Decentralized Control of a String of Vehicles

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1 The design of decentralized control strategies

Based on the conclusion of the first part of this study, the second part addresses the issue of how to design simple decentralized control strategies which will stabilize the system, and assure rapid decay of disturbances. The proposed approach is based on the observation that control strategies which use feedbacks from a finite number of near neighbors can be associated with corresponding continuous systems governed by partial differential equations in space and time. Thus the problem is reduced to finding solutions which lead to partial differential equations which exhibit decaying energy.

2 Formulation with velocity control

Suppose the actual position of the j^{th} vehicles

$$x_{a_j} = x_{d_j} + d_j$$

where

$$\dot{x}_{d_j} = v \ , \ x_{d_{j+1}} - x_{d_j} = \Delta x$$

Assume each vehicle can control its velocity, based on relative distance measurements of its neighbors. Then

$$\dot{x}_{a_j} = v_j + u_j$$

where

$$u_j = a_j^+(x_{a_{j+1}} - x_{a_j} - \Delta x) + a_j^-(x_{a_{j-1}} - x_{a_j} + \Delta x)$$

so that

$$\dot{d}_j = \dot{x}_{a_j} - \dot{x}_{d_j} = a_j^+(d_{j+1} - d_j) + a_j^-(d_{j-1} - d_j)$$

If each vehicle can measure its own absolute position one can have

$$\dot{d}_j = -a_0 d_j + a_j^+(d_{j+1} - d_j) + a_j^-(d_{j-1} - d_j)$$

More generally,

$$\dot{d}_j = -a_0 d_j + \sum a_k(d_{j+k} - d_j)$$

If d_j is a maximum and $a_k \geq 0$

$$\dot{d}_j \leq -a_0 d_j$$

Also set

$$\hat{d}(\omega) = \Delta x \sum_j d_j e^{-i\omega x_j}$$

Then with $\xi = \omega \Delta x$

$$\dot{\hat{d}} = -a_0 \hat{d} + \sum_k a_k(e^{ik\xi} - 1) = \alpha \hat{d}$$

where

$$\alpha = -a_0 - \sum a_k(1 - e^{ik\xi})$$

The solution is

$$\hat{d}(t) = e^{\alpha t} \hat{d}(0)$$

This decays if

$$Re(\alpha) = -a_0 - \sum a_k(1 - \cos k\xi) \leq 0$$

which is satisfied if

$$a_k \geq 0 \text{ for all k}$$

Also

$$d_j = \frac{1}{2\pi} \int_{-\frac{\pi}{\Delta x}}^{\frac{\pi}{\Delta x}} \hat{d}(\omega) e^{i\omega x_j} d\omega$$

3 Formulation with acceleration control

More realistically suppose that each vehicle can only control its acceleration. Accordingly suppose that

$$\dot{v}_j = \sum a_k(v_{j+k} - v_j) + \sum b_k(d_{j+k} - d_j)$$
$$\dot{d}_j = v_j$$

Then

$$\ddot{d}_j = \sum a_k(\dot{d}_{j+k} - \dot{d}_j) + \sum b_k(d_{j+k} - d_j)$$

and

$$\ddot{\hat{d}} - \alpha\dot{\hat{d}} - \beta\hat{d} = 0$$

where

$$\alpha = -\sum a_k(1 - e^{ik\xi}) \ , \ \beta = -\sum b_k(1 - e^{ik\xi})$$

The solution is

$$\hat{d} = a_1 e^{\lambda_1 t} + b_1 e^{\lambda_2 t}$$

where λ_1, λ_2 are the roots of

$$\lambda^2 - \alpha\lambda - \beta = 0$$

so

$$\lambda = \frac{\alpha}{2} \pm \sqrt{\frac{\alpha^2}{4} + \beta}$$

Then

$$\alpha = \lambda_1 + \lambda_2, \beta = -\lambda_1\lambda_2$$

and for stability

$$Re(\lambda_1) \leq 0, Re(\lambda_2) \leq 0$$

so

$$Re(\alpha) \leq 0$$

With no velocity feedback $\alpha = 0$, yielding

$$\lambda = \pm\sqrt{\beta}$$

Then since β has a non zero imaginary part there will be an unstable mode. With no position feedback $\beta = 0$, yielding

$$\lambda = \alpha, 0$$

Thus even if $Re(\alpha) < 0$ there is a neutral mode.

