SHOCK CAPTURING ALGORITHMS

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THEORY OF SHOCK CAPTURING SCHEMES

☞ **TWO INGREDIENTS**

➢ SCALAR MAXIMUM PRINCIPLE
(LOCAL EXTREMUM DIMINISHING (LED)
SCHEMES)

➢ FLUX FOR SYSTEMS
BASED ON DISCRETE SHOCK STRUCTURE
(ONE POINT STATIONARY SHOCK)

☞ ACCURACY AND STABILITY

- ➢ It has been known (since Godunov) that linear positive schemes are first order accurate .
- ➢ This difficulty can be circumvented by introducing nonlinear switches (Boris and Book, Harten)
- ➢ We only need positivity near extrema . Hence we can switch to a higher order scheme away from extrema .
- ➢ Since the slope $\frac{\partial v}{\partial x}$ changes sign at a extremum, the switch can be based on slope comparisons .

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☞ NEED FOR OSCILLATION CONTROL

Consider a right running wave

$$\frac{\partial v}{\partial t} + a\frac{\partial v}{\partial x} = 0, \quad a > 0$$

If $\dfrac{\partial v}{\partial x}$ is discretized by CENTRAL DIFFERENCES

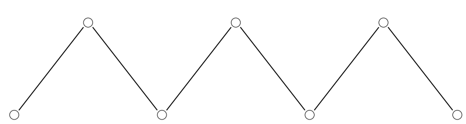
$$D_x v = \frac{v_{j+1} - v_{j-1}}{2\Delta x}$$

then an ODD-EVEN mode

$$v_j = (-1)^j$$

gives

$$D_x v = 0$$

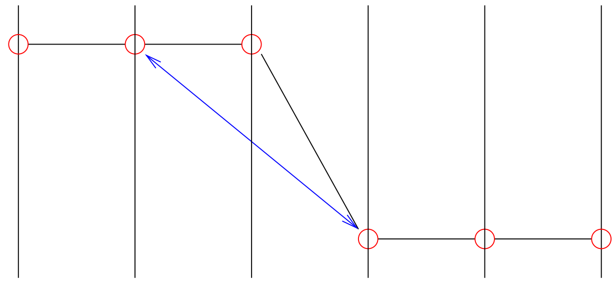


The ODD-EVEN mode is a stationary solution.

ODD-EVEN DECOUPLING must be removed.

$\Longrightarrow$ ARTIFICIAL DIFFUSION or UPWINDING

☞ PROPAGATION OF A STEP DISCONTINUITY



Consider propagation of a step as a right running wave, for which

$$\frac{\partial v}{\partial t} + a\frac{\partial v}{\partial x} = 0, \quad a > 0$$

The CENTRAL DIFFERENCE formula

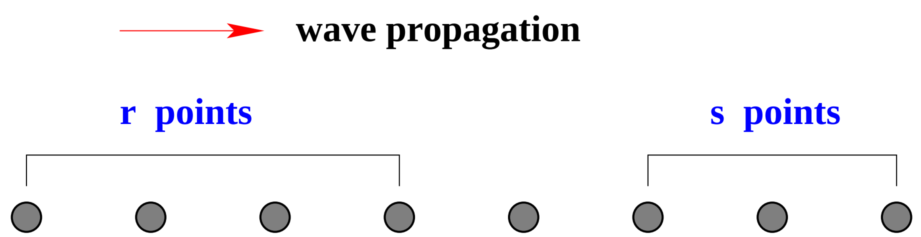
$$D_x v = \frac{v_{j+1} - v_{j-1}}{2\Delta x}$$

gives

$$D_x v_j < 0, \quad \frac{\partial v_j}{\partial t} > 0$$

$\Longrightarrow$ OVER SHOOT

☞ ISERLES' BARRIER THEOREM



Suppose that the transport equation

$$\frac{\partial v}{\partial t} + a\frac{\partial v}{\partial x} = 0, \quad a > 0$$

is approximated by a semi-discrete scheme with r upwind points and s downwind points .

Then the maximum order of accuracy of a stable scheme is

$$\min(r + s, 2r, 2s + 2)$$

This is a generalization of an earlier result of Engquist and Osher that the maximum order of accuracy of a stable upwind semi-discrete scheme is two.

It may also be compared to Dahlquist's result that A-stable linear multistep schemes for ODEs are at most second order accurate.

☞ STABILITY IN THE L_∞ NORM

Consider the nonlinear conservation law for one dependent variable with diffusion

$$\frac{\partial v}{\partial t} + \frac{\partial}{\partial x} f(v) = \frac{\partial}{\partial x} \mu(v) \frac{\partial v}{\partial x} \tag{1}$$

In a true solution, maxima do not increase and minima do not decrease. Hence L_∞ stability is appropriate.

Consider the general discrete scheme

$$v_i^{n+1} = \sum_j a_{ij} v_j^n \tag{2}$$

For consistency with (1) (no source term)

$$\sum_j a_{ij} = 1 \tag{3}$$

For L_∞ stability

$$|v_i^{n+1}| \leq \sum_j |a_{ij}| |v_j^n| \leq \sum_j |a_{ij}| \; \|v^n\|_\infty$$

