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On the Stability of the Flux Reconstruction Schemes on Quadrilateral Elements for the Linear Advection Equation

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Abstract The flux reconstruction (FR) approach to high-order methods has proved to be a promising alternative to traditional discontinuous Galerkin (DG) schemes since they facilitate the adoption of explicit time-stepping methods suitable for parallel architectures like GPUs. The FR approach provides a parameterized family of schemes through which various classical schemes like nodal-DG and spectral difference methods can be recovered. Further, the parameters can be varied to obtain schemes with a maximum stable time-step, or minimum dispersion or dissipation errors etc., providing us a single powerful framework unifying high-order discontinuous Finite Element Methods. There have been various studies on the accuracy and the stability of these schemes and in particular, a subset of the FR schemes known as ESFR or VCJH schemes have been shown to be stable in 1D and on simplex elements in 2D and 3D for the linear advection as well as the advection–diffusion equations. However, the stability of the FR schemes on tensor product quadrilateral elements has remained an open question. Although it is the most natural extension of the 1D FR approach, it has posed a significant challenge, especially for general quadrilateral elements. In this paper, we investigate the stability of the VCJH-type FR schemes for linear advection on Cartesian quadrilateral meshes and show that the schemes could become unstable under certain conditions. However, we find that the VCJH scheme recovering the DG method is stable on all Cartesian meshes. Although we restrict ourselves to Cartesian meshes in order to circumvent the algebraic complexity posed by the variation of the Jacobian matrix inside general tensor-product quadrilateral elements, our analysis offers significant insight into the possible origins of instability in the FR approach on general quadrilaterals.

Keywords Flux reconstruction · High-order methods · Stability · Nodal discontinuous Galerkin method · Quadrilaterals

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1 Introduction

High-order Discontinuous Galerkin-type methods have been growing in popularity due to their promise of increased computational efficiency and flexibility. Several variants of the classical DG method have been developed recently, such as nodal DG [1,2], spectral difference [3,4], flux reconstruction [5,6] and lifting collocation penalty (LCP) methods [7]. The flux reconstruction approach has been unique in the sense that it provides a unifying framework for such discontinuous finite element methods for utilization with explicit time-stepping schemes. Its implementation is perfectly suited for highly parallel architectures like graphics processing units (GPUs) and is one of the few high-order methods that is naturally adaptable to large GPU clusters. In addition, the FR approach, similarly to the DG methods, offers flexibility not only in its capacity to handle complex meshes but also in the plethora of choices it provides in terms of time-stepping methods, strategies for controlling dispersion and dissipation errors [8], multigrid convergence acceleration techniques etc. A detailed investigation of the connections between the FR approach and DG methods, particularly in the similar context of tensor product formulations has been performed by De Grazia et al. [9]

The Flux Reconstruction schemes were originally proposed by Huynh [5] after he observed the similarities between the nodal DG and spectral difference schemes. Vincent et al. [6] studied various properties of the FR framework and proposed new correction functions for the reconstruction process, now referred to as VCJH (Vincent–Castonguay–Jameson–Huynh) correction functions. They also showed the stability of the VCJH schemes in 1D for linear advection based on a similar analysis by Jameson in [10]. Furthermore, Jameson et al. [11] studied the non-linear stability of the FR approach in 1D. Castonguay et al. [12] extended the approach to triangular elements and proposed an energy stable family of correction functions for triangles. Further extensions to advection diffusion problems on triangles and tetrahedral elements along with proofs of stability on those elements were provided by Williams et al. [13–15].

However, the stability of these schemes on tensor product elements like quadrilaterals and hexahedra has not been studied successfully. Even the simplest bilinear quadrilateral elements pose a challenge due to the variation of the Jacobian inside each element unlike in 1D and on simplexes. In fact, direct extension of the 1D approach to the proof of stability does not seem possible. In this paper, we see that, even in the case of rectangular Cartesian meshes, investigating stability requires a somewhat different approach from that used for 1D and simplex elements. We get additional terms which affect stability, each of which is scaled by the VCJH parameter, thereby giving us valuable insight into the behavior of these schemes on general quadrilateral elements. We show that under certain conditions it is possible that a certain energy norm of the solution could grow irrespective of the time-step used. In particular we observe that uniform Cartesian meshes provide a stable platform for the FR approach, while meshes with large growth rates of element sizes[1] could lead to instability, at least temporarily. The instability is observed to be a strictly multidimensional phenomenon. However, since the unstable terms are scaled by the VCJH parameter, the FR approach that recovers DG, corresponding to a zero value of the VCJH parameter[2] is found to be stable on any Cartesian meshes.

[1] Meshes where adjacent elements are of largely different sizes.

[2] FR approaches with non-zero values of the VCJH parameter can be shown to recover filtered DG schemes [17]. The FR approach with the VCJH parameter equal to zero recovers the DG method with the exact mass matrix.

We start in Sect. 2 by providing a brief description of the FR approach for linear advection equation on quadrilateral elements as well as the 1D VCJH correction functions. In Sect. 3 we discuss the stability of these schemes while restricting ourselves to Cartesian meshes. Finally, in “Appendix”, we discuss certain properties of the energy norm we use.

2 Flux Reconstruction Methodology

Before we go on to assess the stability of the VCJH-type flux reconstruction (FR) approach on quadrilateral elements, let us first explain the approach for the linear advection equation on general linear quadrilateral elements.

2.1 Preliminaries

Consider the 2D conservation law

$$\frac{\partial u}{\partial t} + \nabla \cdot \boldsymbol{f} = 0 \quad \text{in } \Omega, \tag{2.1}$$

where Ω is a bounded connected subset of $\mathbb{R}^2$ with boundary Γ composed of a finite union of parts of hyperplanes. Further, $\boldsymbol{f}$ is a linear flux of the form

$$\boldsymbol{f} = \boldsymbol{a}u \quad \text{with} \quad \boldsymbol{f} = \begin{pmatrix} F \\ G \end{pmatrix} \quad \text{and} \quad \boldsymbol{a} = \begin{pmatrix} a \\ b \end{pmatrix}. \tag{2.2}$$

Consider a partition $\mathcal{T}_N$ of Ω into N non-empty, non-overlapping, conforming quadrilateral elements Ω_k with boundaries Γ_k such that $\Gamma_k = \bigcup_{i=1}^{4} \mathcal{F}_k^i$ where $\mathcal{F}_k^i$ are straight lines representing the faces (or edges) of the element Ω_k. Furthermore, we restrict ourselves to non-mortar elements, i.e., if $\mathcal{F}_k^i \cap \Gamma_{k'} \neq \emptyset$ for $k' \neq k$, then $\mathcal{F}_k^i \cap \Gamma_j = \emptyset, \ \forall j \neq k, k'$ and $\mathcal{F}_k^i \cap \Gamma = \emptyset$.

To facilitate a uniform implementation of the method, each element Ω_k can be mapped to a square reference domain defined by $\Omega_S = \left\{(\xi, \eta) | -1 \leq \xi, \eta \leq 1\right\}$ as follows:

$$\mathbf{x_k} = \boldsymbol{\Theta}_\mathbf{k}(\xi, \eta) = \sum_{i=1}^{4} \mathcal{N}_i(\xi, \eta)\mathbf{v}_\mathbf{k}^\mathbf{i} \tag{2.3}$$

Here $\mathbf{x_k}$ represents the physical co-ordinates (x, y) of an arbitrary point in the element Ω_k, $\mathbf{v}_\mathbf{k}^\mathbf{i}$ denote the physical co-ordinates of the 4 vertices of Ω_k and $\mathcal{N}_i(\xi, \eta)$ are bilinear shape functions defined on Ω_S. Figure 1 shows an example of such a mapping. Further, let the Jacobian matrix associated with $\boldsymbol{\Theta}_\mathbf{k}$ be denoted by $\mathbf{J_k}$ and its determinant by J_k. $\mathbf{J_k}$ varies from point to point within an element for a general linear quadrilateral, unlike linear simplex elements.

In addition, we also transform the physical quantities u and $\boldsymbol{f}$ to the reference domain using the following equations:

$$\hat{u}_k = J_k u_k \tag{2.4}$$

$$\hat{\boldsymbol{f}}_k = J_k \mathbf{J_k^{-1}} \boldsymbol{f}_k \tag{2.5}$$

$$\hat{\nabla} \cdot \hat{\boldsymbol{f}}_k = J_k \nabla \cdot \boldsymbol{f}_k \tag{2.6}$$

This transformation is designed to obtain the same form of the conservation law in the reference domain. Using these equations we can see that the conservation law, i.e., (2.1) can be written in the reference domain as follows

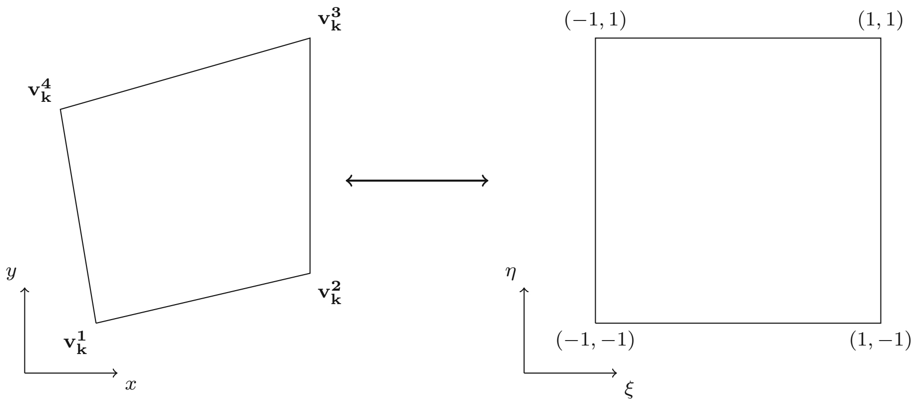

Fig. 1 Mapping between the physical domain (on the *left*) and the reference element (on the *right*)

$$\frac{\partial \hat{u}_k}{\partial t} + \hat{\nabla} \cdot \hat{\boldsymbol{f}}_{\boldsymbol{k}} = 0 \tag{2.7}$$

Since we restrict ourselves to rectangular Cartesian meshes while discussing the stability of the schemes, it is worthwhile to note that the Jacobian matrix is a constant for each element in such a mesh. We could further introduce some additional notation to simplify the algebra. For rectangular Cartesian meshes we have

$$\frac{\partial x_k}{\partial \eta} = \frac{\partial y_k}{\partial \xi} = 0 \tag{2.8}$$

Let $J_{x_k} = \frac{\partial x_k}{\partial \xi}$ and $J_{y_k} = \frac{\partial y_k}{\partial \eta}$. We then have

$$\hat{F}_k = J_{y_k} F_k \quad \hat{G}_k = J_{x_k} G_k \quad \hat{u}_k = J_{x_k} J_{y_k} u_k = J_k u_k \tag{2.9}$$

2.2 FR Procedure

Here we briefly describe the FR procedure as applied to a 2D conservation law with a linear flux on a rectangular Cartesian mesh with linear elements. Details of the implementation of the FR approach on more general quadrilateral elements and fluxes can be found in [16].