3.1 Forward 3-point scheme

Suppose that feedbacks are limited to at most two intervals. With only a forward look a possible solution is

$$\alpha = -2\frac{Ra}{\Delta x}(1 - e^{-i\xi}) \ , \ \beta = -\frac{a^2}{\Delta x^2}(1 - e^{-i\xi})^2$$

which can be realized as

$$\dot{v}_j = -2\frac{Ra}{\Delta x}(v_j - v_{j-1}) - \frac{a^2}{\Delta x^2}(d_j - d_{j-1}) + \frac{a^2}{\Delta x^2}(d_{j-1} - d_{j-2})$$
$$\dot{d}_j = v_j$$

or

$$\ddot{d}_j + 2\frac{Ra}{\Delta x}(\dot{d}_j - \dot{d}_{j-1}) + \frac{a^2}{\Delta x^2}(d_j - 2d_{j-1} + d_{j-2}) = 0$$

Then

$$\lambda^2 + 2\frac{Ra}{\Delta x}(1 - e^{-i\xi}) + \frac{a^2}{\Delta x^2}(1 - e^{-i\xi})^2 = 0$$

and

$$\lambda = A\frac{a}{\Delta x}(1 - e^{-\xi})$$

where

$$A = -R \pm \sqrt{R^2 - 1}$$

If $R < 1$, $|A| = 1$ and

$$A = e^{i\theta}$$

where for the negative sign

$$\pi < \theta < \frac{3\pi}{2}$$

Now for small ξ, $1 - e^{-i\xi} \to i\xi$ so

$$\lambda = Re^{i\phi} \ , \ \frac{3\pi}{2} < \phi < 2\pi$$

representing an unstable mode. When $R = 1$ there are two equal roots

$$\lambda = -\frac{a}{\Delta x}(1 - e^{-\xi})$$

Then there exist solutions of the form

$$\hat{d} = c_1 e^{\lambda t} + c_2 \tau e^{\lambda t}$$

which admit initial growth.

When $R > 1$ both solutions for A are real and negative, while

$$Re\left(a(1 - e^{-i\xi})\right) > 0$$

so both modes decay.

The difference equations approximate a partial differential equation of the form

$$d_{tt} + 2Rad_{xt} + a^2 d_{xx} = 0$$

Setting

$$X = x \ , \ T = t - \frac{R}{a}x$$

The equation reduces to

$$(1 - R^2)d_{TT} + a^2 d_{XX} = 0$$

This is hyperbolic if $R > 1$, with a wave speed

$$c = \frac{a}{\sqrt{1 - R^2}}$$

The general solution is of the form

$$d = f(X - cT) + g(x + cT)$$

corresponding to undamped waves which travel with fixed amplitude.

When $R = 1$ the equation reduces to

$$\left(\frac{\partial}{\partial t} + a\frac{\partial}{\partial x}\right)^2 d = 0$$

which admits solutions of the form

$$d = f(x - at) + (x + at)g(x - at)$$

Then, if $x = at$,

$$d = 2at \ g(x + at)$$

indicating growth. Thus both the discrete and continuous analysis indicate that the condition $R > 1$ is necessary for stability.

3.2 Symmetric 3-point scheme

With looks in both directions a possible solution using feedback only from immediate neighbors is

$$\alpha = -2\frac{Ra}{\Delta x}(1-\cos\xi)\ ,\ \beta = -\frac{a^2}{\Delta x^2}(1-\cos\xi)$$

The roots are now

$$\begin{aligned}\lambda &= \frac{\alpha}{2} \pm \sqrt{\frac{\alpha^2}{4}+\beta} \\ &= -\frac{Ra}{\Delta x}(1-\cos\xi) \pm \frac{a}{\Delta x}\sqrt{R^2(1-\cos\xi)^2-(1-\cos\xi)}\end{aligned}$$

If $R > 0$ then $Re(\alpha) \leq 0$. Also if $R^2(1-\cos\xi) < 1$ then both roots are complex with real part $\frac{\alpha}{2}$. This is always the case if $0 < R < 1$. If $R^2(1-\cos\xi) > 1$ then both roots are real with $Re(\lambda) < 0$ since $\beta < 0$. Thus the scheme is stable if $R > 0$.