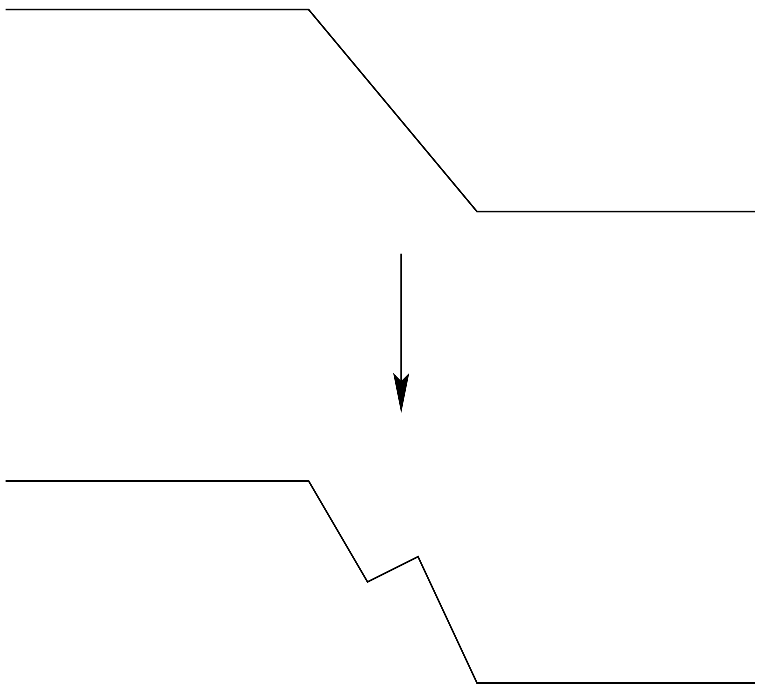
or

$$\sum_j |a_{ij}| \leq 1 \tag{4}$$

(3) and (4) together can be satisfied only if

$$a_{ij} \geq 0 \qquad \text{(positive coefficients)}$$

☞ LOCAL OSCILLATION

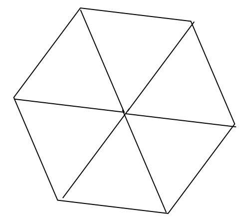


L_∞ stability still allows the possibility of local oscillations appearing in the solution.

☞ LED and ELED SCHEMES

A semi-discrete scheme is LOCAL EXTREMUM DIMINISHING (LED) if local maxima cannot increase and local minima cannot decrease.

A scheme in the form



$$\frac{dv_i}{dt} = \sum_{j \neq i} a_{ij}(v_j - v_i)$$

is LED if

$a_{ij} \geq 0, a_{ij} = 0$ if i and j are not neighbors.
(compact stencil)

Such schemes are first order accurate at extrema .

A scheme will be called ESSENTIALLY LOCAL EXTREMUM DIMINISHING (ELED) if in the limit as the mesh width $\Delta x \to 0$

local maxima are non-increasing

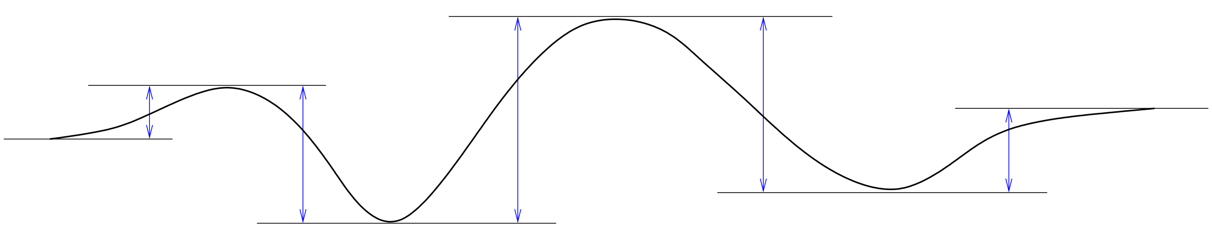
local minima are non-decreasing

Such a scheme can be second or higher order accurate at smooth extrema .

☞ EQUIVALENCE OF LOCAL EXTREMUM DIMINISHING (LED) AND TVD SCHEMES (ONE DIMENSIONAL CASE)

With fixed end values

$$\mathrm{TV}(v) = \int_{-\infty}^{\infty} \left| \frac{dv}{dx} \right| dx$$
$$\sim 2 \left(\sum maxima - \sum minima \right)$$



NONINCREASING LOCAL MAXIMA and NONDECREASING LOCAL MAXIMA imply NONINCREASING TOTAL VARIATION so LED SCHEMES are TVD SCHEMES .

