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Connections between the filtered discontinuous Galerkin method and the flux reconstruction approach to high order discretizations

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ABSTRACT

The purpose of this paper is to provide new insights on the connections that exist between the discontinuous Galerkin method (DG), the flux reconstruction method (FR) and the recently identified energy stable flux reconstruction method (ESFR) when solving time dependent conservation laws. All these schemes appear to be quite similar and it is important to understand how they are related. In this paper, we first review results on the stability of the discontinuous Galerkin method and extend it to the filtered discontinuous Galerkin method. We then consider the flux reconstruction approach and show its connections with DG. In particular, we show how the Energy Stable Flux Reconstruction method introduced by Vincent et al. is equivalent to a filtered DG method, hence giving a new proof of its stability. Also, it allows the use of the method without having to know the special form of the flux correction polynomials. Finally, we underline some fundamental differences that exist between FR and DG.

1. Introduction

High order numerical methods for unstructured grids have seen a large number of developments over the last few decades. The pioneering work of Reed and Hill [1] in the 70's led to the original discontinuous Galerkin method (DG) based on a variational form of the equations. In a series of papers, Cockburn and Shu formulated and developed the discontinuous Galerkin method for conservation laws [2–6]. They also provided extensive theoretical results. However, the computational cost of the original discontinuous Galerkin approach forced researchers to look at somewhat cheaper or simpler alternatives. In their book [7], Hesthaven and Warbuton give a thorough exposition of a nodal variant of the discontinuous Galerkin method. Kopriva and Kolias [8] introduced the staggered grid method, based on the differential form of the equation, later renamed spectral difference (SD) and thoroughly studied by Liu et al. [9] and Wang et al. [10]. Other methods include the popular spectral volume method due to Wang [11].

Recently, Huynh introduced a flux reconstruction (FR) framework [12,13] with which he was able both to recover some existing schemes and to formulate some new variations. Jameson used this framework to recast the Fourier stable spectral difference method and to show its energy stability in a Sobolev type norm [14] for all orders of accuracy. Vincent, Castonguay and Jameson later extended this work to identify a class of FR schemes [15] among Huynh's family of schemes, which are energy stable for all orders of accuracy.

All these numerical methods may appear to be quite similar in both their formulation and the results they provide. It therefore seems legitimate to ask what are the connections that exist between all the various schemes. Huynh started to answer this question by showing that the family of FR schemes contains both the nodal DG and the SD methods. This paper goes further and shows how the entire class of Energy Stable Flux Reconstruction schemes identified by Vincent et al. can be recast as a discontinuous Galerkin method for which a linear filtering operator is applied on the residual. However, this paper also shows that some differences exist between the schemes and that some flux reconstruction methods cannot be described as a filtered discontinuous Galerkin method. Conversely, there exist linearly filtered discontinuous Galerkin methods that cannot be expressed in the flux reconstruction framework.

In Section 2, we describe the classical discontinuous Galerkin method for linear advection and give an energy based proof of stability. We also show how appropriate filters applied to the residual preserve energy stability. In Section 3, we introduce a simple formulation of the flux reconstruction method and show how one can recover a DG scheme by using Radau polynomials for the flux correction function. We then consider the special case of the Energy Stable Flux Reconstruction and show how it can be formulated in terms of a filtered discontinuous Galerkin method, hence giving a new proof of it's stability. Section 4 highlights some fundamental differences that exist between discontinuous Galerkin and flux reconstruction approaches.

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2. Filtered discontinuous Galerkin method for 1D linear advection

In this part, we describe a discontinuous Galerkin (DG) method to solve the following one dimensional linear advection equation on the domain $\Omega = [L,R]$. We then consider the effect of filtering on the stability of the method.

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0, \quad a \text{ is a constant.} \tag{1}$$

2.1. Discontinuous Galerkin method for linear advection

The DG method focuses on finding an approximate weak solution to Eq. (1). To do so, the domain is decomposed in N elements

$$\Omega = \bigcup_{k=0}^{N-1}[x_k, x_{k+1}], \quad L = x_0 < x_1 < \ldots < x_N = R$$
$$= \bigcup_{k=0}^{N-1}\Omega_k$$

on which the solution is approximated by polynomials of degree p:

$$u_k^h = \sum_{i=1}^{p+1} u_k^i \phi_i,$$

where ϕ_i is a basis of $\mathbb{R}_p[X]$, the space of degree p polynomials with real coefficients. We define $\mathbf{u}_k = \left[u_k^1 \cdots u_k^{p+1}\right]^T$.

Now, we require the residual $R_h = \frac{\partial u_h}{\partial t} + a\frac{\partial u_h}{\partial x}$ to be orthogonal to a set of smooth test functions. In particular, any polynomial is in this set, leading to the following equations:

$$\forall j, \quad \int_{\Omega_k} R_h \cdot \phi_j dx = 0.$$

Integrating by parts, replacing the boundary terms by the numerical flux and integrating by parts once more, one derives a discontinuous Galerkin method in its strong form:

$$\forall j, \quad \int_{\Omega_k} \phi_j \cdot \frac{\partial u_k^h}{\partial t}dx + \int_{\Omega_k} \phi_j \cdot a\frac{\partial u_k^h}{\partial x}dx = \left[\left((au_k^h) - (au)^\star\right)\cdot\phi_j\right]_{x_k}^{x_{k+1}}. \tag{2}$$

In this expression, $(au)^\star$ is the numerical flux at cell interfaces. More precisely, $(au)^\star(x_{k+1}) = (au)^\star_{k,k+1}$ is the flux between cell k and $k+1$.

The set of $p+1$ equations given by (2) can be recast as a matrix system:

$$\mathbf{M}^k\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{u}_k + a\mathbf{S}^k\mathbf{u}_k = \left[\left((au_k^h) - (au)^\star\right)\boldsymbol{\Phi}\right]_{x_k}^{x_{k+1}}, \tag{3}$$

where $\mathbf{M}^k$ and $\mathbf{S}^k$ are the local mass matrix and stiffness matrix

$$\mathbf{M}_{ij}^k = \int_{\Omega_k}\phi_i\phi_j dx,$$
$$\mathbf{S}_{ij}^k = \int_{\Omega_k}\phi_i\frac{\mathrm{d}\phi_j}{\mathrm{d}x}dx$$

and $\boldsymbol{\Phi} \in \mathbb{R}^{p+1}$ is defined by $\boldsymbol{\Phi}(x) = [\phi_1(x) \ \ldots, \ \phi_{p+1}(x)]^T$. Using this notation, $u_k^h(x) = \mathbf{u}_k^T\boldsymbol{\Phi}(x)$.