In order to build a scheme of $(p+1)$th order accuracy, we start by selecting a set of $(p+1)^2$ points on the reference domain as our solution points. A possible choice for the solution points is the tensor product of the 1D Gauss–Legendre points on the square domain (see Fig. 2). We then represent our transformed solution within each element, i.e., $\hat{u}_k$, using a tensor product of pth degree Lagrange polynomial basis defined on these solution points.

$$\hat{u}^D = \sum_{i=0}^{p} \sum_{j=0}^{p} l_i(\xi) l_j(\eta) \hat{u}^D_{ij}, \tag{2.10}$$

where $l_i(\xi)$ and $l_j(\eta)$ are the 1D Lagrange polynomials associated with the solution points ξ_i and η_j respectively and $\hat{u}^D_{ij}$ is the value of the transformed solution at (ξ_i, η_j). Note that we have dropped the subscript k in order to keep the notation from getting clumsy. Since $\hat{u}^D$ is a transformed quantity, it is understood to be associated with a certain generic element Ω_k.

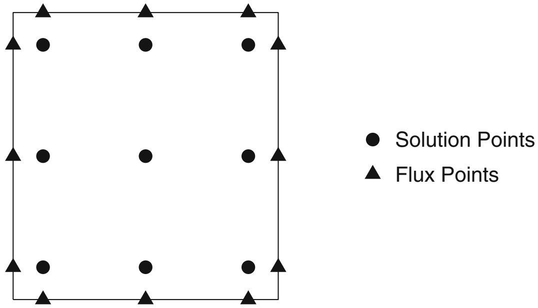

Fig. 2 Figure showing the solution and flux points in the reference element for a $p = 2$ scheme

Also, similar to a discontinuous Galerkin method, we allow our solution u to be discontinuous across the elements. Therefore, we represent such discontinuous quantities with a superscript D .

We also have $p + 1$ flux points along each boundary edge of the quadrilateral element. These flux points are chosen to align with the solution points in the reference domain, i.e., we would choose them to be the 1D Gauss–Legendre points along each edge if we are using such solution points. The total continuous flux $\hat{\boldsymbol{f}}_{\boldsymbol{k}}$ can be written as a sum of a discontinuous component and a correction component.

$$\hat{\boldsymbol{f}}_{\boldsymbol{k}} = \hat{\boldsymbol{f}}_{\boldsymbol{k}}^{D} + \hat{\boldsymbol{f}}_{\boldsymbol{k}}^{C} \tag{2.11}$$

The discontinuous component, $\hat{\boldsymbol{f}}_{\boldsymbol{k}}^{D}$ is the transformed version of the flux computed directly from the solution values at the solution points and is represented using the same pth degree Lagrange polynomial basis we used for the solution points. Therefore, in each element we have

$$\hat{F}^D = \sum_{i=0}^{p}\sum_{j=0}^{p} l_i(\xi) l_j(\eta) \hat{F}_{ij}^D \quad \text{and} \quad \hat{G}^D = \sum_{i=0}^{p}\sum_{j=0}^{p} l_i(\xi) l_j(\eta) \hat{G}_{ij}^D \tag{2.12}$$

where

$$\hat{F}_{ij}^D = J_{y_k} F(u_{ij}) \quad \text{and} \quad \hat{G}_{ij}^D = J_{x_k} G(u_{ij}) \tag{2.13}$$

The correction component of the flux is computed along 1D lines in both the ξ and η directions and can be concisely written as follows

$$\hat{F}^C = -h_L(\xi) \sum_{j=0}^{p} \left(\left(\hat{\boldsymbol{f}}^{*} - \hat{\boldsymbol{f}}^{D} \right) \cdot \hat{\boldsymbol{n}} \right)_{L_j} l_j(\eta) + h_R(\xi) \sum_{j=0}^{p} \left(\left(\hat{\boldsymbol{f}}^{*} - \hat{\boldsymbol{f}}^{D} \right) \cdot \hat{\boldsymbol{n}} \right)_{R_j} l_j(\eta) \tag{2.14}$$

$$\hat{G}^C = -h_L(\eta) \sum_{j=0}^{p} \left(\left(\hat{\boldsymbol{f}}^{*} - \hat{\boldsymbol{f}}^{D} \right) \cdot \hat{\boldsymbol{n}} \right)_{B_j} l_j(\xi) + h_R(\eta) \sum_{j=0}^{p} \left(\left(\hat{\boldsymbol{f}}^{*} - \hat{\boldsymbol{f}}^{D} \right) \cdot \hat{\boldsymbol{n}} \right)_{T_j} l_j(\xi) \tag{2.15}$$

where if one wants to recover the VCJH schemes, h_L and h_R denote the left and right 1D VCJH correction functions of degree $p + 1$ respectively. L, R, B, T represent the left ($\xi = -1$), right ($\xi = 1$), bottom ($\eta = -1$) and top ($\eta = 1$) edges respectively. $(.)_{L_j}$ denotes the value at the jth flux point on the left boundary. $\hat{\boldsymbol{f}}^{*}$ represents the transformed common interface flux value. $\hat{\boldsymbol{f}}^{D}$ on the boundaries is obtained through an extrapolation operation.

Finally, l_j denotes the jth member of the 1D Lagrange basis of degree p defined on the edge and the summation is over the flux points on the corresponding edge.

Remark 2.1 Note that we have used h_L and h_R as the correction functions for $\hat{G}^C$ as well because the correction along the η direction is performed in the same 1D sense as that in the ξ direction.

Remark 2.2 In the above equations, note that the corrections coming in from left and bottom edges have a negative sign associated with them, unlike in 1D, because we use $\hat{\boldsymbol{f}} \cdot \hat{\boldsymbol{n}}$. Since the outward-facing normal vector $\hat{\boldsymbol{n}}$ has a negative sign on the left and bottom edges, we need to compensate for it with an additional negative sign.

Also, for brevity of notation, we let

$$\begin{aligned} \Delta_{L_j} &= \left(\left(\hat{\boldsymbol{f}}^* - \hat{\boldsymbol{f}}^D\right) \cdot \hat{\boldsymbol{n}}\right)_{L_j} \quad & \Delta_{R_j} &= \left(\left(\hat{\boldsymbol{f}}^* - \hat{\boldsymbol{f}}^D\right) \cdot \hat{\boldsymbol{n}}\right)_{R_j} \\ \Delta_{B_j} &= \left(\left(\hat{\boldsymbol{f}}^* - \hat{\boldsymbol{f}}^D\right) \cdot \hat{\boldsymbol{n}}\right)_{B_j} \quad & \Delta_{T_j} &= \left(\left(\hat{\boldsymbol{f}}^* - \hat{\boldsymbol{f}}^D\right) \cdot \hat{\boldsymbol{n}}\right)_{T_j} \end{aligned} \tag{2.16}$$

In order to compute the transformed common interface flux $\hat{\boldsymbol{f}}^*$, we first need to extrapolate the solution values to the flux points on the boundary. For example, the 1D-edge polynomial formed by the extrapolated transformed solution on the left boundary is computed as follows

$$\hat{u}_L^D = \sum_{i=0}^{p} \sum_{j=0}^{p} l_i(-1) l_j(\eta) \hat{u}_{ij}^D \tag{2.17}$$

We then transform this $\hat{u}_L^D$ to the physical domain and compute the flux from the physical solution values at the boundary flux points. Let f_-^D, u_-^D denote the boundary fluxes and solution values calculated as above from the current element and f_+^D, u_+^D denote the corresponding values from a neighboring element along some edge. We then compute the common numerical interface flux $\boldsymbol{f}^*$ at the flux points on the edge using a Lax–Friedrichs approach. i.e.,

$$\boldsymbol{f}^* = \{\{\boldsymbol{f}^D\}\} + \frac{\lambda}{2}\left(\max_{u \in [u_-^D, u_+^D]} \left|\frac{\partial \boldsymbol{f}}{\partial u} \cdot n\right|\right) [\![u^D]\!] \tag{2.18}$$

where $\{\{\cdot\}\}$ and $[\![\cdot]\!]$ are the average and jump operators respectively and λ is an upwinding parameter with $0 \le \lambda \le 1$. $\lambda = 1$ gives a fully upwinded scheme while $\lambda = 0$ is essentially the central flux definition. We then have to transform the normal common interface flux from the physical domain back to the reference domain. For example, on the left boundary we can do this using

$$(\hat{\boldsymbol{f}}^* \cdot \hat{\boldsymbol{n}})_{L_j} = J_{L_j} (\boldsymbol{f}^* \cdot \boldsymbol{n})_{L_j} \tag{2.19}$$

where J_{L_j} is the edge-Jacobian at the jth flux point on the left boundary. The edge-Jacobian is an edge-based scaling factor which is just equal to the edge length in the Cartesian case. Therefore (2.19) can be rewritten for the case of Cartesian meshes as

$$\left(\hat{\boldsymbol{f}}^* \cdot \hat{\boldsymbol{n}}\right)_{L_j} = J_y (\boldsymbol{f}^* \cdot \boldsymbol{n})_{L_j} \tag{2.20}$$

where J_y is the edge length of the left (and right) edge. We can then go on and compute the correction component of the flux using (2.14) and (2.15). Once we have both the discontinuous and the correction components of the flux, we can then calculate the transformed solution at the next time step in the kth element using

$$\frac{\partial \hat{u}_k^D}{\partial t} = -\hat{\nabla} \cdot \hat{\boldsymbol{f}}_{\boldsymbol{k}}^{D} - \hat{\nabla} \cdot \hat{\boldsymbol{f}}_{\boldsymbol{k}}^{C} \tag{2.21}$$

Note that the divergence of the total continuous flux is of degree p due to the $(p+1)$th degree VCJH correction functions in the correction component $\hat{\boldsymbol{f}}_{\boldsymbol{k}}^{C}$.