It can be realized as

$$\begin{aligned}\dot{v}_j &= a_1(v_{j-1}-v_j) + a_1(v_{j+1}-v_j) \\ &\quad + b_1(d_{j-1}-d_j) + b_1(d_{j+1}-d_j) \\ \dot{d}_j &= v_j\end{aligned}$$

where

$$a_1 = \frac{Ra}{\Delta x}\ ,\ b_1 = \frac{a^2}{2\Delta x^2}$$

This reduces to

$$\ddot{d}_j - \frac{Ra}{\Delta x}(\dot{d}_{j+1} - 2\dot{d}_j + \dot{d}_{j-1}) - \frac{a^2}{2\Delta x^2}(d_{j+1} - 2d_j + d_{j-1}) = 0$$

which approxiates

$$d_{tt} - 2Ra\Delta x d_{xxt} = a^2 d_{xx}$$

Then multiplying by d_t and integrating over all space

$$\int d_{tt}d_\tau dx - 2Ra\Delta x \int d_{xxt}d_t dx = a^2 \int d_{xx}d_t dx$$

or integrating by parts with d decaying to zero at ∞

$$\int d_{tt}d_t dx + 2Ra\Delta x \int d_{xt}^2 dx + a^2 \int d_x d_{xt} dx = 0$$

or

$$\frac{d}{dt}\int \left(\frac{d_t^2}{2} + a^2\frac{d_x^2}{2}\right) dx = -2Ra\Delta x \int d_{xt}^2 dx$$

guaranteeing decay.

3.3 Termination of the string

A string of finite length can be terminated by a vehicle which only measures the vehicle ahead of it,

$$\begin{aligned}
\dot{v}_j &= a_1(v_{j-1} - v_j) + a_1(v_{j+1} - v_j) + b_1(d_{j-1} - d_j) + b_1(d_{j+1} - d_j) &&, j = 1, n-1 \\
\dot{v}_n &= a_1(v_{j-1} - v_j) + b_1(d_{j-1} - d_j) &&, j = n \\
\dot{d}_j &= v_j &&, j = 1, n
\end{aligned}$$

This is equivalent to setting

$$d_{n+1} = d_n \ , \ v_{n+1} = v_n$$

and

$$\dot{v}_j = a_1(v_{j-1} - v_j) + a_1(v_{j+1} - v_j) + b_1(d_{j-1} - d_j) + b_1(d_{j+1} - d_j) \ , \ j = 1, n$$

where the tail vehicle $n+1$ is a fictitious vehicle. Similarly if the platoon is not explicitly following a leader one can introduce a fictitious lead vehicle

$$v_0 = v_1 \ , \ d_0 = d_1$$

3.4 Energy stability of 3-point control

With these termination rules we can now prove the energy stability of the discrete system by an argument similar to the analysis of the continuous system, using summation by parts instead of integration by parts.

$$\ddot{d}_j - a_1(\dot{d}_{j-1} - 2\dot{d}_j + \dot{d}_{j+1}) - b_1(d_{j-1} - 2d_j + d_{j+1}) = 0$$

For convenience, setting $\dot{e}_{j+\frac{1}{2}} = \dot{d}_{j+1} - \dot{d}_j$, $\dot{e}_{j-\frac{1}{2}} = \dot{d}_j - \dot{d}_{j-1}$, and $e_{j+\frac{1}{2}} = d_{j+1} - d_j$, $e_{j-\frac{1}{2}} = d_j - d_{j-1}$ to get

$$\ddot{d}_j - a_1(\dot{e}_{j+\frac{1}{2}} - \dot{e}_{j-\frac{1}{2}}) - b_1(e_{j+\frac{1}{2}} - e_{j-\frac{1}{2}}) = 0$$