2.2. Stability of the method

Consider again the linear advection equation on the domain $[L,R]$. Multiplying (1) by u and integrating over x gives

$$\int_L^R u\frac{\partial u}{\partial t}dx = -a\int_L^R u\frac{\partial u}{\partial x}dx$$

and therefore,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_L^R\frac{u^2}{2}dx = \frac{1}{2}a\left(u_L^2 - u_R^2\right), \quad \text{with } u_L = u(L) \quad \text{and} \quad u_R = u(R).$$

This energy estimate tells us that the L^2 norm of the exact solution u remains bounded for finite boundary values. If one assumes periodic boundary conditions, $u_L = u_R$ and $\frac{\mathrm{d}}{\mathrm{d}t}\int_L^R\frac{u^2}{2}dx = 0$, the total energy remains constant in the domain.

We now focus on the stability of the DG method and show how it satisfies a similar criterion. Multiplying (3) by $\mathbf{u}_k^T$, one obtains

$$\mathbf{u}_k^T\mathbf{M}^k\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{u}_k + a \quad \mathbf{u}_k^T\mathbf{S}^k\mathbf{u}_k = \left[\left((au_k^h) - (au)^\star\right)\mathbf{u}_k^T\boldsymbol{\Phi}\right]_{x_k}^{x_{k+1}}$$
$$= \left[\left((au_k^h) - (au)^\star\right)u_k^h\right]_{x_k}^{x_{k+1}}.$$

Now, using the fact that

$$\mathbf{u}^T\mathbf{S}\mathbf{u} = \int_{x_l}^{x_r}u^h\frac{\partial u^h}{\partial x}dx = \left[\frac{{u^h}^2}{2}\right]_{x_l}^{x_r}$$

and that

$$\mathbf{u}^T\mathbf{M}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{u} = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{u}^T\mathbf{M}\mathbf{u}) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{u}\|_{\mathbf{M}}^2,$$

we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{u}_k\|_{\mathbf{M}}^2 = \left[\left(\frac{a{u_k^h}^2}{2} - (au)^\star u_k^h\right)\right]_{x_k}^{x_{k+1}}, \tag{4}$$

$\|\cdot\|_{\mathbf{M}}$ is the norm associated to the inner product defined by $\mathbf{u},\mathbf{v} \mapsto \mathbf{u}^T\mathbf{M}\mathbf{v}$. To make things clearer, we introduce the notation

$$u_k^- = u_{k-1}^h(x_k),$$
$$u_k^+ = u_k^h(x_k).$$

Eq. (4) becomes

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{u}_k\|_{\mathbf{M}}^2 = \left[\frac{1}{2}a{u_{k+1}^-}^2 - (au)^\star_{k,k+1}u_{k+1}^-\right] - \left[\frac{1}{2}a{u_k^+}^2 - (au)^\star_{k-1,k}u_k^+\right]. \tag{5}$$

Now suppose the numerical flux is taken to be

$$(au)^\star_{k-1,k} = \frac{1}{2}a\left(u_k^+ + u_k^-\right) - \frac{1}{2}\alpha|a|\left(u_k^+ - u_k^-\right), \quad \alpha \in [0,1] \tag{6}$$

then for $\alpha = 0$, we recover a central flux, for $\alpha = 1$ we recover an upwind flux. Plugging this expression of the flux in (5) and summing over all the elements, we get

$$\sum_{k=0}^{N-1}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{u}_k\|_{\mathbf{M}}^2 = -\frac{1}{2}\alpha|a|\sum_{k=1}^{N-1}\left(u_k^+ - u_k^-\right)^2 - \frac{1}{2}\alpha|a|\left(u_0^+ - u_N^-\right)^2.$$

For simplicity, we assumed periodic boundary condition. The terms on the right hand side are negative for $\alpha \geqslant 0$. Therefore

$$\sum_{k=0}^{N-1}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{u}_k\|_{\mathbf{M}}^2 \leqslant 0. \tag{7}$$

Since $\sum\|\mathbf{u}_k\|_{\mathbf{M}}^2$ is a positive quantity decreasing in time, it remains bounded. This concludes the stability proof of the method. Here,

$$\|\mathbf{u}_k\|_{\mathbf{M}}^2 = \int_{x_k}^{x_{k+1}}{u_k^h}^2 dx$$

and therefore, the meaning of Eq. (7) is that the energy of the numerical solution can only decrease in time. We can summarize these results in a more concise manner. The DG method described by Eq. (3) can be written

$$\mathbf{M}\frac{d\mathbf{u}}{dt} + a\mathbf{S}\mathbf{u} = \mathbf{RHS}_{DG}, \tag{8}$$

where $\mathbf{RHS}_{DG}$ depends only on the choice of the numerical flux. The index k was dropped for convenience. We showed that the method was stable if $\mathbf{M}$ was any symmetric positive definite matrix (hence defining an inner product on $\mathbb{R}^n$ and a weighted norm associated to it) and if the numerical fluxes where chosen according to Eq. (6). In the rest of this document, we will always assume the latter to be satisfied.

2.3. Stability of the filtered DG method

In an actual implementation of the method, the DG semi discrete Eq. (8) are written

$$\frac{d\mathbf{u}}{dt} = \mathbf{M}^{-1}(-a\mathbf{S}\mathbf{u} + \mathbf{RHS}_{DG}) = \mathbf{R}_{DG}(\mathbf{u}), \quad \text{the DG residual}$$

and are then marched in time. Various explicit and implicit techniques can then be used at this point. Often times, although the method is shown to be stable for linear equations, wiggles tend to appear when solving nonlinear systems of equations, such as the Burgers equation or the Euler equations. In particular, when the solution contains discontinuities, large spurious oscillations (Gibbs phenomenon) can be observed. One classical method to remedy this problem is to introduce filters applied to the residual. Their goal is to damp the highest modes and limit Gibbs phenomenon (at this time, the notion of *modes* remains undefined). From the previous section, it is extremely easy to show how a large class of linear filters preserve stability (or enhance it in some sense) in the case of linear equations. A linear filter $\mathbf{F}$ applied to the DG residual will lead to the equation

$$\frac{d\mathbf{u}}{dt} = \mathbf{F}\cdot\mathbf{R}_{DG}(\mathbf{u}) = \mathbf{F}\cdot\mathbf{M}^{-1}(-a\mathbf{S}\mathbf{u} + \mathbf{RHS}_{DG}),$$

which is equivalent to solving

$$\mathbf{M}\cdot\mathbf{F}^{-1}\frac{d\mathbf{u}}{dt} + a\mathbf{S}\mathbf{u} = \mathbf{RHS}_{DG} \iff \widetilde{\mathbf{M}}\frac{d\mathbf{u}}{dt} + a\mathbf{S}\mathbf{u} = \mathbf{RHS}_{DG},$$

where $\widetilde{\mathbf{M}} = \mathbf{M}\cdot\mathbf{F}^{-1}$ is a modified mass matrix. If $\widetilde{\mathbf{M}}$ is symmetric, positive definite, then we showed in Section 2.2 that the resulting scheme would be stable in the norm associated to $\widetilde{\mathbf{M}}$ for linear advection (The proof is the same with $M \leftarrow \widetilde{M}$).