2.3 VCJH Correction Functions

The 1D VCJH correction functions have been described in detail in [6]. Here we provide a brief description of these functions along with a few important properties which will be used in the stability discussion later. The 1D VCJH correction functions h_L and h_R can be written as follows

$$h_L = \frac{(-1)^p}{2}\left[L_p - \frac{\eta_p L_{p-1} + L_{p+1}}{1+\eta_p}\right] \qquad h_R = \frac{1}{2}\left[L_p + \frac{\eta_p L_{p-1} + L_{p+1}}{1+\eta_p}\right] \tag{2.22}$$

where

$$\eta_p = \frac{c}{2}(2p+1)(a_p p!)^2 \tag{2.23}$$

and a_p is the leading coefficient of the pth Legendre polynomial L_p defined on $[-1, 1]$. c is a free parameter referred to as the VCJH parameter. Several different schemes like the DG, spectral difference (SD) and Huynh's G2 scheme [5] can be recovered by varying this parameter. For example, setting $c = 0$ allows us to recover the classical nodal DG method (for linear fluxes).

The VCJH functions have the following properties:

$$h_L(\xi) = h_R(-\xi) \tag{2.24}$$

$$h_L(-1) = 1 \qquad h_L(1) = 0 \tag{2.25}$$

$$\int_{-1}^{1} \frac{dl_i}{d\xi} h_L d\xi = c \frac{d^p l_i}{d\xi^p} \frac{d^{p+1} h_L}{d\xi^{p+1}} \tag{2.26}$$

3 Stability of FR Scheme on Quadrilaterals

In this section we discuss the stability of the FR scheme on quadrilateral elements. We restrict ourselves to rectangular Cartesian meshes with linear quadrilateral elements. This is mainly to avoid the variation of the Jacobian inside the element which makes the algebra difficult to manage. We state our main result in Theorem 3.5. Our aim is to investigate the growth of an appropriate Sobolev norm of the solution and identify factors that can cause instabilities. Before stating the theorem however, we show some intermediate important results through Lemmas.

Lemma 3.1

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega_k} J_k (u_k^D)^2 d\Omega_k = -\int_{\Omega_S} \hat{u}^D \left(\hat{\nabla}\cdot\hat{\boldsymbol{f}}^D\right) d\Omega_S - \int_{\Gamma_S} \hat{u}^D \left(\hat{\boldsymbol{f}}^C \cdot \hat{\boldsymbol{n}}\right) d\Gamma_S$$
$$\underbrace{-c\int_{-1}^{1} \frac{d^{p+1}h_L(\xi)}{d\xi^{p+1}} \frac{\partial^p \hat{u}^D}{\partial \xi^p}\left(\sum_{j=0}^{p} \Delta_{L_j} l_j(\eta)\right) d\eta}_{A_1}$$
$$\underbrace{+c\int_{-1}^{1} \frac{d^{p+1}h_R(\xi)}{d\xi^{p+1}} \frac{\partial^p \hat{u}^D}{\partial \xi^p}\left(\sum_{j=0}^{p} \Delta_{R_j} l_j(\eta)\right) d\eta}_{A_2}$$
$$\underbrace{-c\int_{-1}^{1} \frac{d^{p+1}h_L(\eta)}{d\eta^{p+1}} \frac{\partial^p \hat{u}^D}{d\eta^p}\left(\sum_{j=0}^{p} \Delta_{B_j} l_j(\xi)\right) d\xi}_{A_3}$$
$$\underbrace{+c\int_{-1}^{1} \frac{d^{p+1}h_R(\eta)}{d\eta^{p+1}} \frac{\partial^p \hat{u}^D}{d\eta^p}\left(\sum_{j=0}^{p} \Delta_{T_j} l_j(\xi)\right) d\xi}_{A_4} \tag{3.1}$$

Proof Let us start by rewriting (2.1) in the kth element by observing that the total continuous flux is composed of the discontinuous and the correction components, i.e.,

$$\frac{\partial u_k^D}{\partial t} = -\nabla\cdot\boldsymbol{f}_{\boldsymbol{k}}^D - \nabla\cdot\boldsymbol{f}_{\boldsymbol{k}}^C \tag{3.2}$$

Let J_k be the determinant of the transformation Jacobian $\mathbf{J_k}$. Multiply (3.2) by $J_k u_k^D$ and integrate over Ω_k to get

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega_k} J_k (u_k^D)^2 d\Omega_k = -\int_{\Omega_k} J_k u_k^D \left(\nabla\cdot\boldsymbol{f}_k^D + \nabla\cdot\boldsymbol{f}_k^C\right) d\Omega_k \tag{3.3}$$

Transforming the RHS to the reference domain using (2.4) and (2.6), we get

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega_k} J_k (u_k^D)^2 d\Omega_k = -\int_{\Omega_S} \hat{u}^D \left(\hat{\nabla}\cdot\hat{\boldsymbol{f}}^D + \hat{\nabla}\cdot\hat{\boldsymbol{f}}^C\right) d\Omega_S \tag{3.4}$$

Now consider the second term in the RHS above:

$$-\int_{\Omega_S} \hat{u}^D \left(\hat{\nabla}\cdot\hat{\boldsymbol{f}}^C\right) d\Omega_S = -\int_{\Omega_S} \hat{u}^D \left(\frac{\partial \hat{F}^C}{\partial \xi} + \frac{\partial \hat{G}^C}{\partial \eta}\right) d\Omega_S$$
$$= \int_{-1}^{1}\int_{-1}^{1} \hat{u}^D \frac{dh_L(\xi)}{d\xi} \sum_{j=0}^{p} \left((\hat{\boldsymbol{f}}\cdot\hat{\boldsymbol{n}})^*_{L_j} - \hat{F}^D_{L_j}\right) l_j(\eta) d\xi d\eta$$

$$
\begin{aligned}
&-\int_{-1}^{1}\int_{-1}^{1}\hat{u}^D\frac{dh_R(\xi)}{d\xi}\sum_{j=0}^{p}\Big((\hat{\boldsymbol{f}}\cdot\hat{\boldsymbol{n}})^*_{R_j}-\hat{F}^D_{R_j}\Big)l_j(\eta)d\xi d\eta\\
&+\int_{-1}^{1}\int_{-1}^{1}\hat{u}^D\frac{dh_L(\eta)}{d\eta}\sum_{j=0}^{p}\Big((\hat{\boldsymbol{f}}\cdot\hat{\boldsymbol{n}})^*_{B_j}-\hat{G}^D_{B_j}\Big)l_j(\xi)d\xi d\eta\\
&-\int_{-1}^{1}\int_{-1}^{1}\hat{u}^D\frac{dh_R(\eta)}{d\eta}\sum_{j=0}^{p}\Big((\hat{\boldsymbol{f}}\cdot\hat{\boldsymbol{n}})^*_{T_j}-\hat{G}^D_{T_j}\Big)l_j(\xi)d\xi d\eta \qquad (3.5)
\end{aligned}
$$

Now the first term on the RHS of (3.5) can be rewritten as follows by using integration by parts

$$
\int_{-1}^{1}\int_{-1}^{1}\hat{u}^D\frac{dh_L(\xi)}{d\xi}\sum_{j=0}^{p}\Big((\hat{\boldsymbol{f}}\cdot\hat{\boldsymbol{n}})^*_{L_j}-\hat{F}^D_{L_j}\Big)l_j(\eta)d\xi d\eta=\int_{-1}^{1}\underbrace{\int_{-1}^{1}\hat{u}^D\frac{dh_L(\xi)}{d\xi}d\xi}_{\text{I.B.P}}\sum_{j=0}^{p}\Delta_{L_j}l_j(\eta)d\eta
$$

$$
\begin{aligned}
&=\int_{-1}^{1}\left(\left(\hat{u}^D h_L(\xi)\right)\Big|_{\xi=-1}^{\xi=1}-\int_{-1}^{1}h_L(\xi)\frac{\partial\hat{u}^D}{\partial\xi}d\xi\right)\left(\sum_{j=0}^{p}\Delta_{L_j}l_j(\eta)\right)d\eta\\
&=\int_{-1}^{1}\left(-\hat{u}^D_L(\eta)\right)\left(\sum_{j=0}^{p}\Delta_{L_j}l_j(\eta)\right)d\eta-\int_{-1}^{1}\int_{-1}^{1}\frac{\partial\hat{u}^D}{\partial\xi}h_L(\xi)\left(\sum_{j=0}^{p}\Delta_{L_j}l_j(\eta)\right)d\xi d\eta \qquad (3.6)
\end{aligned}
$$

where the last step was obtained using (2.25). Since our transformed solution $\hat{u}^D$ is represented by a tensor-product Lagrange basis, we can use property (2.26) to obtain

$$
\boxed{\int_{-1}^{1}\frac{\partial\hat{u}^D}{\partial\xi}h_L(\xi)d\xi=c\frac{\partial^p\hat{u}^D}{\partial\xi^p}\frac{d^{p+1}h_L(\xi)}{d\xi^{p+1}}}
$$

where c is the VCJH parameter. Now we can rewrite (3.6) using this property to get

$$
\begin{aligned}
&=\int_{-1}^{1}\left(-\hat{u}^D_L(\eta)\right)\left(\sum_{j=0}^{p}\Delta_{L_j}l_j(\eta)\right)d\eta-c\int_{-1}^{1}\frac{d^{p+1}h_L(\xi)}{d\xi^{p+1}}\frac{\partial^p\hat{u}^D}{\partial\xi^p}\left(\sum_{j=0}^{p}\Delta_{L_j}l_j(\eta)\right)d\eta\\
&=\int_{-1}^{1}\left(-\hat{u}^D_L(\eta)\right)\left(-\hat{\boldsymbol{f}}^C\cdot\hat{\boldsymbol{n}}\right)_L d\eta-c\int_{-1}^{1}\frac{d^{p+1}h_L(\xi)}{d\xi^{p+1}}\frac{\partial^p\hat{u}^D}{\partial\xi^p}\left(\sum_{j=0}^{p}\Delta_{L_j}l_j(\eta)\right)d\eta \qquad (3.7)\\
&=\int_{-1}^{1}\hat{u}^D_L(\eta)\left(\hat{\boldsymbol{f}}^C\cdot\hat{\boldsymbol{n}}\right)_L d\eta-c\int_{-1}^{1}\frac{d^{p+1}h_L(\xi)}{d\xi^{p+1}}\frac{\partial^p\hat{u}^D}{\partial\xi^p}\left(\sum_{j=0}^{p}\Delta_{L_j}l_j(\eta)\right)d\eta
\end{aligned}
$$

□

Remark 3.1 Note that the term $\int_{-1}^{1}\hat{u}^D_L(\eta)\left(\hat{\boldsymbol{f}}^C\cdot\hat{\boldsymbol{n}}\right)_L d\eta$ is integrating along the left boundary from $\eta=-1$ to $\eta=1$. If we were to include this as a part of the integral along the boundaries of the element, we would have to integrate in the opposite direction since we assume the counter-clockwise direction as positive for element boundary integrals.