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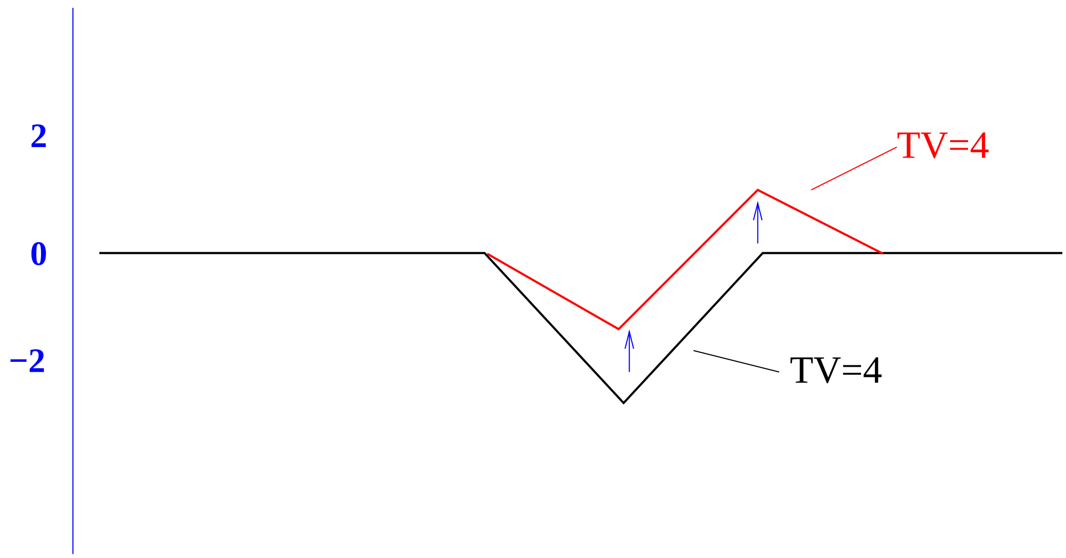
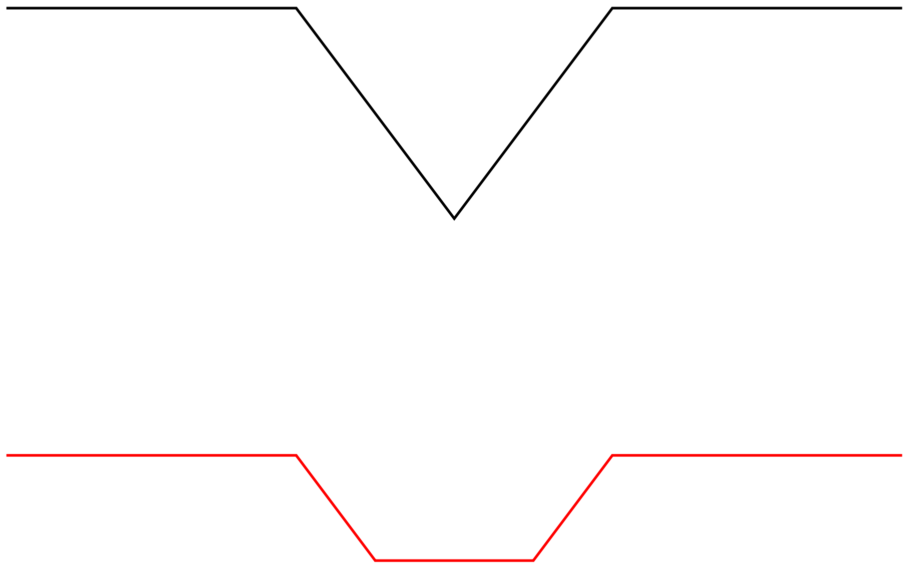
☞ DISTINCTION BETWEEN LED AND TVD SCHEMES



Shift from black to red preserves total variation .

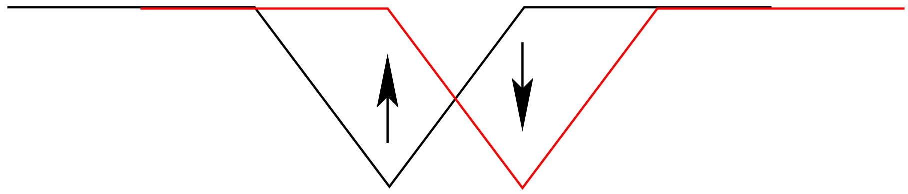
This would be inadmissible by an LED scheme.

☞ CLIPPING



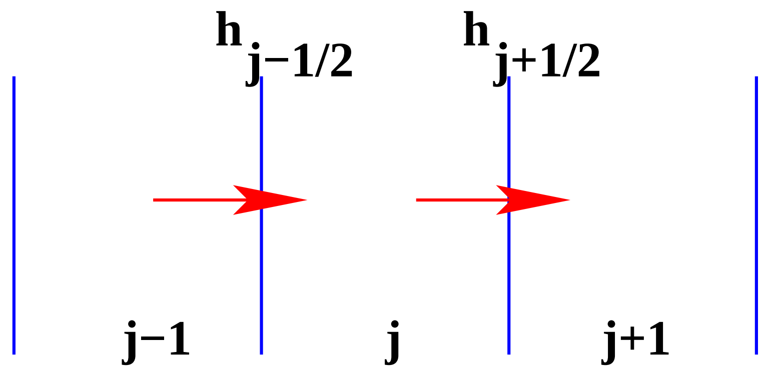
If an extremum is clipped during propagation, it can not be recovered by an LED or TVD scheme.

☞ PROPAGATION OF A PULSE



This is admitted by an LED scheme.

☞ ARTIFICIAL DIFFUSION AND LED SCHEMES



Suppose that the scalar conservation law

$$\frac{\partial v}{\partial t} + \frac{\partial}{\partial x} f(v) = 0$$

is approximated by the semi-discrete scheme

$$\Delta x \frac{dv_j}{dt} + h_{j+\frac{1}{2}} - h_{j-\frac{1}{2}} = 0$$

where the numerical flux is

$$h_{j+\frac{1}{2}} = \frac{1}{2}\left(f_{j+1} + f_j\right) - \alpha_{j+\frac{1}{2}}\left(v_{j+1} - v_j\right)$$

Define a numerical estimate of the wave speed $a(v) = \frac{\partial f}{\partial v}$ as

$$a_{j+\frac{1}{2}} = \begin{cases} \frac{f_{j+1} - f_j}{v_{j+1} - v_j}, & v_{j+1} \neq v_j \\ \\ \frac{\partial f}{\partial v}\Big|_{v_j}, & v_{j+1} = v_j \end{cases}$$