Without any loss of generality, the element $\Omega_k = [x_k, x_{k+1}]$ can be mapped to a reference element $[-1,1]$. In this reference element, the mass matrix is a representation of the bilinear form $(u,v) \mapsto \int_{-1}^{1} uv\,dx$ on a basis of $\mathbb{R}_p[X]$: $\mathcal{B} = \{\phi_1, \phi_2, \ldots, \phi_{p+1}\}$:

$$\mathbf{M}_{ij} = \int_{-1}^{1}\phi_i\phi_j\,dx.$$

In particular, $\mathbf{M} = \mathbf{I}$ the identity matrix if $\mathcal{B} = \overline{\mathcal{P}} = \{\overline{P}_1, \overline{P}_2, \ldots, \overline{P}_{p+1}\}$ the normalized Legendre polynomial basis. Denote $\mathbf{V}_{\mathcal{B},\overline{\mathcal{P}}} = \mathbf{V}$ the transformation matrix from general and unspecified basis $\mathcal{B}$ to basis $\overline{\mathcal{P}}$. Evidently, $\mathbf{M}_{\mathcal{B}} = \mathbf{V}^T\cdot\mathbf{I}\cdot\mathbf{V} = \mathbf{V}^T\mathbf{V}$. Also, $\mathbf{F}_{\mathcal{B}} = \mathbf{V}^{-1}\cdot\mathbf{F}_{\overline{\mathcal{P}}}\cdot\mathbf{V}$, where $\mathbf{F}_{\overline{\mathcal{P}}}$ is the expression of the filter in the normalized modal basis $\overline{\mathcal{P}}$. It follows that the modified mass matrix $\mathbf{M}_{\mathcal{B}}$ takes the form

$$\widetilde{\mathbf{M}}_{\mathcal{B}} = \mathbf{V}^T\mathbf{V}\mathbf{V}^{-1}\mathbf{F}_{\overline{\mathcal{P}}}^{-1}\mathbf{V} = \mathbf{V}^T\mathbf{F}_{\overline{\mathcal{P}}}^{-1}\mathbf{V}.$$

$\widetilde{\mathbf{M}}_{\mathcal{B}}$ is symmetric, positive definite if and only if $\mathbf{F}_{\overline{\mathcal{P}}}$ is symmetric, positive definite as well, leading to a scheme stable for linear advection.

There are many filters satisfying this property. For example, one classical choice is the exponential filter [7] defined by

$$\mathbf{F}_{\overline{\mathcal{P}}} = \begin{pmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_{p+1} \end{pmatrix}, \quad \sigma_i = \exp\left(-\alpha\left(\frac{i-1}{p}\right)^s\right),$$

where α and s are free parameters. The idea is to force the residual to have a decay in its coefficients that is similar to the one observed for smooth functions decompositions (high modes have smaller coefficients). Here, all the terms are smaller than 1 and the energy proof of stability is intuitive. Things can be less intuitive when considering a general positive definite filter $\mathbf{F}_{\overline{\mathcal{P}}}$. Also, in that case, the concept of filtering is not so clear, as various modes can be coupled.

3. Energy Stable Flux Reconstruction scheme as a filtered DG method

3.1. The flux reconstruction method

The formulation given here of the flux reconstruction method follows closely the one given by Huynh [12]. For the linear advection equation $\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0$, the FR method can be described as follows. We consider an element Ω_k mapped to $[-1,1]$. The solution u_k^h can once again be expanded in a polynomial basis. The flux is taken to be $f_k^h = f_k^D + f_k^C$ where

$$\begin{aligned} f_k^D(x) &= au_k(x),\\ f_k^C(x) &= \left[f_{k-1,k}^\star - f_k^D(-1)\right]g_L(x) + \left[f_{k,k+1}^\star - f_k^D(1)\right]g_R(x)\\ &= f_{CL}\cdot g_L(x) + f_{CR}\cdot g_R(x). \end{aligned}$$

"D" stands for *discontinuous*, "C" stands for *correction*. g_L and g_R are flux correction functions. They are chosen to approximate zero in some sense and satisfy

$$\begin{aligned} g_L(-1) &= 1, \quad g_L(1) = 0,\\ g_R(-1) &= 0, \quad g_R(1) = 1. \end{aligned}$$

It follows that f_k is continuous on Ω and for all k

$$f_k(x_k) = f_{k-1}(x_k) = f_{k-1,k}^\star.$$

Its derivative with respect to x on Ω_k is

$$\frac{df_k^h}{dx} = a\frac{du_k^h}{dx} + f_{CL}\frac{dg_L}{dx} + f_{CR}\frac{dg_R}{dx}.$$

We now specify g_L and g_R more precisely by assuming they are in $\mathbb{R}_{p+1}[X]$, the space of real coefficients polynomials of degree at most $p + 1$. As a consequence, $\frac{dg_L}{dx}$ is a polynomial of degree at most p and it can be represented in the same basis as the solution u^h by the vector $\mathbf{g}_L'$. The same can be said about $\frac{dg_R}{dx}$.

We are now in a position to give an explicit vectorial formulation of the FR method

$$\frac{d\mathbf{u}_k}{dt} + a\mathbf{D}^k\mathbf{u}_k + f_{CL}\cdot\mathbf{g}_L' + f_{CR}\cdot\mathbf{g}_R' = 0, \tag{9}$$

where $\mathbf{D}$ is the differentiation matrix defined by $\frac{d\mathbf{u}}{dx} = \mathbf{D}\mathbf{u}$ (using very informal notations). Now, multiplying by the mass matrix introduced earlier one obtains

$$\mathbf{M}^k\frac{d\mathbf{u}_k}{dt} + a\mathbf{S}^k\mathbf{u}_k = -f_{CL}\cdot\mathbf{M}^k\mathbf{g}_L' - f_{CR}\cdot\mathbf{M}^k\mathbf{g}_R'. \tag{10}$$

If g is a polynomial of degree at most $p + 1$ and $\mathbf{g}'$ the vector representation in the basis $\{\phi_1, \phi_2, \ldots, ‘\phi_{p+1}\}$ of g', its derivative with respect to x then

$$\mathbf{M} \cdot \mathbf{g}' = \int_{-1}^{1} g' \boldsymbol{\Phi} \, dx = [g\boldsymbol{\Phi}]_{-1}^{1} - \int_{-1}^{1} g \boldsymbol{\Phi}' \, dx.$$