Writing down similar expressions for the other three terms on the RHS of (3.5), we get

$$
\begin{aligned}
-\int_{\Omega_S} \hat{u}^D \left(\hat{\nabla} \cdot \hat{\boldsymbol{f}}^C\right) d\Omega_S = &-\int_{\Gamma_S} \hat{u}^D \left(\hat{\boldsymbol{f}}^C \cdot \hat{\boldsymbol{n}}\right) d\Gamma_S \\
&- c\int_{-1}^{1} \frac{d^{p+1}h_L(\xi)}{d\xi^{p+1}} \frac{\partial^p \hat{u}^D}{\partial \xi^p} \left(\sum_{j=0}^{p} \Delta_{L_j} l_j(\eta)\right) d\eta \\
&+ c\int_{-1}^{1} \frac{d^{p+1}h_R(\xi)}{d\xi^{p+1}} \frac{\partial^p \hat{u}^D}{\partial \xi^p} \left(\sum_{j=0}^{p} \Delta_{R_j} l_j(\eta)\right) d\eta \\
&- c\int_{-1}^{1} \frac{d^{p+1}h_L(\eta)}{d\eta^{p+1}} \frac{\partial^p \hat{u}^D}{\partial \eta^p} \left(\sum_{j=0}^{p} \Delta_{B_j} l_j(\xi)\right) d\xi \\
&+ c\int_{-1}^{1} \frac{d^{p+1}h_R(\eta)}{d\eta^{p+1}} \frac{\partial^p \hat{u}^D}{\partial \eta^p} \left(\sum_{j=0}^{p} \Delta_{T_j} l_j(\xi)\right) d\xi
\end{aligned}
\tag{3.8}
$$

Substituting these results back into (3.5), we get Lemma 3.1.

Lemma 3.2

$$
\begin{aligned}
\frac{1}{2}\left(\frac{1}{2}\right)\frac{\partial}{\partial t}\int_{\Omega_k} J_k \left(\frac{\partial^p u_k^D}{\partial \xi^p}\right)^2 d\Omega_k = &-\int_{-1}^{1} \frac{\partial^p \hat{u}^D}{\partial \xi^p} \frac{\partial^p}{\partial \xi^p}\left(\frac{\partial \hat{G}^D}{\partial \eta}\right) d\eta \\
&+ \underbrace{\int_{-1}^{1} \frac{\partial^p \hat{u}^D}{\partial \xi^p} \frac{d^{p+1}h_L(\xi)}{d\xi^{p+1}} \left(\sum_{j=0}^{p} \Delta_{L_j} l_j(\eta)\right) d\eta}_{A_1} \\
&- \underbrace{\int_{-1}^{1} \frac{\partial^p \hat{u}^D}{\partial \xi^p} \frac{d^{p+1}h_R(\xi)}{d\xi^{p+1}} \left(\sum_{j=0}^{p} \Delta_{R_j} l_j(\eta)\right) d\eta}_{A_2} \\
&+ \left(\sum_{j=0}^{p} \Delta_{B_j} \frac{\partial^p l_j(\xi)}{\partial \xi^p}\right)\left(-\left.\frac{\partial^p \hat{u}^D}{\partial \xi^p}\right|_{\eta=-1} - \underbrace{c\frac{d^{p+1}h_L(\eta)}{d\eta^{p+1}} \frac{\partial^{2p} \hat{u}^D}{\partial \xi^p \partial \eta^p}}_{B_3}\right) \\
&- \left(\sum_{j=0}^{p} \Delta_{T_j} \frac{\partial^p l_j(\xi)}{\partial \xi^p}\right)\left(\left.\frac{\partial^p \hat{u}^D}{\partial \xi^p}\right|_{\eta=1} - \underbrace{c\frac{d^{p+1}h_R(\eta)}{d\eta^{p+1}} \frac{\partial^{2p} \hat{u}^D}{\partial \xi^p \partial \eta^p}}_{B_4}\right)
\end{aligned}
\tag{3.9}
$$

Proof Multiply (3.2) by J_k and apply the operator $\frac{\partial^p}{\partial \xi^p}$ to the entire equation to get

$$\frac{\partial}{\partial t}\left(J_k \frac{\partial^p u_k^D}{\partial \xi^p}\right) = -\frac{\partial^p}{\partial \xi^p}(J_k \nabla \cdot \boldsymbol{f}_k^D) - \frac{\partial^p}{\partial \xi^p}(J_k \nabla \cdot \boldsymbol{f}_k^C) \tag{3.10}$$

Note that the derivative of u_k^D with respect to ξ is well defined as it is indirectly a function of ξ and η. Now observing that J_k is a constant in a Cartesian mesh, the above equation can then be written as

$$J_k \frac{\partial}{\partial t}\left(\frac{\partial^p u_k^D}{\partial \xi^p}\right) = -\frac{\partial^p}{\partial \xi^p}(J_k \nabla \cdot \boldsymbol{f}_k^D) - \frac{\partial^p}{\partial \xi^p}(J_k \nabla \cdot \boldsymbol{f}_k^C) \tag{3.11}$$

Multiply both sides of (3.11) by $\frac{\partial^p \hat{u}_k^D}{\partial \xi^p}$ and integrate over Ω_k to obtain

$$\frac{1}{2}\frac{\partial}{\partial t}\int_{\Omega_k} J_k \left(\frac{\partial^p u_k^D}{\partial \xi^p}\right)^2 d\Omega_k = -\int_{\Omega_k} \frac{\partial^p u_k^D}{\partial \xi^p}\frac{\partial^p}{\partial \xi^p}(J_k \nabla \cdot \boldsymbol{f}_k^D) d\Omega_k - \int_{\Omega_k} \frac{\partial^p u_k^D}{\partial \xi^p}\frac{\partial^p}{\partial \xi^p}(J_k \nabla \cdot \boldsymbol{f}_k^C) d\Omega_k \tag{3.12}$$

Transforming RHS to reference domain we get

$$\frac{1}{2}\frac{\partial}{\partial t}\int_{\Omega_k} J_k \left(\frac{\partial^p u_k^D}{\partial \xi^p}\right)^2 d\Omega_k = -\int_{\Omega_S} \frac{\partial^p \hat{u}^D}{\partial \xi^p}\frac{\partial^p}{\partial \xi^p}\left(\hat{\nabla} \cdot \hat{\boldsymbol{f}}^D\right) d\Omega_S - \int_{\Omega_S} \frac{\partial^p \hat{u}^D}{\partial \xi^p}\frac{\partial^p}{\partial \xi^p}\left(\hat{\nabla} \cdot \hat{\boldsymbol{f}}^C\right) d\Omega_S \tag{3.13}$$

Let us first consider the 1st term on the RHS of (3.13),

$$\begin{aligned} -\int_{\Omega_S} \frac{\partial^p \hat{u}^D}{\partial \xi^p}\frac{\partial^p}{\partial \xi^p}\left(\hat{\nabla} \cdot \hat{\boldsymbol{f}}^D\right) d\Omega_S &= -\int_{\Omega_S} \frac{\partial^p \hat{u}^D}{\partial \xi^p}\left(\cancelto{0}{\frac{\partial^{p+1} \hat{F}^D}{\partial \xi^{p+1}}} + \frac{\partial^p}{\partial \xi^p}\left(\frac{\partial \hat{G}^D}{\partial \eta}\right)\right) d\Omega_S \\ &= -2\int_{-1}^{1} \frac{\partial^p \hat{u}^D}{\partial \xi^p}\frac{\partial^p}{\partial \xi^p}\left(\frac{\partial \hat{G}^D}{\partial \eta}\right) d\eta \end{aligned} \tag{3.14}$$

where the last step was obtained by observing that the integrand is a constant w.r.t ξ and hence the integral just amounts to 2. Now consider the 2nd term of the RHS of (3.13).

$$\begin{aligned} -\int_{\Omega_S} \frac{\partial^p \hat{u}^D}{\partial \xi^p}\frac{\partial^p}{\partial \xi^p}\left(\hat{\nabla} \cdot \hat{\boldsymbol{f}}^C\right) d\Omega_S &= -\int_{\Omega_S} \frac{\partial^p \hat{u}^D}{\partial \xi^p}\left(\frac{\partial^{p+1} \hat{F}^C}{\partial \xi^{p+1}} + \frac{\partial^p}{\partial \xi^p}\left(\frac{\partial \hat{G}^C}{\partial \eta}\right)\right) d\Omega_S \\ &= +2\int_{-1}^{1} \frac{\partial^p \hat{u}^D}{\partial \xi^p}\frac{d^{p+1} h_L(\xi)}{d\xi^{p+1}}\left(\sum_{j=0}^{p} \Delta_{L_j} l_j(\eta)\right) d\eta \end{aligned}$$

$$-2\int_{-1}^{1} \frac{\partial^p \hat{u}^D}{\partial \xi^p} \frac{d^{p+1} h_R(\xi)}{d\xi^{p+1}} \left(\sum_{j=0}^{p} \Delta_{R_j} l_j(\eta) \right) d\eta$$
$$+2\int_{-1}^{1} \frac{\partial^p \hat{u}^D}{\partial \xi^p} \frac{\partial h_L(\eta)}{\partial \eta} \left(\sum_{j=0}^{p} \Delta_{B_j} \frac{\partial^p l_j(\xi)}{\partial \xi^p} \right) d\eta$$
$$-2\int_{-1}^{1} \frac{\partial^p \hat{u}^D}{\partial \xi^p} \frac{\partial h_R(\eta)}{\partial \eta} \left(\sum_{j=0}^{p} \Delta_{T_j} \frac{\partial^p l_j(\xi)}{\partial \xi^p} \right) d\eta \quad (3.15)$$