Multiplying $\ddot{d}_j$ by $\dot{d}_j$ and summing by parts we have

$$\sum_{j=1}^{n} \ddot{d}_j \dot{d}_j = \underbrace{a_1 \sum \dot{d}_j(\dot{e}_{j+\frac{1}{2}} - \dot{e}_{j-\frac{1}{2}})}_{A} + \underbrace{b_1 \sum \dot{d}_j(e_{j+\frac{1}{2}} - e_{j-\frac{1}{2}})}_{B}$$

where

$$
\begin{aligned}
A &= a_1 \left\{ \dot{d}_1(\dot{e}_{\frac{3}{2}} - \dot{e}_{\frac{1}{2}}) + \dot{d}_2(\dot{e}_{\frac{5}{2}} - \dot{e}_{\frac{3}{2}}) + ... + \dot{d}_n(\dot{e}_{n+\frac{1}{2}} - \dot{e}_{n-\frac{1}{2}}) \right\} \\
&= -a_1 \left\{ \dot{d}_1 \dot{e}_{\frac{1}{2}} + (\dot{d}_2 - \dot{d}_1)\dot{e}_{\frac{3}{2}} + ... + (\dot{d}_n - \dot{d}_{n-1})\dot{e}_{n-\frac{1}{2}} + \dot{d}_n \dot{e}_{n+\frac{1}{2}} \right\} \\
&= -a_1 \left\{ \dot{d}_1(\dot{d}_1 - \dot{d}_0) + (\dot{d}_2 - \dot{d}_1)(\dot{d}_2 - \dot{d}_1) + ... + (\dot{d}_n - \dot{d}_{n-1})(\dot{d}_n - \dot{d}_{n-1}) + \dot{d}_n(\dot{d}_{n+1} - \dot{d}_n) \right\} \\
&= -a_1 \dot{d}_1(\dot{d}_1 - \dot{d}_0) - a_1 \sum_{j=1}^{n-1} \left(\dot{d}_{j+1} - \dot{d}_j \right)^2 - a_1 \dot{d}_n(\dot{d}_{n+1} - \dot{d}_n)
\end{aligned}
$$

and

$$
\begin{aligned}
B &= b_1 \left\{ \dot{d}_1(e_{\frac{3}{2}} - e_{\frac{1}{2}}) + \dot{d}_2(e_{\frac{5}{2}} - e_{\frac{3}{2}}) + ... + \dot{d}_n(e_{n+\frac{1}{2}} - e_{n-\frac{1}{2}}) \right\} \\
&= -b_1 \left\{ \dot{d}_1 e_{\frac{1}{2}} + (\dot{d}_2 - \dot{d}_1)e_{\frac{3}{2}} + ... + (\dot{d}_n - \dot{d}_{n-1})e_{n-\frac{1}{2}} + \dot{d}_n e_{n+\frac{1}{2}} \right\} \\
&= -b_1 \left\{ \dot{d}_1(d_1 - d_0) + (\dot{d}_2 - \dot{d}_1)(d_2 - d_1) + ... + (\dot{d}_n - \dot{d}_{n-1})(d_n - d_{n-1}) + \dot{d}_n(d_{n+1} - d_n) \right\} \\
&= -b_1 \dot{d}_1(d_1 - d_0) - b_1 \sum_{j=1}^{n-1} (\dot{d}_{j+1} - \dot{d}_j)(d_{j+1} - d_j) - b_1 \dot{d}_n(d_{n+1} - d_n)
\end{aligned}
$$

For a finite string with $d_0 = d_1, d_{n+1} = d_n, \dot{d}_0 = \dot{d}_1, \dot{d}_{n+1} = \dot{d}_n$

$$
\sum_{j=1}^{n} \ddot{d}_j \dot{d}_j + b_1 \sum_{j=1}^{n-1} (\dot{d}_{j+1} - \dot{d}_j)(d_{j+1} - d_j) = -a_1 \sum_{j=1}^{n-1} \left(\dot{d}_{j+1} - \dot{d}_j \right)^2
$$

$$
\frac{d}{dt} \left\{ \frac{1}{2} \sum_{j=1}^{n} \dot{d}_j^2 + \frac{b_1}{2} \sum_{j=1}^{n-1} (d_{j+1} - d_j)^2 \right\} = -a_1 \sum_{j=1}^{n-1} \left(\dot{d}_{j+1} - \dot{d}_j \right)^2
$$

The terms on the left can be regarded as representing kinetic energy and potential energy respectively. The term on the right is strictly negative unless $\dot{d}_{j+1} = \dot{d}_j$ for all j, so that every vehicle is traveling at the same speed, and consequently the string must decay to such a state. The same result holds for an infinite string with all sum mations from $j = -\infty$ to ∞.