☞ ARTIFICIAL DIFFUSION AND LED SCHEMES (continued)

then the numerical flux

$$\begin{aligned} h_{j+\frac{1}{2}} &= f_j + \frac{1}{2}\left(f_{j+1} - f_j\right) - \alpha_{j+\frac{1}{2}}\left(v_{j+1} - v_j\right) \\ &= f_j - \left(\alpha_{j+\frac{1}{2}} - \frac{1}{2}a_{j+\frac{1}{2}}\right)\left(v_{j+1} - v_j\right) \end{aligned}$$

and

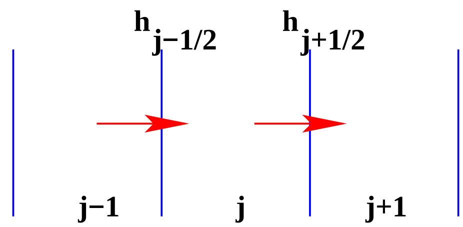
$$\begin{aligned} h_{j-\frac{1}{2}} &= f_j - \frac{1}{2}\left(f_j - f_{j-1}\right) - \alpha_{j-\frac{1}{2}}\left(v_j - v_{j-1}\right) \\ &= f_j - \left(\alpha_{j-\frac{1}{2}} + \frac{1}{2}a_{j-\frac{1}{2}}\right)\left(v_j - v_{j-1}\right) \end{aligned}$$

The semi-discrete scheme then reduces to

$$\begin{aligned} \Delta x\frac{dv_j}{dt} = (\alpha_{j+\frac{1}{2}} - \frac{1}{2}a_{j+\frac{1}{2}})(v_{j+1} - v_j) \\ -(\alpha_{j-\frac{1}{2}} + \frac{1}{2}a_{j-\frac{1}{2}})(v_j - v_{j-1}) \end{aligned}$$

This is LED if $\alpha_{j+\frac{1}{2}} \geq \frac{1}{2}|a_{j+\frac{1}{2}}| \quad \forall j$.

☞ ARTIFICIAL DIFFUSION AND UPWIND BIASING



The least diffusive LED scheme is obtained by setting

$$\alpha_{j+\frac{1}{2}} = \frac{1}{2}\left|a_{j+\frac{1}{2}}\right|$$

to produce the diffusive flux

$$d_{j+\frac{1}{2}} = \frac{1}{2}\left|a_{j+\frac{1}{2}}\right| \Delta v_{j+\frac{1}{2}},$$

where

$$\Delta v_{j+\frac{1}{2}} = v_{j+1} - v_j.$$

This is the upwind scheme since if $a_{j+\frac{1}{2}} > 0$

$$d_{j+\frac{1}{2}} = \frac{1}{2}\frac{f_{j+1} - f_j}{v_{j+1} - v_j}(v_{j+1} - v_j) = \frac{1}{2}(f_{j+1} - f_j)$$

so that $h_{j+\frac{1}{2}} = f_j$,

while if $a_{j+\frac{1}{2}} < 0, \quad h_{j+\frac{1}{2}} = f_{j+1}$.

Thus the upwind scheme is the least diffusive first order accurate LED scheme.

☞ JAMESON-SCHMIDT-TURKEL (JST) SCHEME

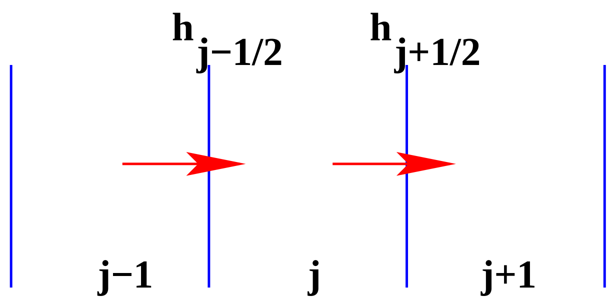
This scheme blends low and high order diffusion .

Suppose that the scalar conservation law

$$\frac{\partial v}{\partial t} + \frac{\partial}{\partial x} f(v) = 0$$

is approximated by the semi-discrete scheme

$$\Delta x \frac{dv_j}{dt} + h_{j+\frac{1}{2}} - h_{j-\frac{1}{2}} = 0$$



In the JST scheme, the numerical flux is

$$h_{j+\frac{1}{2}} = \frac{1}{2}(f_{j+1} + f_j) - d_{j+\frac{1}{2}}$$

where the diffusive flux has the form

$$d_{j+\frac{1}{2}} = \epsilon^{(2)}_{j+\frac{1}{2}} \Delta v_{j+\frac{1}{2}} - \epsilon^{(4)}_{j+\frac{1}{2}} (\Delta v_{j+\frac{3}{2}} - 2\Delta v_{j+\frac{1}{2}} + \Delta v_{j-\frac{1}{2}})$$

with

$$\Delta v_{j+\frac{1}{2}} = v_{j+1} - v_j$$

☞ DESIGN PRINCIPLES FOR THE JST SCHEME

Conservation: integral form

$\Longrightarrow$finite volume scheme

Exact for uniform flow on a curvilinear grid

$\Longrightarrow$constrains discretization, form of diffusion

Steady state independent of Δt

Eliminates Lax-Wendroff, MacCormack schemes

Concurrent computation

Eliminates LU-SGS schemes $\Longrightarrow$ RK schemes

Non-oscillatory shock capturing

$\Longrightarrow$switched artificial diffusion: upwind biasing

At least second order accurate

$\Longrightarrow$first order diffusion coefficient $\sim \Delta_x p$

Constant total enthalpy in steady flow

Eliminates Steger-Warming and other splittings

$\Longrightarrow$diffusion for energy equation $\sim \dfrac{\partial}{\partial x}\epsilon\dfrac{\partial}{\partial x}\rho H$

Simplicity

☞ ORIGINAL JST SCHEME (1980)

Dornier code (Rizzi-Schmidt) solved for $w$vol with MacCormack scheme + added diffusion

$$\sim \delta_x \epsilon \delta_x w \text{vol}, \quad \epsilon \sim \left| \frac{p_{j+1} - 2p_j + p_{j-1}}{p_{j+1} + 2p_j + p_{j-1}} \right|$$

Does not preserve uniform flow on a curvilinear grid

$\Longrightarrow$ move *vol* outside δ_x

$$w^{n+1} = w^n - \frac{\Delta t}{\text{vol}}(Q - D), Q = \text{convectiveterms}$$

Dimensional consistency $\Longrightarrow D \sim \delta_x \frac{\text{vol}}{\Delta t^*} \delta_x w$

where Δt^* is nominal time step

$$\Delta t^* = \frac{\text{vol}}{(Q + cS)_i + (Q + cS)_j}, \quad Q = \vec{q} \cdot \vec{S}$$

Higher order background diffusion was needed for convergence to a steady state.