Now consider the 3rd term on the RHS of (3.15). We can write it as follows:

$$2\int_{-1}^{1} \frac{\partial^p \hat{u}^D}{\partial \xi^p} \frac{\partial h_L(\eta)}{\partial \eta} \left(\sum_{j=0}^{p} \Delta_{B_j} \frac{\partial^p l_j(\xi)}{\partial \xi^p} \right) d\eta = 2\left(\sum_{j=0}^{p} \Delta_{B_j} \frac{\partial^p l_j(\xi)}{\partial \xi^p} \right) \int_{-1}^{1} \frac{\partial^p}{\partial \xi^p} \left(\hat{u}^D \frac{\partial h_L(\eta)}{\partial \eta} \right) d\eta \quad (3.16)$$

$$= 2\left(\sum_{j=0}^{p} \Delta_{B_j} \frac{\partial^p l_j(\xi)}{\partial \xi^p} \right) \frac{\partial^p}{\partial \xi^p} \underbrace{\left(\int_{-1}^{1} \hat{u}^D \frac{\partial h_L(\eta)}{\partial \eta} d\eta \right)}_{\text{I.B.P+VCJH property}} \quad (3.17)$$

$$= 2\left(\sum_{j=0}^{p} \Delta_{B_j} \frac{\partial^p l_j(\xi)}{\partial \xi^p} \right) \left(- \left. \frac{\partial^p \hat{u}^D}{\partial \xi^p} \right|_{\eta=-1} - c \frac{d^{p+1} h_L(\eta)}{d\eta^{p+1}} \frac{\partial^{2p} \hat{u}^D}{\partial \xi^p \partial \eta^p} \right) \quad (3.18)$$

Using the same approach on the last term of Eq. (3.15), we can rewrite (3.15) as

$$\begin{aligned} -\int_{\Omega_S} \frac{\partial^p \hat{u}^D}{\partial \xi^p} \frac{\partial^p}{\partial \xi^p} \left(\hat{\nabla} \cdot \hat{\boldsymbol{f}}^C \right) d\Omega_S &= +2\int_{-1}^{1} \frac{\partial^p \hat{u}^D}{\partial \xi^p} \frac{d^{p+1} h_L(\xi)}{d\xi^{p+1}} \left(\sum_{j=0}^{p} \Delta_{L_j} l_j(\eta) \right) d\eta \\ &- 2\int_{-1}^{1} \frac{\partial^p \hat{u}^D}{\partial \xi^p} \frac{d^{p+1} h_R(\xi)}{d\xi^{p+1}} \left(\sum_{j=0}^{p} \Delta_{R_j} l_j(\eta) \right) d\eta \\ &+ 2\left(\sum_{j=0}^{p} \Delta_{B_j} \frac{\partial^p l_j(\xi)}{\partial \xi^p} \right) \left(- \left. \frac{\partial^p \hat{u}^D}{\partial \xi^p} \right|_{\eta=-1} - c \frac{d^{p+1} h_L(\eta)}{d\eta^{p+1}} \frac{\partial^{2p} \hat{u}^D}{\partial \xi^p \partial \eta^p} \right) \\ &- 2\left(\sum_{j=0}^{p} \Delta_{T_j} \frac{\partial^p l_j(\xi)}{\partial \xi^p} \right) \left(\left. \frac{\partial^p \hat{u}^D}{\partial \xi^p} \right|_{\eta=1} - c \frac{d^{p+1} h_R(\eta)}{d\eta^{p+1}} \frac{\partial^{2p} \hat{u}^D}{\partial \xi^p \partial \eta^p} \right) \end{aligned} \quad (3.19)$$

Substituting the above results back into (3.13), we get the desired result stated in Lemma 3.2.
□

Lemma 3.3

$$\frac{1}{2}\left(\frac{1}{2}\right)\frac{\partial}{\partial t}\int_{\Omega_k} J_k\left(\frac{\partial^p u_k^D}{\partial \eta^p}\right)^2 d\Omega_k = -\int_{-1}^{1}\frac{\partial^p \hat{u}^D}{\partial \eta^p}\frac{\partial^p}{\partial \eta^p}\left(\frac{\partial \hat{F}^D}{\partial \xi}\right)d\xi$$
$$+\underbrace{\int_{-1}^{1}\frac{\partial^p \hat{u}^D}{\partial \eta^p}\frac{d^{p+1}h_L(\eta)}{d\eta^{p+1}}\left(\sum_{j=0}^{p}\Delta_{B_j}l_j(\xi)\right)d\xi}_{A_3}$$
$$-\underbrace{\int_{-1}^{1}\frac{\partial^p \hat{u}^D}{\partial \eta^p}\frac{d^{p+1}h_R(\eta)}{d\eta^{p+1}}\left(\sum_{j=0}^{p}\Delta_{T_j}l_j(\xi)\right)d\xi}_{A_4}$$
$$+\left(\sum_{j=0}^{p}\Delta_{L_j}\frac{\partial^p l_j(\eta)}{\partial \eta^p}\right)\left(-\frac{\partial^p \hat{u}^D}{\partial \eta^p}\bigg|_{\xi=-1}-\underbrace{c\frac{d^{p+1}h_L(\xi)}{d\xi^{p+1}}\frac{\partial^{2p}\hat{u}^D}{\partial \xi^p\partial \eta^p}}_{B_1}\right)$$
$$-\left(\sum_{j=0}^{p}\Delta_{R_j}\frac{\partial^p l_j(\eta)}{\partial \eta^p}\right)\left(\frac{\partial^p \hat{u}^D}{\partial \eta^p}\bigg|_{\xi=1}-\underbrace{c\frac{d^{p+1}h_R(\xi)}{d\xi^{p+1}}\frac{\partial^{2p}\hat{u}^D}{\partial \xi^p\partial \eta^p}}_{B_2}\right) \tag{3.20}$$

Proof We can obtain this by applying the operator $\frac{\partial^p}{\partial \eta^p}$ to (3.2) and arguing as in the proof of Lemma 3.2 □

Lemma 3.4

$$\frac{1}{2}\left(\frac{1}{4}\right)\frac{\partial}{\partial t}\int_{\Omega_k} J_k\left(\frac{\partial^{2p} u_k^D}{\partial \xi^p\partial \eta^p}\right)^2 d\Omega_k = +\underbrace{\frac{\partial^{2p}\hat{u}^D}{\partial \xi^p\partial \eta^p}\frac{d^{p+1}h_L(\xi)}{d\xi^{p+1}}\left(\sum_{j=0}^{p}\Delta_{L_j}\frac{\partial^p l_j(\eta)}{\partial \eta^p}\right)}_{B_1}$$
$$-\underbrace{\frac{\partial^{2p}\hat{u}^D}{\partial \xi^p\partial \eta^p}\frac{d^{p+1}h_R(\xi)}{d\xi^{p+1}}\left(\sum_{j=0}^{p}\Delta_{R_j}\frac{\partial^p l_j(\eta)}{\partial \eta^p}\right)}_{B_2}$$
$$+\underbrace{\frac{\partial^{2p}\hat{u}^D}{\partial \xi^p\partial \eta^p}\frac{d^{p+1}h_L(\eta)}{d\eta^{p+1}}\left(\sum_{j=0}^{p}\Delta_{B_j}\frac{\partial^p l_j(\xi)}{\partial \xi^p}\right)}_{B_3}$$
$$-\underbrace{\frac{\partial^{2p}\hat{u}^D}{\partial \xi^p\partial \eta^p}\frac{d^{p+1}h_R(\eta)}{d\eta^{p+1}}\left(\sum_{j=0}^{p}\Delta_{T_j}\frac{\partial^p l_j(\xi)}{\partial \xi^p}\right)}_{B_4} \tag{3.21}$$

Proof Multiply (3.2) by J_k and apply $\frac{\partial^{2p}}{\partial\xi^p\partial\eta^p}$ to the equation to get

$$J_k \frac{\partial}{\partial t}\left(\frac{\partial^{2p} u_k^D}{\partial\xi^p\partial\eta^p}\right) = -\cancelto{0}{\frac{\partial^{2p}}{\partial\xi^p\partial\eta^p}(J_k\nabla\cdot \boldsymbol{f}_k^D)} - \frac{\partial^{2p}}{\partial\xi^p\partial\eta^p}(J_k\nabla\cdot \boldsymbol{f}_k^C) = -\frac{\partial^{2p}}{\partial\xi^p\partial\eta^p}(J_k\nabla\cdot \boldsymbol{f}_k^C) \tag{3.22}$$

Multiply the above equation by $\frac{\partial^{2p} u_k^D}{\partial\xi^p\partial\eta^p}$ and integrate over Ω_k to get

$$\frac{1}{2}\frac{\partial}{\partial t}\int_{\Omega_k} J_k\left(\frac{\partial^{2p} u_k^D}{\partial\xi^p\partial\eta^p}\right)^2 d\Omega_k = -\int_{\Omega_k}\left(\frac{\partial^{2p} u_k^D}{\partial\xi^p\partial\eta^p}\right)\frac{\partial^{2p}}{\partial\xi^p\partial\eta^p}(J_k\nabla\cdot \boldsymbol{f}_k^C)d\Omega_k \tag{3.23}$$

Transforming the RHS to the reference domain, we get

$$\frac{J_k}{2}\frac{\partial}{\partial t}\int_{\Omega_k}\left(\frac{\partial^{2p} u_k^D}{\partial\xi^p\partial\eta^p}\right)^2 d\Omega_k = -\int_{\Omega_S}\left(\frac{\partial^{2p} \hat{u}_k^D}{\partial\xi^p\partial\eta^p}\right)\frac{\partial^{2p}}{\partial\xi^p\partial\eta^p}(\hat{\nabla}\cdot \hat{\boldsymbol{f}}^C)d\Omega_S \tag{3.24}$$

Upon substituting the expression for $\hat{\boldsymbol{f}}^C$ and noting the integrands are essentially constants, we get (3.21). We can now move on to state the main result of this paper. □

Theorem 3.5 *If the FR scheme for a 2D conservation law with periodic boundary conditions is used in conjunction with the Lax–Friedrichs formulation for the common interface flux*

$$\boldsymbol{f}^* = \left\{\{\boldsymbol{f}^D\}\right\} + \frac{\lambda}{2}\left(\max_{u\in[u_-^D,u_+^D]}\left|\frac{\partial \boldsymbol{f}}{\partial u}\cdot n\right|\right)[[u^D]] \tag{3.25}$$

with $0 \le \lambda \le 1$, *and if a non-negative value of the VCJH parameter c is used, then it can be shown that for a linear advective flux and a uniform Cartesian mesh, the following holds*

$$\frac{d}{dt}\|u^D\|^2_{W_\delta^{2p,2}} \le 0 \tag{3.26}$$

for a modified Sobolev norm defined as follows

$$\|u^D\|^2_{W_\delta^{2p,2}} = \sum_{k=1}^{N}\left(\int_{\Omega_k} J_k\left[(u_k^D)^2 + \frac{c}{2}\left(\left(\frac{\partial^p u_k^D}{\partial\xi^p}\right)^2 + \left(\frac{\partial^p u_k^D}{\partial\eta^p}\right)^2\right) + \frac{c^2}{4}\left(\frac{\partial^{2p} u_k^D}{\partial\xi^p\partial\eta^p}\right)^2\right]d\Omega_k\right) \tag{3.27}$$

For non-uniform Cartesian meshes, if a FR approach recovering DG ($c = 0$) *is used, then the scheme is guaranteed to be stable. If* $c > 0$*, it is possible for the scheme to be unstable under certain conditions stated towards the end of the proof.*
Note: For brevity of proof, we discuss the properties of the above norm in "Appendix".