For a finite string with imposed vales of the lead vehicle $d_0, \dot{d}_0$, and terminated by a fictitious vehicle, $d_{n+1} = d_n, \dot{d}_{n+1} = \dot{d}_n$,

$$
\frac{d}{dt} \left\{ \frac{1}{2} \sum_{j=1}^{n} \dot{d}_j^2 + \frac{b_1}{2} \left(d_1^2 + \sum_{j=1}^{n-1} (d_{j+1} - d_j)^2 \right) \right\} = (a_1 \dot{d}_0 + b_1 d_0)\dot{d}_1 - a_1 \left\{ \dot{d}_1^2 + \sum_{j=1}^{n-1} \left(\dot{d}_{j+1} - \dot{d}_j \right)^2 \right\}
$$

3.5 Symmetric 5-point scheme

With looks in both directions, and using feedback from immediate two neighbors, another possible solution is

$$\alpha = -2\frac{Ra}{\Delta x^2}(1-\cos\xi)\ ,\ \beta = -\frac{a^2}{\Delta x^4}(1-\cos\xi)^2$$

The roots are now

$$\lambda = A(R \pm \sqrt{R^2-1})$$

where

$$A = -\frac{a}{\Delta x^2}(1-\cos\xi)$$

Since A is real and negative, the scheme is stable if $R > 0$.

The scheme can be realized as

$$\begin{aligned}\dot{v}_j &= a_1(v_{j-1}-v_j) + a_1(v_{j+1}-v_j)\\ &\quad + b_1(d_{j-1}-d_j) + b_1(d_{j+1}-d_j)\\ &\quad + b_2(d_{j-2}-d_j) + b_2(d_{j+2}-d_j)\\ \dot{d}_j &= v_j\end{aligned}$$

where

$$a_1 = \frac{Ra}{\Delta x^2}\ ,\ b_1 = \frac{4a^2}{\Delta x^4}\ ,\ b_2 = -\frac{a^2}{\Delta x^4}$$

This reduces to

$$\begin{aligned}&\dot{v}_j - \frac{Ra}{\Delta x^2}(v_{j-1}-2v_j+v_{j-1}) + \frac{a^2}{\Delta x^4}(d_{j-2}-4d_{j-1}+6d_j-4d_{j+1}+d_{j+2}) = 0\\ &\dot{d}_j = v_j\end{aligned}$$

or

$$\ddot{d}_j - \frac{Ra}{\Delta x^2}(\dot{d}_{j-1}-2\dot{d}_j+\dot{d}_{j+1}) - \frac{a^2}{\Delta x^4}(d_{j-2}-4d_{j-1}+6d_j-4d_{j+1}+d_{j+2}) = 0$$

which approxiates

$$d_{tt} - 2Rad_{xxt} + a^2 d_{xxxx} = 0$$

Then multiplying by d_t and integrating by parts

$$\begin{aligned}0 &= \int d_{tt}d_\tau dx - 2Ra\int d_{xxt}d_t dx + a^2\int d_{xxxx}d_t dx\\ &= \int d_{tt}d_\tau dx + 2Ra\int d_{xt}^2 dx + a^2\int d_{xx}d_{xxt}dx\end{aligned}$$

$$\frac{d}{dt}\int\left(\frac{d_t^2}{2}+\frac{d_{xx}^2}{2}\right)dx = -2Ra\int d_{xt}^2 dx$$

When $R = 2$ the partial differential equation becomes

$$d_{tt} - 2ad_{xxt} + a^2 d_{xxxx} = \left(\frac{\partial}{\partial t} - a\frac{\partial^2}{\partial x^2}\right)^2 d = 0$$

3.6 Termination of the string with 5 point control

In this case the string can be bounded by a pair of fictitious vehicles

$$v_{n+1} = v_n \ , \ v_{n+2} = v_n$$

$$d_{n+1} = d_n \ , \ d_{n+2} = d_n$$

with a similar pair at the front. This corresponds to each vehicle using the same feedback law for each neighbor it actually has.

3.7 Energy analysis with 5-point symmetric control

As before the energy stability of the discrete system can be demonstrated by an analysis similar to the analyses of the corresponding continuous system, replacing integration by parts by summation by parts, assuming termination with fictitious vehicles.