This had to be switched off in the vicinity of a shock to prevent oscillations.

☞ JST SCHEME

Let $a_{j+\frac{1}{2}}$ be an estimate of the wave speed $\frac{\partial f}{\partial v}$

$$a_{j+\frac{1}{2}} = \frac{f_{j+1} - f_j}{v_{j+1} - v_j}$$

or

$$\frac{\partial f}{\partial v}|_{v=v_j} \quad \text{if} \quad v_{j+1} = v_j$$

Theorem : The JST scheme is LED if whenever v_j or v_{j+1} is an extremum

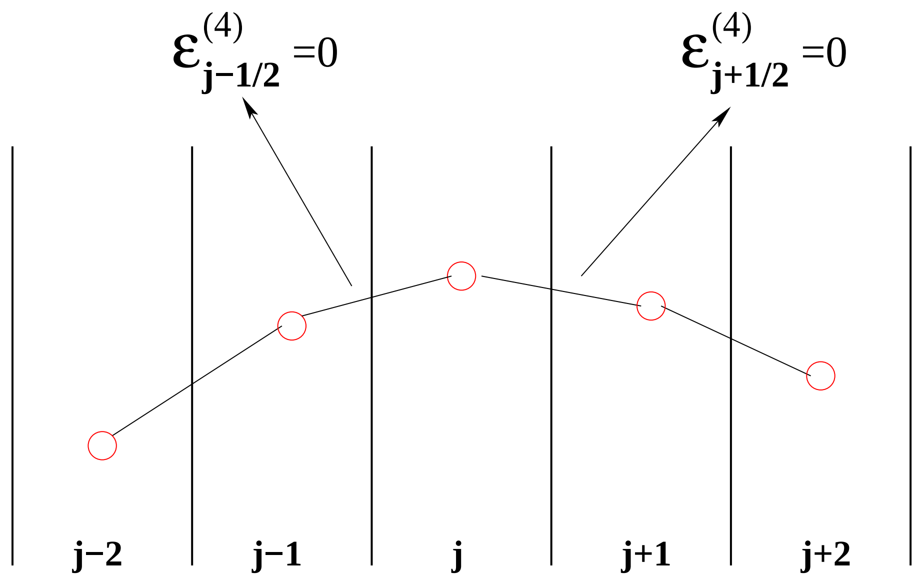
$$\epsilon^{(2)}_{j+\frac{1}{2}} \geq \frac{1}{2}|a_{j+\frac{1}{2}}|, \quad \epsilon^{(4)}_{j+\frac{1}{2}} = 0$$

Proof : At an extremum the scheme reduces to

$$\Delta_x \frac{dv_j}{dt} = \left(\epsilon^{(2)}_{j+\frac{1}{2}} - \frac{1}{2}a_{j+\frac{1}{2}}\right)\Delta v_{j+\frac{1}{2}} - \left(\epsilon^{(2)}_{j-\frac{1}{2}} + \frac{1}{2}a_{j-\frac{1}{2}}\right)\Delta v_{j-\frac{1}{2}}$$

where each term in parenthesis ≥ 0.

☞ JST SCHEME AT A MAXIMUM



The condition that $\epsilon^{(4)}_{j+\frac{1}{2}} = 0$ if v_j or v_{j+1} is an extremum

$$\Longrightarrow \epsilon^{(4)}_{j+\frac{1}{2}} = \epsilon^{(4)}_{j-\frac{1}{2}} = 0.$$

Hence the scheme reduces to a 3-point scheme and

$$\frac{dv_j}{dt} \leq 0$$

if

$$\epsilon^{(2)}_{j+\frac{1}{2}} \geq \frac{1}{2}|a_{j+\frac{1}{2}}|, \quad \epsilon^{(2)}_{j-\frac{1}{2}} \geq \frac{1}{2}|a_{j-\frac{1}{2}}|,$$

since then the coefficients multiplying $(v_{j+1} - v_j)$ and $(v_{j-1} - v_j)$ are both ≥ 0.

☞ SWITCH FOR JST SCHEME

Define

$$R(u,v) = \left| \frac{u-v}{|u|+|v|} \right|^q, \quad q \geq 1$$

Set

$$Q_{j+\frac{1}{2}} = R(\Delta v_{j+\frac{3}{2}}, \Delta v_{j-\frac{1}{2}})$$

$$\epsilon^{(2)}_{j+\frac{1}{2}} = \alpha_{j+\frac{1}{2}} Q_{j+\frac{1}{2}}$$

$$\epsilon^{(4)}_{j+\frac{1}{2}} = \beta_{j+\frac{1}{2}} (1 - Q_{j+\frac{1}{2}})$$

Then the scheme is LED if

$$\alpha_{j+\frac{1}{2}} \geq \frac{1}{2} |a_{j+\frac{1}{2}}|.$$

If

$$\beta_{j+\frac{1}{2}} = \frac{1}{2} \alpha_{j+\frac{1}{2}}$$

the JST scheme reduces to the SLIP scheme.