Proof Multiply (3.9) and (3.20), i.e., Lemmas 3.2 and 3.3 by c, (3.21), i.e., Lemma 3.4 by c^2 and add them to (3.1). Note that terms that are marked ($A_1, A_2, A_3, A_4, B_1, B_2, B_3, B_4$) cancel out and we get

$$\frac{1}{2}\frac{d}{dt}\left(\int_{\Omega_k} J_k(u_k^D)^2 d\Omega_k + \frac{c}{2}\int_{\Omega_k} J_k\left(\left(\frac{\partial^p u_k^D}{\partial\xi^p}\right)^2 + \left(\frac{\partial^p u_k^D}{\partial\eta^p}\right)^2\right)d\Omega_k + \frac{c^2}{4}\int_{\Omega_k} J_k\left(\frac{\partial^{2p} u_k^D}{\partial\xi^p\partial\eta^p}\right)^2 d\Omega_k\right)$$

$$= -\int_{\Omega_S}\hat{u}^D\left(\hat{\nabla}\cdot\hat{\boldsymbol{f}}^D\right)d\Omega_S - \int_{\Gamma_S}\hat{u}^D(\hat{\boldsymbol{f}}^C\cdot\hat{\boldsymbol{n}})d\Gamma_S - c\int_{-1}^{1}\frac{\partial^p\hat{u}^D}{\partial\eta^p}\frac{\partial^p}{\partial\eta^p}\left(\frac{\partial\hat{F}^D}{\partial\xi}\right)d\xi$$

$$-c\int_{-1}^{1}\frac{\partial^p\hat{u}^D}{\partial\xi^p}\frac{\partial^p}{\partial\xi^p}\left(\frac{\partial\hat{G}^D}{\partial\eta}\right)d\eta$$
$$-c\frac{\partial^p\hat{u}^D}{\partial\xi^p}\bigg|_{\eta=-1}\left(\sum_{j=0}^{p}\Delta_{B_j}\frac{\partial^p l_j(\xi)}{\partial\xi^p}\right)-c\frac{\partial^p\hat{u}^D}{\partial\xi^p}\bigg|_{\eta=1}\left(\sum_{j=0}^{p}\Delta_{T_j}\frac{\partial^p l_j(\xi)}{\partial\xi^p}\right)$$
$$-c\frac{\partial^p\hat{u}^D}{\partial\eta^p}\bigg|_{\xi=-1}\left(\sum_{j=0}^{p}\Delta_{L_j}\frac{\partial^p l_j(\eta)}{\partial\eta^p}\right)-c\frac{\partial^p\hat{u}^D}{\partial\eta^p}\bigg|_{\xi=1}\left(\sum_{j=0}^{p}\Delta_{R_j}\frac{\partial^p l_j(\eta)}{\partial\eta^p}\right)\tag{3.28}$$

Now let us consider the 3rd term of the RHS of (3.28) which can be written as follows for a linear flux (note that we leave out the factor c in order to just focus on the algebraic manipulations)

$$\begin{aligned}-\int_{-1}^{1}\frac{\partial^p\hat{u}^D}{\partial\eta^p}\frac{\partial^p}{\partial\eta^p}\left(\frac{\partial\hat{F}^D}{\partial\xi}\right)d\xi &= -\int_{-1}^{1}\frac{\partial^p\hat{u}^D}{\partial\eta^p}\frac{\partial}{\partial\xi}\left(\frac{\partial^p\hat{F}^D}{\partial\eta^p}\right)d\xi = -\hat{a}\int_{-1}^{1}\frac{\partial^p\hat{u}^D}{\partial\eta^p}\frac{\partial}{\partial\xi}\left(\frac{\partial^p\hat{u}^D}{\partial\eta^p}\right)d\xi\\
&= -\frac{\hat{a}}{2}\int_{-1}^{1}\frac{\partial}{\partial\xi}\left(\frac{\partial^p\hat{u}^D}{\partial\eta^p}\frac{\partial^p\hat{u}^D}{\partial\eta^p}\right)d\xi = -\frac{1}{2}\int_{-1}^{1}\frac{\partial}{\partial\xi}\left(\frac{\partial^p\hat{u}^D}{\partial\eta^p}\frac{\partial^p\hat{F}^D}{\partial\eta^p}\right)d\xi\\
&= -\frac{1}{2}\left(\frac{\partial^p\hat{u}^D}{\partial\eta^p}\frac{\partial^p\hat{F}^D}{\partial\eta^p}\right)\bigg|_{\xi=-1}^{\xi=1}\\
&= -\frac{1}{2}\left(\frac{\partial^p\hat{u}^D}{\partial\eta^p}\frac{\partial^p\hat{F}^D}{\partial\eta^p}\right)\bigg|_{\xi=1}+\frac{1}{2}\left(\frac{\partial^p\hat{u}^D}{\partial\eta^p}\frac{\partial^p\hat{F}^D}{\partial\eta^p}\right)\bigg|_{\xi=-1}\\
&= -\frac{1}{2}\left(\frac{\partial^p\hat{u}^D}{\partial\eta^p}\frac{\partial^p(\hat{\boldsymbol{f}}^D\cdot\hat{\boldsymbol{n}})}{\partial\eta^p}\right)_R-\frac{1}{2}\left(\frac{\partial^p\hat{u}^D}{\partial\eta^p}\frac{\partial^p(\hat{\boldsymbol{f}}^D\cdot\hat{\boldsymbol{n}})}{\partial\eta^p}\right)_L\end{aligned}\tag{3.29}$$

Remark 3.2 In the above argument we have used the fact that we have a linear advective flux $\hat{\boldsymbol{f}}_k^D=\mathbf{J_k^{-1}a}\hat{u}^D$. If $\mathbf{J_k^{-1}a}$ is written as $\hat{\mathbf{a}}$, and $\hat{\mathbf{a}}=[\hat{a}\quad\hat{b}]^T$, then we have $\hat{F}^D=\hat{a}\hat{u}^D$. In the final step, we use the fact that on the left boundary, $\hat{\boldsymbol{n}}_L=[-1\quad 0]^T$, which implies $\hat{F}_L^D=-(\hat{\boldsymbol{f}}^D\cdot\hat{\boldsymbol{n}})_L$.

Similarly the 4th term of the RHS of (3.28) can be written as

$$\begin{aligned}-\int_{-1}^{1}\frac{\partial^p\hat{u}^D}{\partial\xi^p}\frac{\partial^p}{\partial\xi^p}\left(\frac{\partial\hat{G}^D}{\partial\eta}\right)d\eta &= -\frac{1}{2}\int_{-1}^{1}\frac{\partial}{\partial\eta}\left(\frac{\partial^p\hat{u}^D}{\partial\xi^p}\frac{\partial^p\hat{G}^D}{\partial\xi^p}\right)d\eta\\
&= -\frac{1}{2}\left(\frac{\partial^p\hat{u}^D}{\partial\xi^p}\frac{\partial^p\hat{G}^D}{\partial\xi^p}\right)\bigg|_{\eta=-1}^{\eta=1}\\
&= -\frac{1}{2}\left(\frac{\partial^p\hat{u}^D}{\partial\xi^p}\frac{\partial^p(\hat{\boldsymbol{f}}^D\cdot\hat{\boldsymbol{n}})}{\partial\xi^p}\right)_T-\frac{1}{2}\left(\frac{\partial^p\hat{u}^D}{\partial\xi^p}\frac{\partial^p(\hat{\boldsymbol{f}}^D\cdot\hat{\boldsymbol{n}})}{\partial\xi^p}\right)_B\end{aligned}\tag{3.30}$$

Now note that $\Delta_L=((\hat{\boldsymbol{f}}^*-\hat{\boldsymbol{f}}^D)\cdot\hat{\boldsymbol{n}})_L$. Therefore we can rewrite the 7th term of (3.28) as follows

$$\frac{\partial^p\hat{u}^D}{\partial\eta^p}\bigg|_{\xi=-1}\left(\sum_{j=0}^{p}\Delta_{L_j}\frac{\partial^p l_j(\eta)}{\partial\eta^p}\right)=\left(\frac{\partial^p\hat{u}^D}{\partial\eta^p}\frac{\partial^p}{\partial\eta^p}\left((\hat{\boldsymbol{f}}^*-\hat{\boldsymbol{f}}^D)\cdot\hat{\boldsymbol{n}}\right)\right)_L\tag{3.31}$$

We can write similar expressions for the other terms on the RHS of (3.28). Therefore when we substitute (3.29), (3.30) and these above results into (3.28) and sum over all the elements, we get

$$\begin{aligned}\frac{d}{dt}\|u^D\|^2 &= \Theta_{adv}\\ &\quad + c\sum_{k=1}^{N}\left(\left[\frac{1}{2}\frac{\partial^p \hat{u}_R^D}{\partial\eta^p}\frac{\partial^p\left(\hat{\boldsymbol{f}}^D\cdot\hat{\boldsymbol{n}}\right)_R}{\partial\eta^p} - \frac{\partial^p \hat{u}_R^D}{\partial\eta^p}\frac{\partial^p\left(\hat{\boldsymbol{f}}\cdot\hat{\boldsymbol{n}}\right)_R^*}{\partial\eta^p}\right]_k\right.\\ &\quad + \left[\frac{1}{2}\frac{\partial^p \hat{u}_L^D}{\partial\eta^p}\frac{\partial^p\left(\hat{\boldsymbol{f}}^D\cdot\hat{\boldsymbol{n}}\right)_L}{\partial\eta^p} - \frac{\partial^p \hat{u}_L^D}{\partial\eta^p}\frac{\partial^p\left(\hat{\boldsymbol{f}}\cdot\hat{\boldsymbol{n}}\right)_L^*}{\partial\eta^p}\right]_k\\ &\quad + \left[\frac{1}{2}\frac{\partial^p \hat{u}_T^D}{\partial\xi^p}\frac{\partial^p\left(\hat{\boldsymbol{f}}^D\cdot\hat{\boldsymbol{n}}\right)_T}{\partial\xi^p} - \frac{\partial^p \hat{u}_T^D}{\partial\xi^p}\frac{\partial^p\left(\hat{\boldsymbol{f}}\cdot\hat{\boldsymbol{n}}\right)_T^*}{\partial\xi^p}\right]_k\\ &\quad \left. + \left[\frac{1}{2}\frac{\partial^p \hat{u}_B^D}{\partial\xi^p}\frac{\partial^p(\hat{\boldsymbol{f}}^D\cdot\hat{\boldsymbol{n}})_B}{\partial\xi^p} - \frac{\partial^p \hat{u}_B^D}{\partial\xi^p}\frac{\partial^p(\hat{\boldsymbol{f}}\cdot\hat{\boldsymbol{n}})_B^*}{\partial\xi^p}\right]_k\right)\end{aligned} \tag{3.32}$$

where Θ_{adv} represents the summation of the first two terms in (3.28) over all the elements. This summation over elements of the two terms can be rewritten as a summation over all element boundaries as shown by Castonguay et al. [12]. These terms arise while applying the FR procedure to the linear advection equation on triangles, and they have shown this term to be non-positive, i.e., $\Theta_{adv} \le 0$. Although they proved this on triangles, since the summation over elements can be converted to one over element boundaries, the argument remains exactly the same for the case of quadrilaterals and we therefore omit this proof.

