i-1}z)x^{n-i} + b_0 x^n - b_n \\ &= \sum_{i=1}^{n} a_i x^{n-i} + a_0 x^n - p(z) \\ &= p(x) - p(z)\end{aligned}$$

Note also that where the $b_i$ are evaluated for $p_n(z)$

$$q_{n-1}(z) = p_n'(z)$$

since differentiating

$$(x-z)q_{n-1}(x) = p_n(x) - p_n(z)$$

gives

$$q_{n-1}(x) + (x-z)q_{n-1}'(x) = p_n'(x)$$

We can sum $q_{n-1}(z)$ by the same rule

$$c_0 = b_0$$

$$c_i = b_i + zc_{i-1}$$

$$\ldots$$

$$q_{n-1}(z) = c_{n-1}$$

Newton's method for several variables

To solve

$$f_i(x) = 0$$

we have

$$f_i^{n+1} = f_i^n + \sum \frac{\partial f_i}{\partial x_j}\left(x_j^{n+1} - x_j^n\right) + \text{higher order terms}$$

Thus to make

$$f_i^{n+1} = 0$$

let $x_j^{n+1} - x_j^n$ satisfy

$$f_i^n + \sum \frac{\partial f_i}{\partial x_j}\left(x_j^{n+1} - x_j^n\right) = 0$$

Horner's rule and synthetic division

Consider

$$\frac{p_n(x)}{x - z} = \frac{a_0 x^n + a_1 x^{n-1} \cdots + a_n}{x - z}$$

Then

$$\begin{aligned} p_n(x) - (x - z)a_0 x^{n-1} &= (a_1 + b_0 z)x^{n-1} + a_2 x^{n-2} \cdots \\ &= b_1 x^{n-1} + a_2 x^{n-2} \cdots \\ &= p_{n-1}(x) \quad \text{say} \end{aligned}$$

Also

$$\begin{aligned} p_{n-1}(x) - (x-z)b_1 x^{n-2} &= (a_2 + b_1 z)x^{n-2} + a_3 x^{n-3} \cdots \\ &= b_2 x^{n-2} + a_3 x^{n-3} \cdots \\ &= p_{n-2}(x) \end{aligned}$$

Finally

$$p_1(x) = b_{n-1}x + a_n$$

$$p_1(x) - (x-z)b_{n-1} = a_n + b_{n-1}z = b_n$$

so

$$p_n(x) - (x-z)q_{n-1}(x) = b_n = p_n(z)$$

where

$$q_{n-1}(x) = b_0 x^{n-1} + b_1 x^{n-2} \cdots + b_{n-1}$$

<u>Horner's rule and derivatives</u>

Since

$$p_n(x) = p_n(z) + (x-z)q_{n-1}(x)$$

and repeating the same rule

$$q_{n-1}(x) = q_{n-1}(z) + (x-z)q_{n-2}(x)$$

where

$$q_{n-2}(x) = \sum_{i=0}^{n-2} c_i x^{n-2-i}$$

we have

$$\begin{aligned} p_n(x) &= p_n(z) + (x-z)q_{n-1}(z) + (x-z)^2 q_{n-2}(x) \\ &= p_n(z) + (x-z)q_{n-1}(z) + (x-z)^2 q_{n-2}(z) + q_0 \end{aligned}$$

where

$$q_0 = a_0$$

Thus differentiating

$$\left.\frac{d}{dx^k} p_n(x)\right|_{x=z} = k!\ q_{n-k}(z)$$

Repeated application of Horner's rule thus gives the derivatives.

# 2 Rules for locating roots

The roots of a high order polynomial must be found by iteration, since it was proved by Galois that for polynomials of order $> 4$, there is no procedure for finding the roots with a finite number of algebraic operations, such as multiplications root extractions as in $2^{nd}$ order case where the roots of $x^2 + 2ax + b$ are $-a \pm \sqrt{a^2 - b^2}$. An iterative method may need a starting guess so it is useful to locate a root approximately. For locating real roots see Isaacson & Keller, p. 126.

A method of locating complex roots is to note that

$$w = P_n(z) = (z - z_1)(z - z_2)\cdots(z - z_n)$$

Any $w = \arg(z - z_1) + \arg(z - z_2) + \ldots + \arg(z - z_n)$. Thus a loop in the $z$ plane enclosing $n$ roots will cause $\arg w$ to increase by $2n\pi$, i.e. $w$ encircles origin $n$ times.

