Decentralized Control of a String of Vehicles

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1 Introduction

The benefits of drafting are well known in stock car racing and also bicycle racing and wind tunnel testing also confirms that indeed vehicles in a closely spaced string experience a drag reduction. This is true even for the lead vehicle. Moreover, the volume of traffic on a freeway will be increased if the vehicles are more closely spaced. This motivates the study of how to control the spacing of a string of vehicles. In a finite string the optimal control law would vary with the position of the vehicle in the string. The problem is simplified, however, if one considers the control of an infinite string of vehicles. The situation observed by every vehicle is then the same, with the consequence that the optimal control law must also be the same for every vehicle. The following analysis examines a simplified abstract model of an infinite string of vehicles in order to discover how rapidly the optimal feedback from the other vehicles decays with the separation from a given vehicle.

2 Plant model

To simplify the analysis it is assumed that the velocity of each vehicle can be directly controlled. Suppose that the entire string of vehicles is traveling at a speed V, and that x_{d_j} is the desired position of the j^{th} vehicles. Then we can represent the absolute position of the j^{th} vehicle as

$$x_{a_j} = Vt + x_{d_j} + d_j$$

where d_j is the displacement of the vehicle from its desired position. Then

$$\dot{x}_{a_j} = V + \dot{d}_j$$

and we shall suppose that if u_j is the control for the j^{th} vehicle, then

$$\dot{d}_j = u_j$$

or

$$\dot{x}_{a_j} = V + u_j$$

3 Cost function

In order to measure the performance of the control system we introduce a cost function

$$J = \int_{j=-\infty}^{\infty} \left(M + r \sum u_j^2 \right) dt$$

where

$$\begin{aligned} M &= \sum_{j=-\infty}^{\infty} \left\{ \left((x_{a_j} - x_{d_j}) - (x_{a_{j-1}} - x_{d_{j-1}}) \right)^2 + \alpha \left(x_{a_j} - Vt - x_{d_j} \right)^2 \right\} \\ &= \sum_{j=-\infty}^{\infty} \left\{ (d_j - d_{j-1})^2 + \alpha d_j^2 \right\} \\ &= \dots\ d_{-1}^2 - 2d_{-1}d_0 + d_0^2 + \alpha d_{-1}^2 \\ &\qquad + d_0^2 - 2d_0 d_1 + d_1^2 + \alpha d_0^2 \\ &\qquad\qquad + d_1^2 - 2d_1 d_2 + d_2^2 + \alpha d_2^2 \ \dots \\ &= d^T Q d \end{aligned}$$

where Q is a Toeplitz matrix with the form

$$Q = \begin{pmatrix} \cdot & \cdot & \cdot & & & & \\ & \cdot & \cdot & \cdot & & & \\ & & -1 & 2+\alpha & -1 & & \\ & & & \cdot & \cdot & \cdot & \\ & & & & \cdot & \cdot & \cdot \end{pmatrix}$$

The element in the diagonals are

$$\begin{aligned} q(0) &= 2 + \alpha \\ q(1) &= q(-1) = -1 \\ q(k) &= 0, |k| > 1 \end{aligned}$$

4 Formulation of optimal control

The control problem can now be written in vector matrix notation with infinite vectors and Toeplitz matrices as

$$\dot{d} = Ad + Bu$$

where

$$A = 0 \ , \ \ B = I$$

with the cost function

$$J = \int_0^\infty \left(d^T Q d + u^T R u \right) dt$$

where

$$R = rI$$

The optimal control is

$$u = -R^{-1} B^T P d$$

where P satisfies the limiting solution as $t \to \infty$ of the matrix Riccati equation

$$\dot{P} + A^T P + PA + Q - PBR^{-1} B^T P = 0$$

In this case the equation reduces simply to

$$u = -\frac{1}{r} P d$$

where P is a Toeplitz matrix satisfying

$$P^2 = Q$$

In terms of the diagonal elements $p(k)$ of P, and using indices in brackets instead of subscripts to denote the position in the string,

$$u(j) = -\frac{1}{r} \sum_{k=-\infty}^{\infty} p(k) d(j+k)$$

with a feedback $\frac{1}{r} p(k)$ from a vehicle separated by k spacings. Also

$$\sum_{k=-\infty}^{\infty} p(k) p(j-k) = q(j)$$

5 Solution by Fourier Transform

The equations can be conveniently solved by introducing the discrete Fourier transforms

$$\hat{q} = \sum_{k=-\infty}^{\infty} q(k)e^{iwk}, \quad \hat{p} = \sum_{k=-\infty}^{\infty} p(k)e^{iwk}$$

Then

$$\begin{aligned}\hat{q} &= \sum_{j=-\infty}^{\infty} e^{iwj} \sum_{k=-\infty}^{\infty} p(k)p(j-k) \\ &= \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} e^{iw(j-k)} e^{iwk} p(k)p(j-k) \\ &= \hat{p}^2\end{aligned}$$

Also

$$\begin{aligned}\hat{q} &= 2 + \alpha - e^{-i\omega} - e^{i\omega} \\ &= 2(1 - \cos\omega) + \alpha \\ &= 4\sin^2\frac{\omega}{2} + \alpha\end{aligned}$$

In the case that $\alpha = 0$, so that there is no penalty on the individual position of each vehicle,

$$\hat{p} = 2\left|\sin\frac{\omega}{2}\right|$$

In this case the elements $p(k)$ are simply the coefficients of the Fourier series representing $2\left|\sin\frac{\omega}{2}\right|$.