Again $\boldsymbol{\Phi} = [\phi_1 \, \phi_2 \ldots \phi_{p+1}]^T$ and $\boldsymbol{\Phi}' = \left[\frac{d\phi_1}{dx} \, \frac{d\phi_2}{dx} \ldots \frac{d\phi_{p+1}}{dx}\right]^T$. Eq. (10) becomes

$$\begin{aligned} \mathbf{M}^k \frac{d\mathbf{u}_k}{dt} + a\,\mathbf{S}^k \mathbf{u}_k &= f_{CL} \cdot \boldsymbol{\Phi}(-1) - f_{CR} \cdot \boldsymbol{\Phi}(1) \\ &\quad + \int_{-1}^{1} (f_{CL} \cdot g_L + f_{CR} \cdot g_R) \boldsymbol{\Phi}' \, dx \\ &= \left[\left(a u_k^h - (au)^\star \right) \boldsymbol{\Phi} \right]_{-1}^{1} + \int_{-1}^{1} (f_{CL} \cdot g_L + f_{CR} \cdot g_R) \boldsymbol{\Phi}' \, dx \end{aligned}$$

and therefore

$$\mathbf{M} \frac{d\mathbf{u}}{dt} + a\mathbf{S}\mathbf{u} = \mathbf{RHS}_{DG} + \int_{-1}^{1} (f_{CL} \cdot g_L + f_{CR} \cdot g_R) \boldsymbol{\Phi}' \, dx. \quad (11)$$

This equation is extremely interesting and should be put in relation to Eq. (8). It tells us that when considering the flux reconstruction method with polynomial correction functions of degree at most $p + 1$, one recovers the DG method plus an extra term $\int_{-1}^{1} (f_{CL} \cdot g_L + f_{CR} \cdot g_R) \boldsymbol{\Phi}' \, dx$. As pointed out by Huynh [12], we recover exactly the DG method if we define g_R and g_L using Radau polynomials, so that the extra term vanishes.

3.2. Energy Stable Flux Reconstruction as a filtered DG method

We now consider the Energy Stable Flux Reconstruction approach introduced by Vincent et al. [15]. In this section we derive a new formulation of the method based on Jameson's proof of stability of the spectral difference method [14] and show how it can be interpreted as a filtered DG scheme. Suppose there exist a symmetric matrix $\mathbf{K}$ such that $\mathbf{K} \cdot \mathbf{D} = 0$ (we will show later how such a matrix can be easily found). By multiplying (9) by $\mathbf{K}$ one obtains

$$\mathbf{K} \frac{d\mathbf{u}}{dt} = -f_{CL} \cdot \mathbf{K}\mathbf{g}'_L - f_{CR} \cdot \mathbf{K}\mathbf{g}'_R. \quad (12)$$

Adding this new relation to Eq. (11) yields to

$$\begin{aligned} (\mathbf{M} + \mathbf{K}) \frac{d\mathbf{u}}{dt} + a\mathbf{S}\mathbf{u} &= \mathbf{RHS}_{DG} + f_{CL} \cdot \left[\int_{-1}^{1} g_L \boldsymbol{\Phi}' \, dx - \mathbf{K}\mathbf{g}'_L \right] \\ &\quad + f_{CR} \cdot \left[\int_{-1}^{1} g_R \boldsymbol{\Phi}' dx - \mathbf{K}\mathbf{g}'_R \right]. \end{aligned}$$

The FR method proposed by Vincent et al. aims to find g_L and g_R such that the two last terms in square brackets in the above relation vanish. For this particular choice of g_L and g_R, their flux reconstruction method is therefore completely equivalent to solving

$$(\mathbf{M} + \mathbf{K}) \frac{d\mathbf{u}}{dt} + a\mathbf{S}\mathbf{u} = \mathbf{RHS}_{DG}, \quad (13)$$

as long as $(\mathbf{M} + \mathbf{K})$ is invertible. The first observation here is that if one chooses to solve (13) instead of (9), the explicit forms of g_L and g_R need not being given. The second observation is that their flux reconstruction scheme takes the exact form of a discontinuous Galerkin method with modified mass matrix $\widetilde{\mathbf{M}} = \mathbf{M} + \mathbf{K}$:

$$\begin{aligned} \widetilde{\mathbf{M}} \frac{d\mathbf{u}}{dt} + a\mathbf{S}\mathbf{u} = \mathbf{RHS}_{DG} &\iff \mathbf{M} \cdot (\mathbf{I} + \mathbf{M}^{-1}\mathbf{K}) \frac{d\mathbf{u}}{dt} + a\mathbf{S}\mathbf{u} = \mathbf{RHS}_{DG} \\ &\iff \mathbf{M} \cdot \mathbf{F}^{-1} \frac{d\mathbf{u}}{dt} + a\mathbf{S}\mathbf{u} = \mathbf{RHS}_{DG}, \end{aligned}$$

or equivalently

$$\frac{d\mathbf{u}}{dt} = \mathbf{F} \cdot \mathbf{R}_{DG}(\mathbf{u}),$$

where $\mathbf{F}^{-1} = (\mathbf{I} + \mathbf{M}^{-1}\mathbf{K})$. Once again, $\mathbf{F}$ can be interpreted as a linear filtering operator applied on the DG residual, hence proving the stability of the method.

Let us now be more specific about the method introduced by Vincent et al. It is evident that if $\mathbf{D}$ is the differentiation operator for polynomials of degree at most p, then $\mathbf{D}^k$ is the kth derivative operator for these polynomials. In particular, we know that $\mathbf{D}$ is nilpotent and $\mathbf{D}^{p+1} = 0$. Since we want $\mathbf{K}$ to be symmetric and such that $\mathbf{KD} = 0$, the choice $\mathbf{K} \equiv c(\mathbf{D}^p)^T\mathbf{D}^p$ appears immediately. c is a real scaling coefficient. Their work was then to find g_R such that

$$\int_{-1}^{1} g_R \boldsymbol{\Phi}' dx = c(\mathbf{D}^p)^T \mathbf{D}^p \mathbf{g}'_R$$

and to define g_L by symmetry. Various choices of c lead to many known schemes (DG, spectral differences, Huynh's g_2 flux reconstruction…). As mentioned above, these schemes can be recast in the DG framework as filtering operators applied to the residual without obtaining an explicit expression for g_L and g_R. Here, the filter takes the form

$$\mathbf{F} = (\mathbf{I} + c\mathbf{M}^{-1}(\mathbf{D}^p)^T\mathbf{D}^p)^{-1}.$$

It is now possible to derive an explicit expression of $\mathbf{F}$ in the classical Legendre polynomial basis $\mathcal{P} = \{P_0, P_1, \cdots, P_p\}$ (actually we could derive the expression of the filter in any basis, but the values of c would then have to be rescaled to match the ones computed by Vincent).

$$P_p(x) = c_p x^p + c_{p-1} x^{p-1} + \ldots + c_0 = \frac{1}{2^p} \frac{(2p)!}{(p!)^2} x^p + \ldots$$

Therefore,

$$\mathbf{D}^p = \begin{pmatrix} 0 & \cdots & p!c_p \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \quad \text{and} \quad c(\mathbf{D}^p)^T \cdot \mathbf{D}^p = \begin{pmatrix} 0 & & \\ & \ddots & \\ & & c(p!c_p)^2 \end{pmatrix}.$$