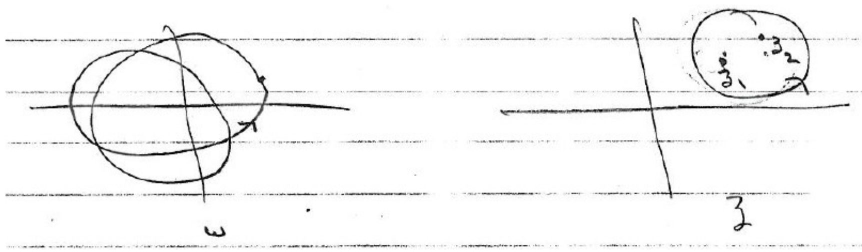


**Figure 1:**

$$\begin{aligned}
\tan \Delta\theta &= \tan(\theta_i - \theta_{i-1}) \\
&= \frac{\tan\theta_i - \tan\theta_{i-1}}{1 + \tan\theta_i \tan\theta_{i-1}}
\end{aligned}$$

$$\begin{aligned}
\Delta\theta &= \arctan \frac{\frac{y_i}{x_i} - \frac{y_{i-1}}{x_{i-1}}}{1 + \frac{y_i}{x_i}\frac{y_{i-1}}{x_{i-1}}} \\
&= \arctan \frac{y_i x_{i-1} - y_{i-1} x_i}{x_i x_{i-1} + y_i y_{i-1}}
\end{aligned}$$

# 3 Bernoulli's method

Given the polynomial

$$P_n(z) = a_0 z^n + a_1 z^{n-1} \ldots + a_n$$

consider the difference equation

$$a_0 x_k + a_1 x_{k-1} \ldots + a_n x_{k-n} = 0$$

where $x_k$ is calculated from $x_{k-1}, \ldots$ obtained at previous steps. Try the solution

$$x_k = z^k$$

Then this is a solution if

$$P_n(z) = 0.$$

The general solution is of the form

$$x_k = \sum c_i z_i^k$$

where the $z_i$ are the roots and the $c_i$ depend on the initially given $x_0, x_1, \ldots x_{n-1}$.

Let the roots be ordered so that

$$|z_1| > |z_2| \cdots > |z_n|$$

Then

$$\begin{aligned}
\frac{x_{k+1}}{x_k} &= \frac{c_1 z_1^{k+1} + c_2 z_2^{k+1} + \cdots + c_n z_n^{k+1}}{c_1 z_1^k + c_2 z_2^k + \cdots + c_n z_n^k} \\
&= z \frac{1 + \frac{c_2}{c_1}\left(\frac{z_2}{z_1}\right)^{k+1} + \cdots \frac{c_n}{c_1}\left(\frac{z_n}{z_1}\right)^{k+1}}{1 + \frac{c_2}{c_1}\left(\frac{z_2}{z_1}\right)^{k} + \cdots \frac{c_n}{c_1}\left(\frac{z_n}{z_1}\right)^{k}}
\end{aligned}$$

If $\left|\frac{z_i}{z_1}\right| < 1$ for $i > 1$ then regardless of the initial values

$$\lim \frac{x_{k+1}}{x_k} = z_1$$

If the dominant roots are a complex conjugate pair

$$z_1 = re^{i\theta}, \quad z_2 = re^{-i\theta}$$

where

$$\left|\frac{z_i}{r}\right| < 1, \quad i > 2$$

then with real initial values

$$c_1 = ce^{i\delta}, \quad c_2 = ce^{-i\delta}$$

$$\begin{aligned} x_k &= 2cr^k \cos(k\theta + \delta) + c_3 z_3^k \cdots + c_n z_n^k \\ &= 2cr^k \left(\cos(k\theta + \delta) + p_k\right) \end{aligned}$$

where

$$\lim_{k\to\infty} p_k = 0$$

Also $2cr^k \cos(k\theta + \delta)$ satisfies the difference equation

$$x_k + Ax_{k-1} + Bx_{k-2} = 0$$

where

$$A = -2r\cos\theta, \quad B = r^2$$

Then in the limit $x_k$ satisfies the same equation, and

$$x_k + Ax_{k-1} + Bx_{k-2} = 0$$

$$x_{k+1} + Ax_k + Bx_{k-1} = 0$$

can be solved for $A, B$. The determinant

$$\begin{vmatrix} x_{k-1} & x_{k-2} \\ x_k & x_{k-1} \end{vmatrix} = 4c^2 r^{2k-2} \sin^2\theta \neq 0$$

since by assumption $\theta \neq 0$.