In this case

$$\ddot{d}_j - a_1(v_{j-1} - 2v_j + v_{j-1}) + b_1(d_{j-2} - 4d_{j-1} + 6d_j - 4d_{j+1} + d_{j+2}) = 0$$

Set

$$\begin{aligned} e_{j+\frac{1}{2}} &= d_{j+1} - d_j \\ f_j &= e_{j+\frac{1}{2}} - e_{j-\frac{1}{2}} = d_{j+1} - 2d_j + d_{j-1} \\ g_{j+\frac{1}{2}} &= f_{j+1} - f_j \end{aligned}$$

Now

$$\sum_{j=1}^{n} \ddot{d}_j \dot{d}_j = \underbrace{a_1 \sum \dot{d}_j(\dot{e}_{j+\frac{1}{2}} - \dot{e}_{j-\frac{1}{2}})}_{A} - \underbrace{b_1 \sum \dot{d}_j(g_{j+\frac{1}{2}} - g_{j-\frac{1}{2}})}_{B}$$

With the termination conditions the first sum yields

$$
\begin{aligned}
A &= a_1 \left\{ \dot{d}_1(\dot{e}_{\frac{3}{2}} - \dot{e}_{\frac{1}{2}}) + \dot{d}_2(\dot{e}_{\frac{5}{2}} - \dot{e}_{\frac{3}{2}}) + ... + \dot{d}_n(\dot{e}_{n+\frac{1}{2}} - \dot{e}_{n-\frac{1}{2}}) \right\} \\
&= -a_1 \left\{ \dot{d}_1 \dot{e}_{\frac{1}{2}} + (\dot{d}_2 - \dot{d}_1)\dot{e}_{\frac{3}{2}} + ... + (\dot{d}_n - \dot{d}_{n-1})\dot{e}_{n-\frac{1}{2}} + \dot{d}_n \dot{e}_{n+\frac{1}{2}} \right\} \\
&= -a_1 \left\{ \dot{d}_1(\dot{d}_1 - \dot{d}_0) + (\dot{d}_2 - \dot{d}_1)(\dot{d}_2 - \dot{d}_1) + ... + (\dot{d}_n - \dot{d}_{n-1})(\dot{d}_n - \dot{d}_{n-1}) + \dot{d}_n(\dot{d}_{n+1} - \dot{d}_n) \right\} \\
&= -a_1 \dot{d}_1(\dot{d}_1 - \dot{d}_0) - a_1 \sum_{j=1}^{n-1} \left(\dot{d}_{j+1} - \dot{d}_j \right)^2 - a_1 \dot{d}_n(\dot{d}_{n+1} - \dot{d}_n) \\
&= -a_1 \sum_{j=1}^{n-1} \left(\dot{d}_{j+1} - \dot{d}_j \right)^2
\end{aligned}
$$

Using summation by parts twice, the second sum yields

$$
\begin{aligned}
B &= b_1 \sum_{j=1}^{n-1} g_{j+\frac{1}{2}}(\dot{d}_{j+1} - \dot{d}_j) \\
&= -b_1 \sum_{j=1}^{n-1} (f_{j+1} - f_j)(\dot{e}_{j+\frac{1}{2}}) \\
&= -b_1 \sum_{j=2}^{n-1} f_j(\dot{e}_{j+\frac{1}{2}} - \dot{e}_{j-\frac{1}{2}}) \\
&= -b_1 \sum_{j=2}^{n-1} (d_{j+1} - 2d_j + d_{j-1})(\dot{d}_{j+1} - 2\dot{d}_j + \dot{d}_{j-1})
\end{aligned}
$$

Thus

$$
\frac{d}{dt} \left\{ \frac{1}{2} \sum_{j=1}^{n} \dot{d}_j^2 + \frac{b_1}{2} \sum_{j=2}^{n-1} (d_{j+1} - 2d_j + d_{j-1})^2 \right\} = -a_1 \sum_{j=1}^{n-1} \left(\dot{d}_{j+1} - \dot{d}_j \right)^2
$$

The terms on the left can be regarded as kinetic energy and potential energy respectively, and the system must decay to a state where $\dot{d}_j = constant$ for all j.

4 Conclusion

The analysis of the previous section indicates that stable decentralized control systems can be devised with feedbacks from no more than two neighbors ahead and behind any given vehicle. These correspond to partial differential equations which exhibit energy decay with time. Energy stability can also be proved for the actual discrete systems.