Also,

$$\int_{-1}^{1} P_i^2 dx = \frac{2}{2i+1} \quad \text{leading to} \quad \mathbf{M}^{-1} = \begin{pmatrix} \frac{1}{2} & & & \\ & \frac{3}{2} & & \\ & & \ddots & \\ & & & \frac{2p+1}{2} \end{pmatrix}.$$

Hence,

$$\mathbf{I} + c\mathbf{M}^{-1}(\mathbf{D}^p)^T\mathbf{D}^p = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & 1 + c\frac{2p+1}{2}(p!c_p)^2 \end{pmatrix}$$

and eventually

$$\mathbf{F} = \left(\mathbf{I} + c\mathbf{M}^{-1}(\mathbf{D}^p)^T\mathbf{D}^p\right)^{-1} = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & \frac{1}{1 + c\frac{2p+1}{2}(p!c_p)^2} \end{pmatrix}.$$

The filter can then be transformed to the computational basis $F_{\mathcal{B}} = \mathbf{V}_{\mathcal{B},\mathcal{P}}^{-1} \cdot F \cdot \mathbf{V}_{\mathcal{B},\mathcal{P}}$. As pointed out in Section 2.3, the resulting scheme is stable provided that $\mathbf{F}$ is symmetric, positive definite. This is the case if

$$1 + c\frac{2p+1}{2}(p!c_p)^2 > 0 \iff c > c_- = \frac{-2}{(2p+1)\cdot(p!c_p)^2}.$$

If $c_- < c < 0$ the effect of the filter is to amplify the highest mode of the residual. If $c = 0$ the filter reduces to the identity matrix and we recover the unfiltered DG method. Finally, if $c > 0$ the action of the filter is to damp the highest mode of the residual. Vincent et al. identified a few values of c that recover some interesting schemes.

- **Discontinuous Galerkin** – $c_{DG} = 0$.
  In this case, the filter reduces to the identity matrix and has no action on the residual. Therefore, the DG method remains unchanged.
- **spectral difference** – $c_{SD} = \frac{2p}{(2p+1)(p+1)(p!c_p)^2}$.

  Here we recover the stable SD scheme identified by Huynh [12]. In the non-normalized Legendre basis, and for this particular value of c, the filter takes the form

$$\mathbf{F} = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & \frac{p+1}{2p+1} \end{pmatrix}$$

- **Huynh's g_2 Scheme**- $c_{HU} = \frac{2(p+1)}{(2p+1)p(p!c_p)^2}$.

  This time, we recover the g_2 scheme introduced by Huynh in his original paper on the flux reconstruction method and found to be particularly stable [12]. Again we can give the explicit form of the filter in the non-normalized Legendre basis

$$\mathbf{F} = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & \frac{p}{2p+1} \end{pmatrix}.$$

- **Special case** – $c_\infty \mapsto \infty$
  This time, the largest mode of the residual is completely annihilated by the filter. It takes the form

$$\mathbf{F} = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & 0 \end{pmatrix}.$$

  Large losses in accuracy are expected for this particular scheme

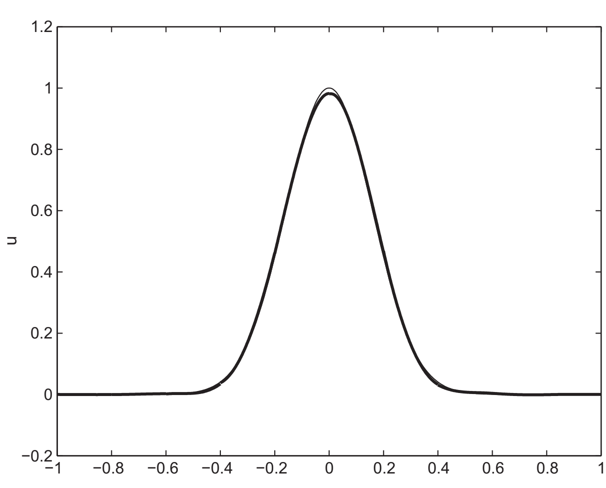


Fig. 1. Reference solution – DG $c = 0$.

It shall be noted that the last term of the filter decreases as c increases. Therefore, the larger c, the more dissipative is the scheme.

3.2.1. Numerical examples

We now consider the linear advection of a Gaussian bump and verify we recover Vincent's results. The domain of interest $\Omega = [-1,1]$ is decomposed in 10 elements of equal length. The advection speed is $a = 1$. The initial condition is given by

$$u(x,0) = e^{-20x^2}.$$

Periodic boundary conditions are applied at both ends of the domain. For the DG implementation, we consider the case $p = 3$ and the collocation points are taken to be the Gauss–Lobatto points. Time integration is done explicitly via a third order Runge Kutta scheme [16]. Results are presented at $t = 20$.

3.2.2. Upwind flux

Here, the numerical flux defined in Eq. (6) is considered for $\alpha = 1$. We therefore recover a fully upwind flux

$$(au)^\star_{k-1,k} = \frac{1}{2}a(u_k^+ + u_k^-) - \frac{1}{2}|a|(u_k^+ - u_k^-),$$

Fig. 1 is a plot of the solution at $t = 20$ for $c = 0$ (unfiltered discontinuous Galerkin). Fig. 2 shows the solution for 4 interesting values of c. $c = c_-/2$ is a value close to the stability limit found above. $c = c_{HU}$ and $c = c_{SD}$ lead respectively to the recovery of Huynh's g_2 flux reconstruction scheme and to the stable spectral difference scheme. Eventually, $c \mapsto \infty$ is a particular case where the last mode of the residual is completely cancelled. All the results obtained by filtering the DG residual match exactly the ones obtained by Vincent using the flux reconstruction approach, hence confirming the preceding theoretical results.

3.2.3. Central flux

Now, the flux defined in Eq. (6) is taken with $\alpha = 0$, leading to a central flux

$$(au)^\star_{k-1,k} = \frac{1}{2}a(u_k^+ + u_k^-).$$

Results are presented in Fig. 3. Once again, our plots match exactly the ones obtained by Vincent.

4. Further analysis of the schemes

The stable method proposed by Vincent, Castonguay and Jameson recovers many of the flux reconstruction schemes introduced by Huynh. We just showed how it is included in a larger class of stable schemes: the filtered DG schemes. Thus, two questions arise naturally:

- Can all the flux reconstruction schemes be expressed in the form of a filtered DG?
- Can any linearly filtered DG scheme be transformed into flux reconstruction form (i.e. for a given filter, can we always find g_L and g_R such that the flux reconstruction method and the filtered DG are equivalent)?

The goal of this section is to give a formal answer to these questions.