# 4 Finding the roots of a polynomial by Newton's method

To find a root of $P_n(x)$ by Newton's method we set

$$x_{n+1} = x_n - \frac{P_n(x_n)}{P_n'(x_n)}$$

where to evaluate $P_n(x_n),\ P_n'(x_n)$ we carry out the operations for the first 2 columns of the array for nested multiplications.

After finding a roots, we can use the same method to obtain divide out $(x - s_1)$,

$$P_n(x) = (x - s_1)Q_{n-1}(x)$$

where

$$Q_{n-1} = b_0 x^{n-1} + b_1 x^{n-2} \cdots + b_{n-1}$$

with the $b_k$ evaluated at $\xi = s_1$.

Then we repeat the process to find another root. Note that if $P_n(x)$ has real coefficients then $P_n(x)$ and $P_n'(x)$ are both real if $x$ is real, so Newton's method can only find a complex root if the initial guess is complex.

# 5 Bairstow's method

To avoid searching for complex roots we can search for quadratic factors. Bairstow's method applies Newton's method for finding the factors. Let

$$P_n(x) = (x^2 + sx + t)Q_{n-2}(x) + xR_1(s,t) + R_0(s,t).$$

Then for zero remainder we must have

$$R_1(s,t) = 0$$

$$R_0(s,t) = 0$$

This is 2 nonlinear equations for 2 unknowns which may be solved by Newton's method. We need

$$J = \begin{bmatrix} \frac{\partial R_1}{\partial s} & \frac{\partial R_1}{\partial t} \\ \frac{\partial R_0}{\partial s} & \frac{\partial R_0}{\partial t} \end{bmatrix}$$

To get these indirectly let

$$Q_{n-2}(x) = (x^2 + sx + t)Q_{n-4}(x) + xR_3(s,t) + R_4(s,t)$$

so that

$$P_n(x) = (x^2 + sx + t)^2 Q_{n-4}(x) + (x^2 + sx + t)(xR_3 + R_2) + xR_1 + R_0$$

Then differentiating with respect to $s$, $t$ and setting $x$ to a root $z_i$ of $x^2 + sx + t$

$$z_i(z_i R_3 + R_2) + z_i \frac{\partial R_1}{\partial s} + \frac{\partial R_0}{\partial s} = 0$$

$$z_i R_3 + R_2 + z_i \frac{\partial R_1}{\partial t} + \frac{\partial R_0}{\partial t} = 0$$

Since $z^2 = -(sz + t)$ these can be written

$$z_i \left( \frac{\partial R_1}{\partial s} + R_2 - sR_3 \right) + \left( \frac{\partial R_0}{\partial s} + tR_3 \right) = 0$$

$$z_i \left( \frac{\partial R_1}{\partial t} + R_3 \right) + \left( \frac{\partial R_0}{\partial t} + R_2 \right) = 0$$

Since this holds for two roots $z_1$, $z_2$ each bracketed coefficients must vanish, giving

$$\frac{\partial R_1}{\partial s} = sR_3 - R_2, \quad \frac{\partial R_0}{\partial s} = tR_3$$

$$\frac{\partial R_1}{\partial t} = -R_3, \quad \frac{\partial R_0}{\partial t} = -R_2$$

The polynomials $R_0$, $R_1$, $R_2$ and $R_3$ are found by carrying out two synthetic divisions by the quadratic factor $x^2 + sx + t$. This is most easily accomplished by equating coefficients to obtain a recursion rule.

We have

$$\begin{aligned}
(a_0x^n + a_1x^{n-1} \cdots + a_n) &= (x^2 + sx + t)(b_0x^{n+2} \cdots + b_{n-2}) \\
&= b_0x^n + b_1x^{n-1} + b_2x^{n-2} \ldots + b_{n-2}x^2 \\
&\qquad +sb_0x^{n-1} + sb_1x^{n-2} \ldots + sb_{n-2}x \\
&\qquad\qquad +tb_0x^{n-2} \ldots + tb_n
\end{aligned}$$

Thus

$$b_0 = a_0$$

$$b_1 + sb_0 = a_1$$

$$b_2 + sb_1 + tb_0 = a_2$$

and

$$b_n + sb_{n-1} + tb_{n-2} = a_n$$