Let us call the rest of the terms in the RHS of (excluding the multiplicative factor, c) (3.32) as Θ_{extra} in order to rewrite it as

$$\frac{d}{dt}\|u^D\|^2 = \Theta_{adv} + c\Theta_{extra} \tag{3.33}$$

Now it remains to show that $\Theta_{extra} \le 0$.

Remark 3.3 If $c = 0$ as in the case of the DG-recovering FR approach, then the contribution of the Θ_{extra} term is zero, therefore guaranteeing stability.

Remark 3.4 Note that

$$\frac{\partial^p \hat{u}_L^D}{\partial\eta^p} = \frac{\partial^p \hat{u}_{\xi=-1}^D}{\partial\eta^p}$$

is the pth degree edge derivative of the 1D polynomial formed on the flux points using the extrapolated values of $\hat{u}^D$ on the left boundary of the reference domain. Since $\hat{u}_{\xi=-1}^D$ is a pth degree polynomial in η, its pth derivative with respect to η is a constant. Similar arguments can be made about the flux derivatives in the term Θ_{extra} as well.

In order to sum these quantities across elements, we first transform these to the physical domain using the notation and transformation equations discussed in Sect. 2.1. Transforming the pth derivative in general quadrilateral meshes can be tedious, but since we have rectangular Cartesian meshes, we can easily write Θ_{extra} in terms of quantities in the physical domain as

$$\begin{aligned}\Theta_{extra} = \sum_{k=1}^{N} \Bigg(& J_{y_k}^{2p+1} J_k \left[\frac{1}{2} \frac{\partial^p u_R^D}{\partial y^p} \frac{\partial^p F_R^D}{\partial y^p} - \frac{\partial^p u_R^D}{\partial y^p} \frac{\partial^p (\boldsymbol{f} \cdot \boldsymbol{n})_R^*}{\partial y^p} \right]_k \\ & + J_{y_k}^{2p+1} J_k \left[-\frac{1}{2} \frac{\partial^p u_L^D}{\partial y^p} \frac{\partial^p F_L^D}{\partial y^p} - \frac{\partial^p u_L^D}{\partial y^p} \frac{\partial^p (\boldsymbol{f} \cdot \boldsymbol{n})_L^*}{\partial y^p} \right] \\ & + J_{x_k}^{2p+1} J_k \left[\frac{1}{2} \frac{\partial^p u_T^D}{\partial x^p} \frac{\partial^p G_T^D}{\partial x^p} - \frac{\partial^p u_T^D}{\partial x^p} \frac{\partial^p (\boldsymbol{f} \cdot \boldsymbol{n})_T^*}{\partial x^p} \right] \\ & + J_{x_k}^{2p+1} J_k \left[-\frac{1}{2} \frac{\partial^p u_B^D}{\partial x^p} \frac{\partial^p G_B^D}{\partial x^p} - \frac{\partial^p u_B^D}{\partial x^p} \frac{\partial^p (\boldsymbol{f} \cdot \boldsymbol{n})_B^*}{\partial x^p} \right] \Bigg) \end{aligned} \tag{3.34}$$

This expression can be rewritten as a sum over all the edges instead of all the elements. Consider one such summation along an interior vertical edge. Let $-$ and $+$ subscripts denote the element on the left and right. For the element on the left, this edge is its right boundary and for the right element, it is the left boundary. Also, note that for a Cartesian mesh with no mortar elements, the J_y for these left and right elements are the same, since it is the edge length of their common boundary. Adding the 2 terms coming from each element from this edge, we get

$$J_y^{2p+1} \left[\frac{J_-}{2} \frac{\partial^p u_-^D}{\partial y^p} \frac{\partial^p F_-^D}{\partial y^p} - J_- \frac{\partial^p u_-^D}{\partial y^p} \frac{\partial^p (\boldsymbol{f} \cdot \boldsymbol{n})_-^*}{\partial y^p} - \frac{J_+}{2} \frac{\partial^p u_+^D}{\partial y^p} \frac{\partial^p F_+^D}{\partial y^p} - J_+ \frac{\partial^p u_+^D}{\partial y^p} \frac{\partial^p (\boldsymbol{f} \cdot \boldsymbol{n})_+^*}{\partial y^p} \right] \tag{3.35}$$

where J_- is the determinant of the Jacobian of the left element and J_+ the determinant of the Jacobian of the right element. In the Cartesian case, these are just the areas of the left and right elements respectively. Now, we use the fact that f is a linear advective flux, i.e., $F^D = au^D$ and $G^D = bu^D$. Also, from the definition of the Lax–Friedrichs flux, we have,

$$(\boldsymbol{f} \cdot \boldsymbol{n})_-^* = \frac{1}{2} a(u_-^D + u_+^D) + \frac{\lambda}{2} |a| (u_-^D - u_+^D) \tag{3.36}$$

$$(\boldsymbol{f} \cdot \boldsymbol{n})_+^* = -\frac{1}{2} a(u_-^D + u_+^D) + \frac{\lambda}{2} |a| (u_+^D - u_-^D) \tag{3.37}$$

Substituting these results in (3.35)), we get

$$J_y^{2p+1} \left[-\frac{\lambda J_-}{2} |a| \left(\frac{\partial^p u_-^D}{\partial y^p} \frac{\partial^p u_-^D}{\partial y^p} - \frac{\partial^p u_-^D}{\partial y^p} \frac{\partial^p u_+^D}{\partial y^p} \right) - \frac{\lambda J_+}{2} |a| \left(\frac{\partial^p u_+^D}{\partial y^p} \frac{\partial^p u_+^D}{\partial y^p} - \frac{\partial^p u_-^D}{\partial y^p} \frac{\partial^p u_+^D}{\partial y^p} \right) \right] \tag{3.38}$$

which can be further simplified as

$$J_y^{2p+1} \left[-\frac{\lambda}{2} |a| \left(\frac{\partial^p u_-^D}{\partial y^p} - \frac{\partial^p u_+^D}{\partial y^p} \right) \left(J_- \frac{\partial^p u_-^D}{\partial y^p} - J_+ \frac{\partial^p u_+^D}{\partial y^p} \right) \right] \tag{3.39}$$

We can see that, if $J_- = J_+$, this reduces to

$$J_y^{2p+1} \left[-\frac{\lambda J_-}{2} |a| \left(\frac{\partial^p u_-^D}{\partial y^p} - \frac{\partial^p u_+^D}{\partial y^p} \right)^2 \right] \tag{3.40}$$

Therefore, if we have a uniform Cartesian mesh in a periodic domain, it is clear that this sum will be negative, i.e., $\Theta_{extra} \leq 0$ and therefore Theorem 3.5 holds. Also, if $c = 0$, as stated earlier the Θ_{extra} terms will not contribute, thus giving a stable scheme, irrespective of whether the Cartesian mesh is uniform. However, if $c \neq 0$ and J_+ and J_- are largely different, we can see that the term in (3.39) can become a large positive number, possibly

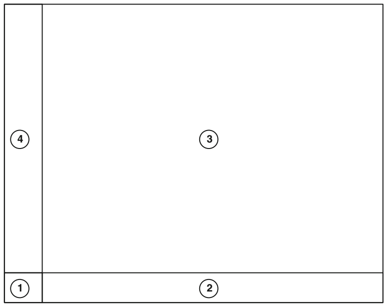

Fig. 3 An example of a non-uniform Cartesian mesh with 4 elements that could cause an instability

causing instability. Therefore, it is important to control the growth rate of the elements in the mesh.

Note that the y-derivative of the horizontal flux and the x-derivative of the vertical flux are the terms that show up in the above Θ_{extra} terms. Also, note that a is the horizontal component of the wavespeed. Therefore, for a vertical edge to contribute towards instability, we need:

1. Non-zero horizontal advection (or wavespeed).
2. Variation of the solution along the vertical edge - in fact the pth derivative of the solution must be non-vanishing along the vertical edge.
3. The elements on the left and right of the vertical edge must be of (largely) different sizes, therefore having different values of the determinant of the Jacobian J.

Similarly, we can list the requirements for a horizontal edge to contribute towards instability. Finally, although it is possible to have positive contributions towards instability from some of the edges, there will also be negative contributions from both Θ_{adv} and Θ_{extra} from other edges. Therefore, while not all cases cause instability, large growth rates of elements should be avoided in order to preempt this issue. Apart from controlling the growth rate of elements, since the possibly unstable terms are scaled by c, smaller values of c are more favorable in the context of stability. A simple example of a Cartesian mesh where such instability could occur is shown in Fig. 3. In this example, there is a high growth rate of elements in both the horizontal and vertical directions and therefore satisfies the third condition for all the element boundaries.

4 Conclusions

An investigation of the linear stability of the FR approach on quadrilaterals has been performed for the first time. It has been shown that it is possible for the FR approach to develop instabilities when applied to the linear advection equation on quadrilateral elements by using the case of Cartesian meshes. In the case of triangles, a straightforward extension of the one dimensional FR approach was not possible, and hence a family of energy stable FR schemes (ESFR) were built particularly for that case by Castonguay et al. [12]. In contrast, for quadrilaterals, the tensor product nature of the geometry allows for a simple extension of the 1D FR process as discussed in this paper. However, we have shown that this extension can lead to linear instabilities under certain conditions.