Proposition 1. *There exist FR schemes that cannot be expressed as filtered DG schemes (linearly or nonlinearly)*

Proof. The DG method can be expressed as

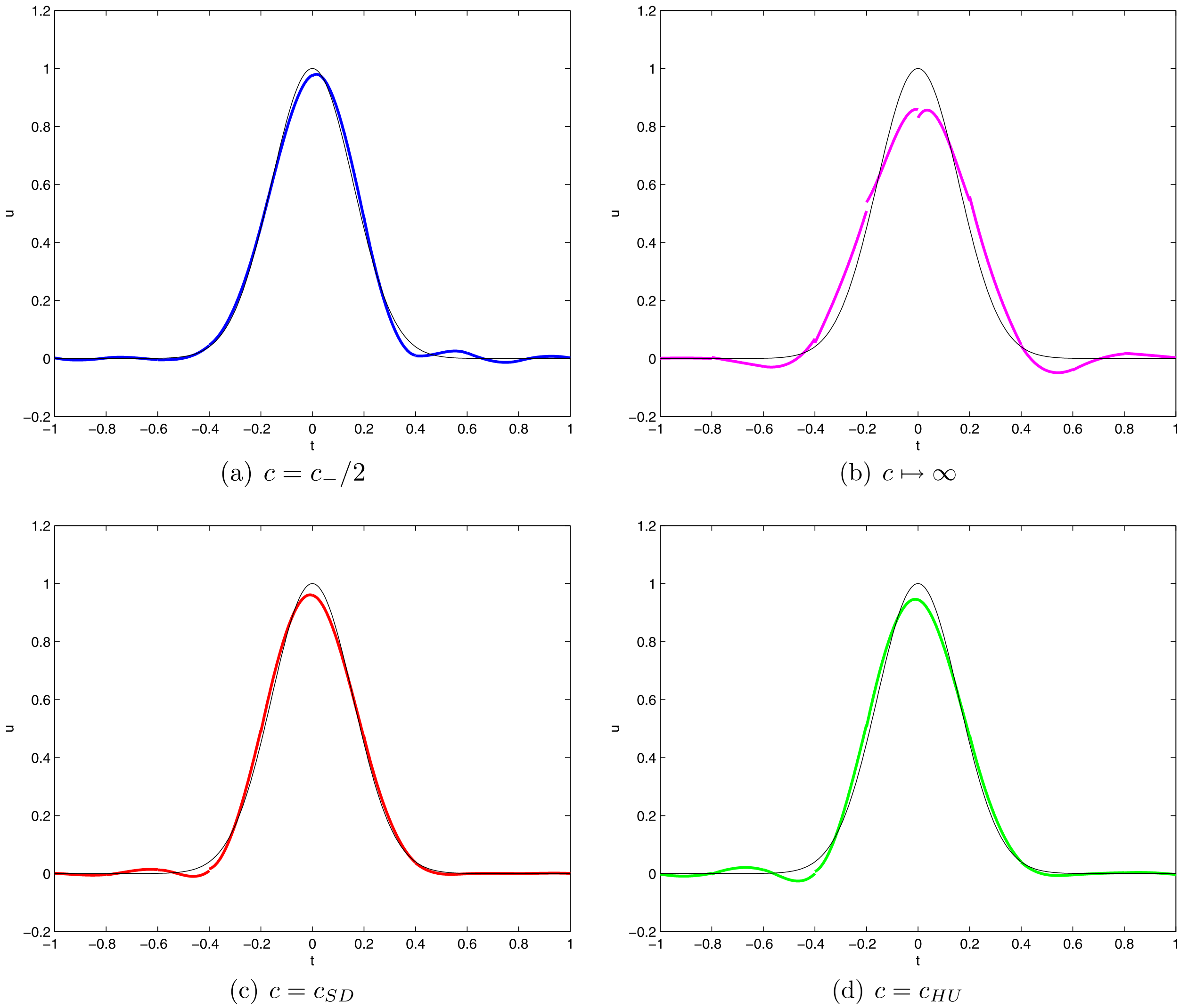


Fig. 2. Plot of the solution at $t = 20$ for various values of c for an upwind flux.

$$\frac{d\mathbf{u}}{dt} = \mathbf{R}_{DG}(\mathbf{u}), \tag{14}$$

while the FR method can be written in compact form

$$\frac{d\mathbf{u}}{dt} = \mathbf{R}_{FR}(\mathbf{u}). \tag{15}$$

We say the FR method is a filtered DG method if there exist a linear or nonlinear operator **F** (a filter) such that

(i) $\mathbf{F}(0) = 0$.
(ii) **F** is independent of **u**.
(iii) For any **u**, $\mathbf{R}_{FR}(\mathbf{u}) = \mathbf{F}(\mathbf{R}_{DG}(\mathbf{u}))$.

We show in the appendix at the end of this document that the DG method admits spurious non constant steady solutions u^h to the linear advection equations such that $f_{CR} = (-1)^p f_{CL} \neq 0$. For this particular non constant solution, the DG residual is zero ($\mathbf{R}_{DG}(\mathbf{u}) = 0$) although this is a non physical result (resulting from the odd/even decoupling phenomenon when using a central flux). If a FR method is obtained by filtering the DG residual, then for this particular solution, we should have $\mathbf{R}_{FR}(\mathbf{u}) = 0$. We know that

$$\begin{aligned}\mathbf{R}_{FR}(\mathbf{u}) &= \mathbf{R}_{DG}(\mathbf{u}) + \mathbf{M}^{-1}\int_{-1}^{1}(f_{CL}\cdot g_L + f_{CR}\cdot g_R)\boldsymbol{\Phi}'\,dx \\ &= 0 + \mathbf{M}^{-1}\int_{-1}^{1}(f_{CL}\cdot g_L + f_{CR}\cdot g_R)\boldsymbol{\Phi}'\,dx.\end{aligned}$$