(described in the following slides)

☞ JST SCHEME (ELED)

Define the switching function

$$R(u,v) = \left| \frac{u - v}{\max(|u| + |v|, \epsilon \Delta x^r)} \right|^q$$

Set

$$Q_{j+\frac{1}{2}} = R(\Delta v_{j+\frac{3}{2}}, \Delta v_{j-\frac{1}{2}})$$

$$\epsilon^{(2)}_{j+\frac{1}{2}} = \alpha_{j+\frac{1}{2}} Q_{j+\frac{1}{2}}, \quad \alpha_{j+\frac{1}{2}} \geq \frac{1}{2}|a_{j+\frac{1}{2}}|$$

$$\epsilon^{(4)}_{j+\frac{1}{2}} = \frac{1}{2}|a_{j+\frac{1}{2}}|(1 - Q_{j+\frac{1}{2}})$$

$$q \geq 2, \quad r = \frac{3}{2}$$

Theorem :

(1) The JST scheme is ELED ;

(2) It is second order accurate at smooth extrema .

Note :

Also with this switch , the JST scheme is a variation of the SYMMETRIC LIMITED POSITIVE (SLIP) scheme.

☞ **SYMMETRIC LIMITED POSITIVE (SLIP) SCHEME (Int. J. of CFD, 4,1995, 171-218)**

Limited averages

Define $L(u,v)$ with the following properties

$$\begin{array}{lll} P1: & L(u,v) & = L(v,u) \\ P2: & L(\alpha u, \alpha v) & = \alpha L(u,v) \\ P3: & L(u,u) & = u \\ P4: & L(u,v) & = 0 \text{ if } u \text{ and } v \text{ have opposite signs} \end{array}$$

Example

$$\text{minmod}(u,v) = \frac{1}{2}(\text{sign}(u) + \text{sign}(v)) \min(|u|,|v|)$$

Limited flux

Now define the numerical flux

$$h_{j+\frac{1}{2}} = \frac{1}{2}(f_{j+1} + f_j) - d_{j+\frac{1}{2}}$$

where

$$d_{j+\frac{1}{2}} = \alpha_{j+\frac{1}{2}} \{ \Delta v_{j+\frac{1}{2}} - \underbrace{L(\Delta v_{j+\frac{3}{2}}, \Delta v_{j-\frac{1}{2}})\}}_{\text{anti-diffusion}}$$

$$\Delta v_{j+\frac{1}{2}} = v_{j+1} - v_j$$

This is LED if $\alpha_{j+\frac{1}{2}} \geq \frac{1}{2}|a_{j+\frac{1}{2}}|$.

☞ WELL KNOWN FLUX LIMITERS

Define

$$S(u,v) = \frac{1}{2}(\text{sign}(u) + \text{sign}(v))$$

so that

$$S(u,v) = \begin{cases} 1 & \text{when } u > 0 \text{ and } v > 0 \\ 0 & \text{when } u \text{ and } v \text{ have oppisite signs} \\ -1 & \text{when } u < 0 \text{ and } v < 0 \end{cases}$$

Well know limiters include

Minmod :

$$L(u,v) = S(u,v)\min(|u|,|v|)$$

van Leer :

$$L(u,v) = S(u,v)\frac{2|u|\ |v|}{|u| + |v|}$$

Superbee :

$$L(u,v) = S(u,v)\max\{\min(2|u|,|v|), \min(|u|,2|v|)\}$$

☞ CHARACTERIZATION OF LIMITERS

Define

$$\Phi(r) = L(1, r) = L(r, 1)$$

so that by P2 , setting $\alpha = \frac{1}{u}$,

$$L(1, \frac{v}{u}) = \frac{1}{u} L(u, v)$$

Hence

$$L(u, v) = \Phi\left(\frac{v}{u}\right) u$$

and similarly

$$L(u, v) = \Phi\left(\frac{u}{v}\right) v$$

Also by P4

$$\Phi(r) \geq 0 \;\; \text{since} \;\; \Phi(r) = 0 \;\; \text{when} \;\; r < 0.$$

☞ PROOF THAT SLIP SCHEME IS LED

Define

$$r^+ = \frac{\Delta v_{j+\frac{3}{2}}}{\Delta v_{j-\frac{1}{2}}}, \quad r^- = \frac{\Delta v_{j-\frac{3}{2}}}{\Delta v_{j+\frac{1}{2}}}$$

Then

$$L(\Delta v_{j+\frac{3}{2}}, \Delta v_{j-\frac{1}{2}}) = \Phi(r^+)\Delta v_{j-\frac{1}{2}}$$

and

$$L(\Delta v_{j+\frac{1}{2}}, \Delta v_{j-\frac{3}{2}}) = \Phi(r^-)\Delta v_{j+\frac{1}{2}}.$$

The semi-discrete scheme now reduces to

$$\begin{aligned}
\Delta x \frac{dv_j}{dt} &= -\tfrac{1}{2}a_{j+\frac{1}{2}}\Delta v_{j+\frac{1}{2}} - \tfrac{1}{2}a_{j-\frac{1}{2}}\Delta v_{j-\frac{1}{2}} \\
&\quad +\alpha_{j+\frac{1}{2}}(\Delta v_{j+\frac{1}{2}} - \Phi(r^+)\Delta v_{j-\frac{1}{2}}) \\
&\quad -\alpha_{j-\frac{1}{2}}(\Delta v_{j-\frac{1}{2}} - \Phi(r^-)\Delta v_{j+\frac{1}{2}}) \\
&= \{\alpha_{j+\frac{1}{2}} - \tfrac{1}{2}a_{j+\frac{1}{2}} + \alpha_{j-\frac{1}{2}}\Phi(r^-)\}\Delta v_{j+\frac{1}{2}} \\
&\quad -\{\alpha_{j-\frac{1}{2}} + \tfrac{1}{2}a_{j-\frac{1}{2}} + \alpha_{j+\frac{1}{2}}\Phi(r^+)\}\Delta v_{j-\frac{1}{2}}
\end{aligned}$$