Since $\left|\sin\frac{\omega}{2}\right|$ is even, only cosine terms appear,

$$2\left|\sin\frac{\omega}{2}\right| = a(0) + \sum_{k=1}^{\infty} a(k)\cos\omega k$$

where

$$a(0) = \frac{1}{\pi}\int_0^{\pi} 2\sin\frac{\omega}{2}d\omega = -\frac{1}{\pi}\left[4\cos\frac{\omega}{2}\right]_0^{\pi} = \frac{4}{\pi}$$

and for $k > 0$

$$\begin{aligned}
a(k) &= \frac{2}{\pi}\int_0^\pi 2\sin\frac{\omega}{2}\cos k\omega d\omega \\
&= \frac{2}{\pi}\int_0^\pi [\sin(k+\frac{1}{2})\omega - \sin(k-\frac{1}{2})\omega] d\omega \\
&= -\frac{2}{\pi}\left[\frac{\cos(k+\frac{1}{2})\omega}{k-\frac{1}{2}} - \frac{\cos(k-\frac{1}{2})\omega}{k-\frac{1}{2}}\right]_0^\pi \\
&= \frac{2}{\pi}\left(\frac{1}{k+\frac{1}{2}} - \frac{1}{k-\frac{1}{2}}\right) \\
&= -\frac{2}{\pi}\frac{1}{k^2-\frac{1}{4}}
\end{aligned}$$

Here the coefficients decay algebraically with the separation distance. This is because $\hat{p}(w)$ is discontinuous at $\omega = 0$. When $\alpha > 0$, however, $\hat{p}(w)$ is continuous.

It is then possible to estimate the decay of the coefficients using contour integration because $\hat{p}(\omega)$ is analytic in a neighborhood of the origin. It turns out that the decay is exponential as shown below.

To evaluate

$$a_k = \frac{1}{\pi}\int_0^\pi \sqrt{\alpha + 2(1-\cos\omega)}\cos k\omega d\omega$$

note this is even and is the real part of

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ik\omega}\sqrt{\alpha + 2(1-\cos\omega)} d\omega$$

Extending the integrand to the complex plane as

$$e^{ik(\omega+i\tau)}\sqrt{\alpha + 2(1-\cos(\omega+i\tau))}$$

it is analytic for small enough τ.

Hence the integral around the contour in the diagram is zero, and since contributions on the vertical segments cancel, we can evaluate a_n as the real part of

$$c_k = e^{-k\tau}\int_{-\pi}^{\pi} e^{ik\omega}\sqrt{\alpha + 2(1-\cos(\omega+i\tau))} d\omega$$

A bound on τ can be estimated by finding the range of the imaginary axis for which

$$\alpha + 2(1-\cos i\tau) > 0$$

or

$$\cosh \tau < 1 + \frac{\alpha}{2}$$

Since when $\tau > 0$

$$\cosh \tau = \frac{1}{2}(e^{\tau} + e^{-\tau}) < e^{\tau}$$

This is satisfied if

$$e^{\tau} < 1 + \frac{\alpha}{2}$$

or

$$\tau < \log(1 + \frac{\alpha}{2})$$

Also

$$\begin{aligned} &\int_{-\pi}^{\pi} e^{ik\omega} \sqrt{\alpha + 2(1 - \cos(\omega + i\tau))} d\omega \\ =& \int_{-\pi}^{\pi} e^{ik\omega} \sqrt{\alpha + 2 - e^{i\omega - \tau} - e^{-i\omega + \tau}} d\omega \end{aligned}$$

and this has an absolute value bounded by

$$\begin{aligned} &\int_{-\pi}^{\pi} \sqrt{\alpha + 2 - e^{i\omega - \tau} - e^{-i\omega + \tau}} d\omega \\ =& 2\pi \sqrt{\alpha + 2 + 2\cosh \tau} \end{aligned}$$

Hence

$$|a_k| \leq e^{-k\tau} \sqrt{\alpha + 2 + 2\cosh \tau}$$

Substituting $e^{\tau} < 1 + \frac{\alpha}{2}$,

$$|a_k| < \frac{2\sqrt{1 + \frac{\alpha}{2}}}{(1 + \frac{\alpha}{2})^k}$$

6 Conclusion

The main conclusions of this analysis are first that the optimal control problems is greatly simplified when the string is extended to an infinite number of identical vehicles, because then the optimal feedbacks are the same for every vehicle, depending only on the number of vehicles separating any given pair of vehicles. Second, for the abstract model considered here, the optimal feedbacks decay rapidly as the number of vehicles separating the pair is increased. This suggests that good performance may be attainable with a decentralized control strategy in which each vehicle only senses the position and velocity of a small number of near neighbors.