Therefore,

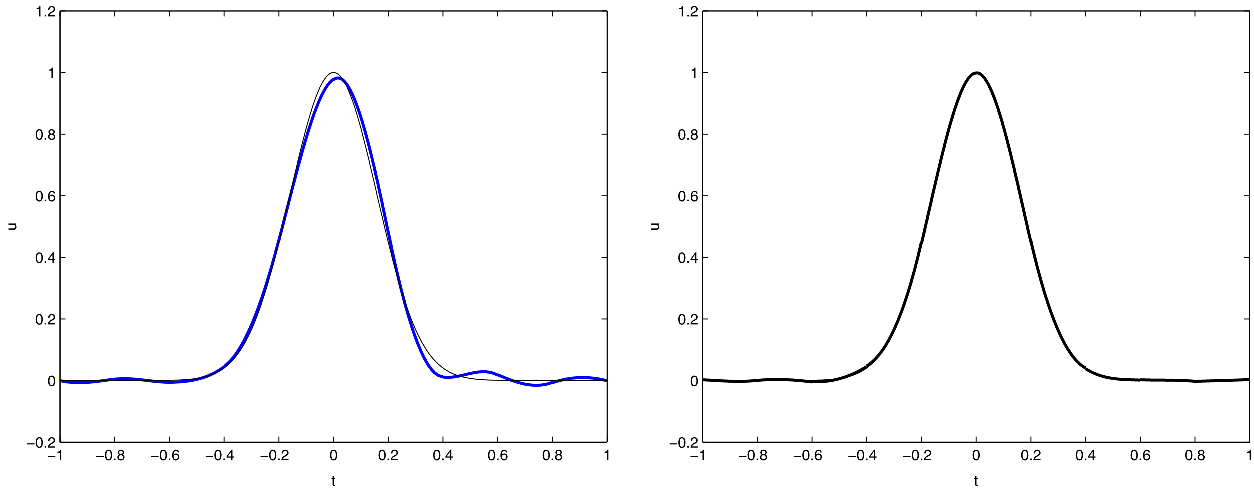
$$\begin{aligned} & \mathbf{R}_{FR}(\mathbf{u}) = 0 \\ \Leftrightarrow\quad & \mathbf{M}^{-1}\int_{-1}^{1}(f_{CL}\cdot g_L + f_{CR}\cdot g_R)\,\boldsymbol{\Phi}'\ \mathrm{dx} = 0 \\ \Leftrightarrow \forall i,\quad & \int_{-1}^{1}(f_{CL}\cdot g_L + f_{CR}\cdot g_R)\,\phi'_i\ \mathrm{dx} = 0 \\ \Leftrightarrow \forall i,\quad & \cancel{f_{CL}}\int_{-1}^{1}(g_L + (-1)^p g_R)\,\phi'_i\ \mathrm{dx} = 0 \qquad (f_{CL} \neq 0)\end{aligned}$$

Consider the case $p = 2$ and $\{\phi_i\} = \{P_i\}$ is the Legendre polynomial basis. Now suppose

$$g_L = \tfrac{1}{8}(1-x)^3,$$
$$g_R = \tfrac{1}{8}(1+x)^3.$$

These correction functions would be the g_3 functions for $K = 3$ in Huynh's paper [12]. They satisfy

$$g_L(-1) = 1,\quad g_L(1) = 0,$$
$$g_R(-1) = 0,\quad g_R(1) = 1,$$



(a) $c = c_-/2$ (b) $c = 0$ (c) $c = c_{SD}$ (d) $c = c_{HU}$

Fig. 3. Plot of the solution at $t = 20$ for various values of c for an central flux.

but for $i = 1,\ P_1' = 1$ and

$$\int_{-1}^{1}(g_L + g_R)\,dx = 1 \neq 0 \ \Rightarrow\ \mathbf{R}_{FR}(\mathbf{u}) \neq 0.$$

We found a flux reconstruction method such that there exist a solution u for which $\mathbf{R}_{DG}(\mathbf{u}) = 0$ but $\mathbf{R}_{FR}(\mathbf{u}) \neq 0$. We exhibited a particular FR method that cannot be expressed as a filtered DG method. □

Proposition 2. *There exist filtered DG schemes that cannot be recovered by a flux reconstruction approach. Also while all FR schemes are conservative, some filtered DG are not.*

Proof. The flux reconstruction and discontinuous Galerkin residuals are related by

$$\mathbf{R}_{FR}(\mathbf{u}) = \mathbf{R}_{DG}(\mathbf{u}) + \mathbf{M}^{-1}\int_{-1}^{1}(f_{CL}\cdot g_L + f_{CR}\cdot g_R)\boldsymbol{\Phi}'\,dx$$

Therefore, FR modifies the DG residual by the addition of an extra term. However, this extra term cannot affect the lowest mode of the residual. Indeed, consider again $\mathcal{P} = \{P_i\}$ the Legendre polynomial basis.

$$P_0 = 1, \quad P_0' = 0$$

It immediately follows that

$$\int_{-1}^{1}(f_{CL}\cdot g_L + f_{CR}\cdot g_R)P_0'\,dx = 0$$

The conclusion of this proof is then straightforward. Let $\mathbf{F}$ be a linear filter. $\mathbf{F}$ can be decomposed as $\mathbf{F} = \mathbf{I} + \mathbf{G}$ where $\mathbf{I}$ is the identity matrix.

$$\mathbf{F}\cdot\mathbf{R}_{DG}(\mathbf{u}) = \mathbf{R}_{FR}(\mathbf{u})$$
$$\iff \mathbf{G}\cdot\mathbf{R}_{DG}(\mathbf{u}) = \mathbf{M}^{-1}\int_{-1}^{1}(f_{CL}\cdot g_L + f_{CR}\cdot g_R)\boldsymbol{\Phi}'\,dx.$$

Take $\mathbf{G}$ diagonal with $G_{11} \neq 0$. Then

$$\mathbf{M}\cdot\mathbf{G}\cdot\mathbf{R}_{DG}(\mathbf{u}) = \begin{pmatrix}\times\\ \times\\ \vdots\\ \times\end{pmatrix}, \quad \times \text{ can be a non zero entry}$$

but

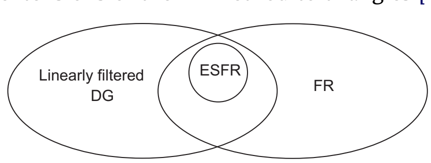
$$\int_{-1}^{1}(f_{CL}\cdot g_L + f_{CR}\cdot g_R)\boldsymbol{\Phi}'\,\mathrm{dx} = \begin{pmatrix} 0 \\ \times \\ \vdots \\ \times \end{pmatrix},$$

which shows that the above equality cannot be satisfied all the time by any flux reconstruction method. We should mention that it is extremely easy to find $\mathbf{u}$ such that the $\times$'s are strictly non zero. The fact that FR cannot alter the lowest mode means that the average value in a cell is always the one obtained by a regular DG scheme, hence proving conservation. The above example would imply that conservation is no more respected for that particular filtered DG. Of course, choosing such a DG filter that modifies the lowest mode would be a bad choice and it would not be done in practice. A more fundamental question that has yet to be answered is "can any conservative linearly filtered DG scheme be transformed into flux reconstruction form?" □

We therefore answered the two questions posed at the beginning of this section. Not all filtered DG methods can be expressed in a flux reconstruction framework, and reciprocally, not all flux reconstruction schemes can be casted as a filtered DG method.

5. Conclusions

In this paper, connections between the filtered discontinuous Galerkin method, the flux reconstruction and Energy Stable Flux reconstructions methods have been established and help understand the working mechanisms of the various methods. We showed how a large class of filtered DG methods are energy stable. We also gave a new derivation of ESFR that led to its formulation in terms of a filtered DG method, giving a new and elegant proof of its energy stability property. Finally, we highlighted differences between the flux reconstruction and the filtered DG methods. In particular, we have demonstrated that neither method is a subset of the other. However, we showed that their intersection is not empty since the ESFR scheme is both a filtered DG and a FR method. This study can easily be extended to simplex elements and we refer the reader to the work of Castonguay et al. for a detailed discussion on the ESFR method on triangles [17]. Wang and Hyunh also gave interesting extensions of the FR method to triangles [18,19].