Since $\Phi(r) \geq 0$, the coefficient of $\Delta v_{j+\frac{1}{2}}$ is non-negative and the coefficient of $\Delta v_{j-\frac{1}{2}}$ is non-positive if

$$\alpha_{j+\frac{1}{2}} \geq \frac{1}{2}|a_{j+\frac{1}{2}}|, \quad \forall j.$$

☞ FLUX LIMITING VIA A SWITCH

Define the limited average as the arithmetic average multiplied by a switch :

$$L(u,v) = \frac{1}{2} D(u,v)(u+v)$$

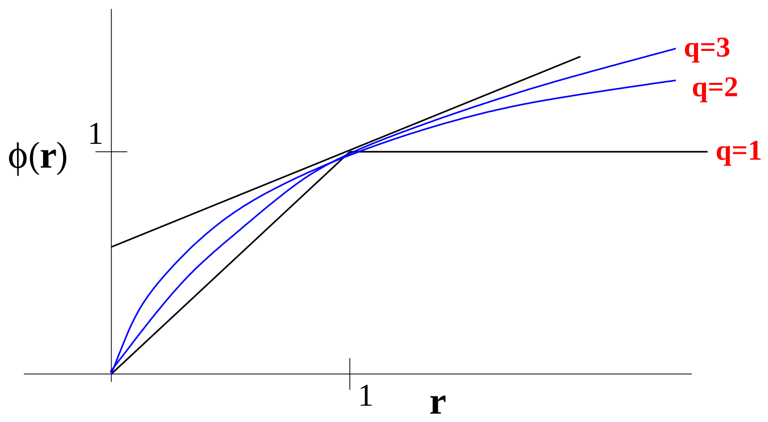
where $0 \leq D(u,v) \leq 1$ and $D(u,v) = 0$ if u and v have opposite signs.

This is realized by $D(u,v) = 1 - \left| \frac{u-v}{|u|+|v|} \right|^q$

q=1 gives Minmod
q=2 gives the van Leer limiter since
$\frac{1}{2}\left(1 - \left(\frac{u-v}{u+v}\right)^2\right)(u+v) = \frac{2uv}{u+v}$

As $q \to \infty$, $L(u,v)$ approaches a limit set by the arithmetic mean if u and v have the same sign and zero if they have opposite signs.

The corresponding switch $\Phi(r) = L(1,r)$ is sketched below.



☞ EQUIVALENCE OF JST AND SLIP SCHEMES

With the limited average

$$L(u, v) = \frac{1}{2}(1 - R(u, v))(u + v)$$

set

$$Q_{j+\frac{1}{2}} = R(\Delta v_{j+\frac{3}{2}}, \Delta v_{j-\frac{1}{2}})$$

Then the SLIP scheme can be written as

$$\begin{aligned} d_{j+\frac{1}{2}} &= \alpha_{j+\frac{1}{2}}\{\Delta v_{j+\frac{1}{2}} - \frac{1}{2}(1 - Q_{j+\frac{1}{2}})(\Delta v_{j+\frac{3}{2}} + \Delta v_{j-\frac{1}{2}})\} \\ &= \alpha_{j+\frac{1}{2}} Q_{j+\frac{1}{2}} \Delta v_{j+\frac{1}{2}} \\ &\quad -\frac{1}{2}\alpha_{j+\frac{1}{2}}(1 - Q_{j+\frac{1}{2}})(\Delta v_{j+\frac{3}{2}} - 2\Delta v_{j+\frac{1}{2}} + \Delta v_{j-\frac{1}{2}}) \end{aligned}$$

This is the Jameson-Schmidt-Turkel (JST) scheme.

Thus the JST scheme with an appropriate switch is a variation of a SLIP scheme and is ELED .

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☞ SOFT LIMITER

To prevent a reduction to first order accuracy at smooth extrema , redefine $L(u,v)$ with a threshold as

$$L(u,v) = \frac{1}{2}(1 - R(u,v))(u+v)$$

where $R(u,v) = \left| \dfrac{u-v}{\max\{(|u|+|v|), \epsilon \Delta x^r\}} \right|^q$

Now in a smooth region $\Delta v_{j+\frac{3}{2}} - \Delta v_{j-\frac{1}{2}} = \mathcal{O}(\Delta x^2)$

Take $r = \frac{3}{2}$. Then $R(\Delta v_{j+\frac{3}{2}}, \Delta v_{j-\frac{1}{2}}) = \mathcal{O}(\Delta x^{\frac{q}{2}})$ and

$$\begin{aligned} d_{j+\frac{1}{2}} &= \alpha_{j+\frac{1}{2}}(\Delta v_{j+\frac{1}{2}} - \frac{1}{2}\Delta v_{j+\frac{3}{2}} - \frac{1}{2}\Delta v_{j-\frac{1}{2}}) \\ &\quad + \alpha_{j+\frac{1}{2}} R(\Delta v_{j+\frac{3}{2}}, \Delta v_{j-\frac{1}{2}})(\Delta v_{j+\frac{3}{2}} + \Delta v_{j-\frac{1}{2}}) \\ &= \mathcal{O}(\Delta x^3) + \mathcal{O}(\Delta x^{\frac{q}{2}+1}) \end{aligned}$$

Also $L(u,v) = 0$ if $|u| + |v| > \epsilon \Delta x^r$ and have opposite signs. It follows that if v_j is a maximum ,
$\Delta x \frac{dv_j}{dt} < K \Delta x^r$, with $r = \frac{3}{2}$, $\frac{dv_j}{dt} < K \Delta x^{\frac{1}{2}}$.