Although only Cartesian meshes have been considered, the analysis yields significant insights into the possible origins of instability. In particular, it has been found that a high growth rate of elements can lead to an unstable scheme under certain conditions on the solution and fluxes. This is important in the context of boundary layer meshes where such high growth rates of elements are not uncommon. It is also found that the terms that could lead to instabilities are scaled by the VCJH parameter c, and therefore larger values of c must be used carefully. In particular, the FR approach that recovers the DG method, i.e., the case of $c = 0$ is stable for use with any Cartesian mesh.

Apart from providing insights into causes for instability, the approach used to investigate stability might be particularly helpful for similar studies in three dimensions or for non-linear fluxes. Further studies could also explore other two dimensional approaches that give rise to stable schemes on quadrilateral elements.

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5 Appendix: Modified Sobolev Norm

Let us reconsider the modified partial Sobolev norm used in Theorem 3.5.

$$\|u^D\|^2_{W^{2p,2}_\delta} = \sum_{k=1}^{N} \left(\int_{\Omega_k} J_k \left[(u_k^D)^2 + \frac{c}{2} \left(\left(\frac{\partial^p u_k^D}{\partial \xi^p} \right)^2 + \left(\frac{\partial^p u_k^D}{\partial \eta^p} \right)^2 \right) + \frac{c^2}{4} \left(\frac{\partial^{2p} u_k^D}{\partial \xi^p \partial \eta^p} \right)^2 \right] d\Omega_k \right) \tag{5.1}$$

Notice that the norm has u^D in the physical domain while the derivatives are with respect to the reference coordinates. We can use the additional notation we introduced for the Cartesian mesh geometry to rewrite this completely in the physical domain as follows

$$\begin{aligned} \|u^D\|^2_{W^{2p,2}_\delta} = \sum_{k=1}^{N} \Bigg(\int_{\Omega_k} J_k \Bigg[(u_k^D)^2 + \frac{c}{2} \left(J_x^{2p} \left(\frac{\partial^p u_k^D}{\partial x^p} \right)^2 + J_y^{2p} \left(\frac{\partial^p u_k^D}{\partial y^p} \right)^2 \right) \\ + \frac{c^2}{4} J_x^{2p} J_y^{2p} \left(\frac{\partial^{2p} u_k^D}{\partial x^p \partial y^p} \right)^2 \Bigg] d\Omega_k \Bigg) \end{aligned} \tag{5.2}$$

From our analysis of stability, we can see that we are mainly interested in $c \geq 0$, since for $c < 0$, the Θ_{extra} term contributes towards instability in the simplest case of a uniform Cartesian mesh. However, as an exercise, it is interesting to investigate the range of c for which the above is a norm.

In (5.2) we write the norm completely in the physical domain. However for algebraic manipulations, it is better to write the norm completely in the reference domain. Since the norm in the domain Ω is a sum of the norms inside each element, it is sufficient to consider the norm in a single element (kth element) as follows

$$\|u_k^D\|^2_{W^{2p,2}_\delta} = \int_{-1}^{1}\int_{-1}^{1} \left[(\hat{u}_k^D)^2 + \frac{c}{2} \left(\left(\frac{\partial^p \hat{u}_k^D}{\partial \xi^p} \right)^2 + \left(\frac{\partial^p \hat{u}_k^D}{\partial \eta^p} \right)^2 \right) + \frac{c^2}{4} \left(\frac{\partial^{2p} \hat{u}_k^D}{\partial \xi^p \partial \eta^p} \right)^2 \right] d\xi d\eta \tag{5.3}$$

Till now the transformed solution $\hat{u}^D$ has been represented using the pth degree tensor-product Lagrange polynomial basis. However, we can equivalently expand our solution in a pth degree tensor product Legendre polynomial basis:

$$\hat{u}^D = \sum_{i=0}^{p} \sum_{j=0}^{p} L_i(\xi) L_j(\eta) \tilde{u}_{ij} \tag{5.4}$$

where $\tilde{u}_{ij}$ represent the modal coefficients. This is usually referred to as the modal form while the Lagrange expansion is referred to as the nodal form of the solution. We can change from one form to the other using the corresponding Vandermonde matrix [18]. An important difference between the Lagrange and Legendre polynomials is that the nth Legendre polynomial is of degree n unlike the Lagrange polynomials which are all of degree p. Now we substitute the above expression for $\hat{u}^D$ into the norm definition (5.3). The first term can be written as follows

$$\begin{aligned}
\int_{-1}^{1}\int_{-1}^{1} (\hat{u}^D)^2 d\xi d\eta &= \sum_{i=0}^{p}\sum_{m=0}^{p}\sum_{j=0}^{p}\sum_{n=0}^{p} \tilde{u}_{ij}\tilde{u}_{mn} \int_{-1}^{1}\int_{-1}^{1} L_i(\xi) L_m(\xi) L_j(\eta) L_n(\eta) d\xi d\eta \\
&= \sum_{i=0}^{p}\sum_{j=0}^{p} \tilde{u}_{ij}^2 \int_{-1}^{1}\int_{-1}^{1} L_i^2(\xi) L_j^2(\eta) d\xi d\eta \\
&\quad + \sum_{i=0}^{p}\sum_{\substack{m=0\\ m\neq i}}^{p}\sum_{j=0}^{p}\sum_{n=0}^{p} \tilde{u}_{ij}\tilde{u}_{mn} \int_{-1}^{1}\int_{-1}^{1} L_i(\xi) L_m(\xi) L_j(\eta) L_n(\eta) d\xi d\eta \\
&\quad + \sum_{i=0}^{p}\sum_{j=0}^{p}\sum_{\substack{n=0\\ n\neq j}}^{p} \tilde{u}_{ij}\tilde{u}_{in} \int_{-1}^{1}\int_{-1}^{1} L_i^2(\xi) L_j(\eta) L_n(\eta) d\xi d\eta
\end{aligned} \tag{5.5}$$

By using the orthogonality property of the Legendre polynomials, we get

$$\int_{-1}^{1}\int_{-1}^{1} (\hat{u}^D)^2 d\xi d\eta = \sum_{i=0}^{p}\sum_{j=0}^{p} \left(\frac{2}{2i+1}\right)\left(\frac{2}{2j+1}\right) \tilde{u}_{ij}^2 \tag{5.6}$$

Now the pth ξ-derivative can be written in terms of Legendre polynomials as follows

$$\frac{\partial^p \hat{u}^D}{\partial \xi^p} = \frac{d^p L_p(\xi)}{d\xi^p} \sum_{j=0}^{p} L_j(\eta) \tilde{u}_{pj} = a_p p! \sum_{j=0}^{p} L_j(\eta) \tilde{u}_{pj} \tag{5.7}$$

where one may recall that a_p is the leading coefficient of L_p. Note that we have used the fact that the pth derivative of $L_n(\xi)$ for $n < p$ is 0 in the above expression. Therefore we have

$$\int_{-1}^{1}\int_{-1}^{1} \left(\frac{\partial^p \hat{u}^D}{\partial \xi^p}\right)^2 d\xi d\eta = 2(a_p p!)^2 \sum_{j=0}^{p} \int_{-1}^{1} L_j^2(\eta) \tilde{u}_{pj}^2 d\eta = 2(a_p p!)^2 \sum_{j=0}^{p} \left(\frac{2}{2j+1}\right) \tilde{u}_{pj}^2 \tag{5.8}$$

Similarly we have the pth η-derivative

$$\int_{-1}^{1}\int_{-1}^{1} \left(\frac{\partial^p \hat{u}^D}{\partial \eta^p}\right)^2 d\xi d\eta = 2(a_p p!)^2 \sum_{i=0}^{p} \left(\frac{2}{2i+1}\right) \tilde{u}_{ip}^2 \tag{5.9}$$

Now we consider the last term of the norm

$$\frac{\partial^{2p} \hat{u}^D}{\partial \xi^p \partial \eta^p} = \frac{d^p L_p(\xi)}{d\xi^p} \frac{d^p L_p(\eta)}{d\eta^p} \tilde{u}_{pp} = (a_p p!)^2 \tilde{u}_{pp} \tag{5.10}$$

Therefore

$$\int_{-1}^{1}\int_{-1}^{1}\left(\frac{\partial^{2p}\hat{u}^D}{\partial\xi^p\partial\eta^p}\right)^2 d\xi d\eta = 4(a_p p!)^4\tilde{u}_{pp}^2 \tag{5.11}$$

From (5.6), (5.8), (5.9) and (5.11), we can see that the norm inside the kth element can be written as

$$\begin{aligned}\|u_k^D\|^2_{W_\delta^{2p,2}} &= \sum_{i=0}^{p-1}\sum_{j=0}^{p-1}\left(\frac{2}{2i+1}\right)\left(\frac{2}{2j+1}\right)\tilde{u}_{ij}^2 \\ &+ \sum_{j=0}^{p-1}\left[\left(\frac{2}{2p+1}\right)\left(\frac{2}{2j+1}\right)+\frac{c}{2}\,2(a_p p!)^2\left(\frac{2}{2j+1}\right)\right]\tilde{u}_{pj}^2 \\ &+ \sum_{i=0}^{p-1}\left[\left(\frac{2}{2p+1}\right)\left(\frac{2}{2i+1}\right)+\frac{c}{2}\,2(a_p p!)^2\left(\frac{2}{2i+1}\right)\right]\tilde{u}_{ip}^2 \\ &+ \left[\frac{4}{(2p+1)^2}+\frac{2(a_p p!)^2}{2p+1}c+(a_p p!)^4c^2\right]\tilde{u}_{pp}^2\end{aligned} \tag{5.12}$$

In order for this to be a norm, we need the co-efficients of each $\tilde{u}_{ij}$ have to be non negative. Therefore we have the following 2 conditions,

$$\frac{1}{2p+1}+\frac{c}{2}(a_p p!)^2 \geq 0 \Longrightarrow c \geq \frac{-2}{(2p+1)(a_p p!)^2} \tag{5.13}$$

$$\frac{4}{(2p+1)^2}+\frac{2(a_p p!)^2}{2p+1}c+(a_p p!)^4c^2 \geq 0 \tag{5.14}$$

The first condition is the same as the one obtained for the 1D case in [6]. The LHS of the second condition is a convex quadratic with a negative discriminant, therefore condition 2, i.e., (5.14) is always satisfied. Therefore, the condition on c for (5.3) to be a norm is the same as obtained in 1D.

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