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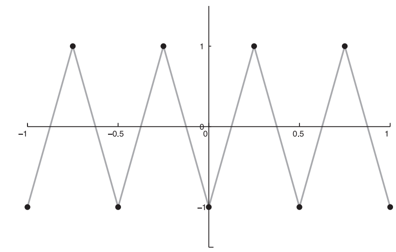
Appendix A. Odd/even decoupling phenomenon in DG

The Odd/even decoupling is a well known phenomenon that appears in finite differences and finite volume methods when using central schemes. In this section we focus on solving the linear advection equation $\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0$. Theoretically, the only steady solution should be obtained for $\frac{\partial u}{\partial x} = 0$, i.e. u = const. However, if one uses a central scheme, there exist a set of spurious non constant solutions that yet lead to a zero residual.

A.1. Example – Central scheme, finite differences

Here, for all i, we have

$$\delta_x(u_i) = \frac{u_{i+1} - u_{i-1}}{2\Delta x} = 0$$



We can show that this phenomenon still exists when considering a DG approach to solving the problem. From now on, we suppose the polynomial basis to be the Legendre polynomials $\mathcal{P} = \{P_0, P_1, \ldots, P_p\}$. These polynomials satisfy the following properties

$$P_i(-1) = (-1)^i,$$
$$P_i(1) = 1, \qquad .$$
$$\int_{-1}^{1} P_i P_j\,\mathrm{dx} = \tfrac{2}{2i+1}\delta_{ij}.$$

For linear advection, the DG method is

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{dt}} = \mathbf{M}^{-1}(-a\mathbf{S}\mathbf{u} + f_{CL}\cdot\boldsymbol{\Phi}(-1) - f_{CR}\cdot\boldsymbol{\Phi}(1)) = \mathbf{R}_{DG}(\mathbf{u}), \quad \text{the DG residual.}$$

Therefore,

$$\mathbf{R}_{DG}(\mathbf{u}) = 0 \iff a\mathbf{S}\mathbf{u} = f_{CL}\cdot\boldsymbol{\Phi}(-1) - f_{CR}\cdot\boldsymbol{\Phi}(1)$$
$$\iff \forall i, \quad \int_{-1}^{1} a u_h' P_i \mathrm{dx} = f_{CL}\cdot P_i(-1) - f_{CR}\cdot P_i(1).$$

In particular, $u_h' \in \mathbb{R}_{p-1}[X]$, so for $i = p$

$$f_{CL}\cdot P_p(-1) - f_{CR}\cdot P_p(1) = 0,$$

leading to

$$f_{CR} = (-1)^p f_{CL}.$$

It follows for other i:

$$\int_{-1}^{1} a u_h' P_i\ \mathrm{dx} = f_{CL}\cdot P_i(-1) - f_{CR}\cdot P_i(1) = f_{CL}(-1)^i - f_{CR} = [(-1)^i - (-1)^p]f_{CL}.$$

Eventually, we have the condition

$$\boxed{\mathbf{R}_{DG}(\mathbf{u}) = 0 \quad \iff \forall i, \quad \int_{-1}^{1} a u_h' P_i\,\mathrm{dx} = [(-1)^i - (-1)^p]f_{CL}}.$$

Note. If we use a fully upwind flux, then

$$a > 0 \Rightarrow f_{CR} = 0,$$
$$a < 0 \Rightarrow f_{CL} = 0.$$

In both case, since $f_{CR} = (-1)^p f_{CL}$, we have $f_{CR} = f_{CL} = 0$. It follows that for all i, $\int_{-1}^{1} a u_h' P_i\,\mathrm{dx} = 0$ implying that u is constant. As it does for finite volume and finite differences, upwinding prevents the odd/even phenomenon.

It is now possible to find an exact expression of the spurious modes. Suppose $f_{CL} \neq 0$ (No full upwinding, $0 \leqslant \alpha < 1$ in Eq. (6)) and $\mathbf{R}_{DG}(\mathbf{u}) = 0$. Suppose u_h' takes the form

$$u_h' = \sum_{i=0}^{p-1}\gamma_i P_i \quad (\gamma_p = 0)$$

As a consequence,

$$\forall i \leqslant p, \ \gamma_i \frac{2a}{2i+1} = [(-1)^i - (-1)^p]f_{CL}$$

If p is even, then

$$\gamma_i = \begin{cases} 0, & i \text{ is even} \\ -\frac{2i+1}{a} f_{CL}, & i \text{ is odd} \end{cases}$$

and if p is odd,

$$\gamma_i = \begin{cases} \frac{2i+1}{a} f_{CL}, & i \text{ is even,} \\ 0, & i \text{ is odd.} \end{cases}$$

Recalling the following property of the Legendre polynomials

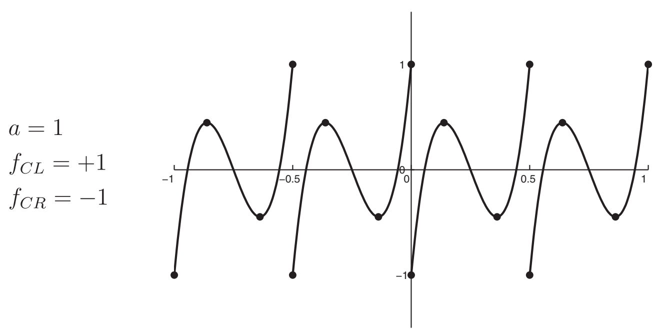
$$\frac{\mathrm{d}}{\mathrm{d}x} P_{n+1} = (2n+1)P_n + (2(n-2)+1)P_{n-2} + (2(n-4)+1)P_{n-4} + \cdots$$

we conclude

$$\mathbf{R}_{DG}(\mathbf{u}^k) = 0 \Longleftrightarrow u_h^k = (-1)^{p+1} \frac{f_{CL}}{a} P_p + \lambda, \quad \lambda \in \mathbb{R}.$$

A.2. Example – Central scheme, p = 3

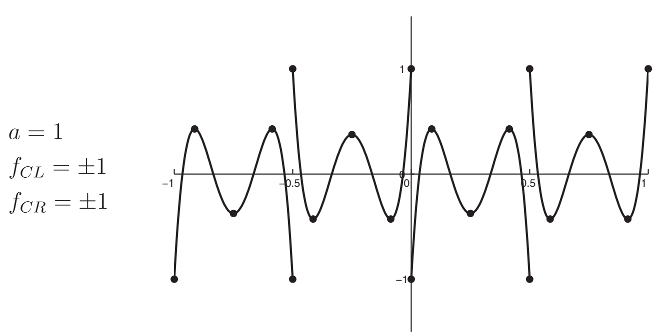
Consider the case



and the solution $u = P_3$

A.3. Example – Central scheme, p = 4

Consider the case



and the solution $u = \mp P_4$

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