Similarly if v_j is a minimum , $\frac{dv_j}{dt} > -K\Delta x^{\frac{1}{2}}$.

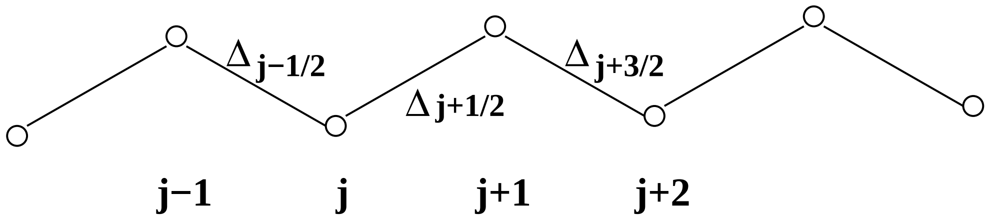
Theorem : the SLIP scheme with the soft limiter and $q \geq 2, r = \frac{3}{2}$ is

(1) second order accurate at smooth extrema ;

(2) essentially local extremum diminishing (ELED).

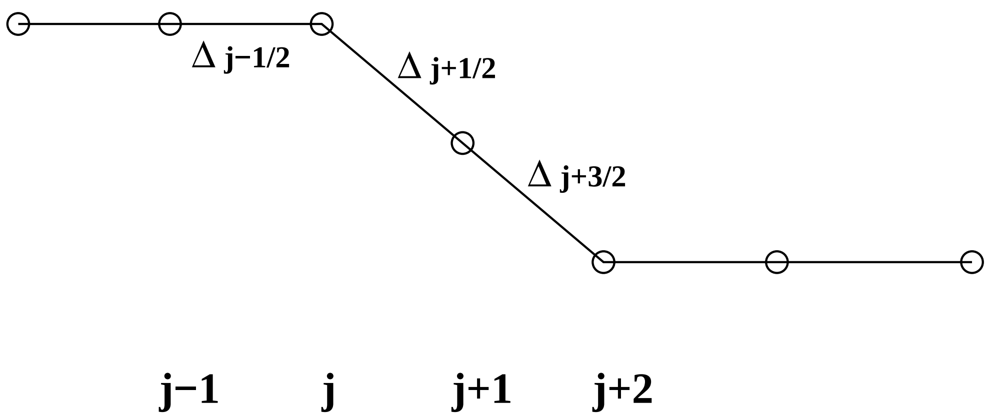
☞ SLIP SCHEME IN PRACTICE

(1) Odd-even oscillation



$\Delta_{j+\frac{3}{2}}$ and $\Delta_{j-\frac{1}{2}}$ have the same sign, opposite to that of $\Delta_{j+\frac{1}{2}}$, so they are not limited, and both reinforce $\Delta_{j+\frac{1}{2}}$ like they would in simple diffusion with fourth differences .

(2) Shock wave

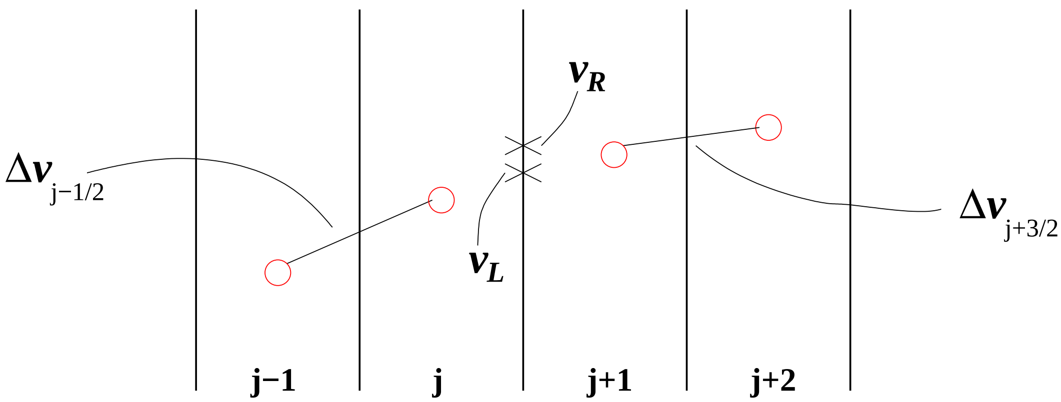


$\Delta_{j-\frac{1}{2}} = 0$ and $L(\Delta_{j+\frac{3}{2}}, \Delta_{j-\frac{1}{2}}) = 0$ so the scheme is exactly like the first order upwind scheme .

☞ SLIP RECONSTRUCTION

Define a limited average $L(u,v)$ satisfying

$$\begin{array}{llll} (1) & L(u,v) & = & L(v,u) \\ (2) & L(\alpha u,\alpha v) & = & \alpha L(u,v) \\ (3) & L(u,u) & = & u \\ (4) & L(u,v) & = & 0 \text{ if } u \text{ and } v \text{ have opposite signs} \end{array}$$



Set

$$\Delta v_{j+\frac{1}{2}} = v_{j+1} - v_j$$

$$v_L = v_j + \frac{1}{2} L(\Delta v_{j+\frac{3}{2}}, \Delta v_{j-\frac{1}{2}})$$

$$v_R = v_{j+1} - \frac{1}{2} L(\Delta v_{j+\frac{3}{2}}, \Delta v_{j-\frac{1}{2}})$$

so that

$$v_R - v_L = \Delta v_{j+\frac{1}{2}} - L(\Delta v_{j+\frac{3}{2}}, \Delta v_{j-\frac{1}{2}})$$

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