WINGLET OPTIMIZATION

OPTIMIZATION FOR INDUCED DRAG

Lift of a wing can be integrated using the circulation strengths of the wake in a plane downstream of the wing, the so-called Trefftz plane. Note that here the span-direction variable, 'y', has been used for the integration since it is in the horizontal plane and lift is normally defined upwards.

$$Lift = \rho V_\infty \int_{-b/2}^{b/2} \Gamma_{(y)} dy$$

$$C_L = \frac{\rho V_\infty}{1/2 \rho V_\infty^2 S} \int_{-b/2}^{b/2} \Gamma_{(y)} dy = \frac{2}{V_\infty S} \int_{-b/2}^{b/2} \Gamma_{(y)} dy = \frac{2}{S} \int_{-b/2}^{b/2} \frac{\Gamma_{(y)}}{V_\infty} dy$$

Likewise, lift-induced drag can be calculated in the Trefftz plane by integrating the product of the local circulation and the local normal velocity to the wake sheet (normalwash.) In this case, the variable of integration is 's' and corresponds to the distance along the wake sheet. If the wing and wake are in a horizontal plane, 's' and 'y' are equivalent. If there is a dihedral angle or a winglet on the wing, they are related but no longer equivalent.

$$Drag_{induced} = \rho \int_{-b/2}^{b/2} \Gamma_{(s)} w_{(s)} ds$$

$$C_{D_{induced}} = \frac{\rho}{1/2 \rho V_\infty^2 S} \int_{-b/2}^{b/2} \Gamma_{(s)} w_{(s)} ds = \frac{2}{V_\infty^2 S} \int_{-b/2}^{b/2} \Gamma_{(s)} w_{(s)} ds = \frac{2}{S} \int_{-b/2}^{b/2} \frac{\Gamma_{(s)}}{V_\infty} \frac{w_{(s)}}{V_\infty} ds$$

In a potential-flow solution, the local circulation of wake column 'i' is also equivalent to the strength of the doublet jump on the wake sheet on that column. This is frequently defined with the variable μ.

$$\Gamma_i = \mu_i$$

The potential jump and normalwash for each column can be expanded into a first-order, Taylor series of the variations in the 'm' design variables.

$$\mu_{i(j)} = \mu_{i(o)} + \sum_{j=1}^{m} \frac{\partial \mu_i}{\partial x_j} \delta x_j$$

$$w_{i(j)} = w_{i(o)} + \sum_{j=1}^{m} \frac{\partial w_i}{\partial x_j} \delta x_j$$

Making the change in notation from Γ to μ, and changing the integral to a summation leads to the following form of the lift equation.

$$C_L = \frac{2}{S} \int_{-b/2}^{b/2} \frac{\Gamma_{(y)}}{V_\infty} dy = \frac{2}{S} \int_{-b/2}^{b/2} \frac{\mu_{(y)}}{V_\infty} dy$$

$$C_L = \frac{2}{S} \sum_{i=-n}^{n} \frac{\mu_i}{V_\infty} dy_i = \frac{2(sym)}{S} \sum_{i=1}^{n} \frac{\mu_i}{V_\infty} dy_i$$

Note that -n to +n goes from left to right wingtip while 1 to n only does the right wing. The second assumes a symmetric case about a vertical center plane and then sym = 2.

The design variables (x_i to x_m) are incorporated in the following version of the lift equation.

$$C_L = \frac{2(sym)}{S} \sum_{i=1}^{n} \left(\frac{\mu_{i(o)}}{V_\infty} + \sum_{j=1}^{m} \frac{\partial \mu_i / V_\infty}{\partial x_j} \delta x_j \right) dy_i$$

If for example, there are three design variables (m = 3), and these variables are x_1, x_2, and x_3, the lift equation can be written.

$$C_L = \frac{2(sym)}{S} \sum_{i=1}^{n} \left(\frac{\mu_{i(o)}}{V_\infty} + \frac{\partial \mu_i / V_\infty}{\partial x_1} \delta x_1 + \frac{\partial \mu_i / V_\infty}{\partial x_2} \delta x_2 + \frac{\partial \mu_i / V_\infty}{\partial x_3} \delta x_3 \right) dy_i$$

Typically one is interested in a specific target C_L and the equation can be rearranged to calculate the difference between the target and the baseline C_L values.

$$C_{Ltar} - \frac{2(sym)}{S} \sum_{i=1}^{n} \left(\frac{\mu_{i(o)}}{V_\infty} \right) dy_i = \frac{2(sym)}{S} \sum_{i=1}^{n} \left(\frac{\partial \mu_i / V_\infty}{\partial x_1} \delta x_1 + \frac{\partial \mu_i / V_\infty}{\partial x_2} \delta x_2 + \frac{\partial \mu_i / V_\infty}{\partial x_3} \delta x_3 \right) dy_i$$

$$C_{Ltar} - C_{Lo} = \frac{2(sym)}{S} \sum_{i=1}^{n} \left(\frac{\partial \mu_i / V_\infty}{\partial x_1} \delta x_1 + \frac{\partial \mu_i / V_\infty}{\partial x_2} \delta x_2 + \frac{\partial \mu_i / V_\infty}{\partial x_3} \delta x_3 \right) dy_i$$

Defining the design-variable vector {x} and the lift influence-coefficient vector [lic]

$$\{x\} = \begin{Bmatrix} \delta x_1 \\ \delta x_2 \\ \delta x_3 \end{Bmatrix} \quad \text{and} \quad [lic] = \left[\sum_{i=1}^{n} \left(\frac{\partial \mu_i / V_\infty}{\partial x_1} \right) dy_i \quad \sum_{i=1}^{n} \left(\frac{\partial \mu_i / V_\infty}{\partial x_2} \right) dy_i \quad \sum_{i=1}^{n} \left(\frac{\partial \mu_i / V_\infty}{\partial x_3} \right) dy_i \right]$$

allows the lift-difference equation to be written in vector form.

$$C_{Ltar} - C_{Lo} = \frac{2(sym)}{S} [lic]\{x\}$$

The induced-drag equation can be rewritten as a summation.

$$C_{Dinduced} = \frac{2}{S} \int_{-b/2}^{b/2} \frac{\mu(s)}{V_\infty} \frac{w(s)}{V_\infty} ds = \frac{2}{S} \sum_{i=-n}^{n} \frac{\mu_i}{V_\infty} \frac{w_i}{V_\infty} ds_i = \frac{2(sym)}{S} \sum_{i=1}^{n} \frac{\mu_i}{V_\infty} \frac{w_i}{V_\infty} ds_i$$

The V_∞ terms have been pulled out front for the following equations to clean up the notation and the influence of the design variables on the induced drag has been included.

$$C_{Dinduced} = \frac{2(sym)}{V_\infty^2 S} \sum_{i=1}^{n} \left(\mu_{i(o)} + \sum_{j=1}^{m} \frac{\partial \mu_i}{\partial x_j} \delta x_j \right) \left(w_{i(o)} + \sum_{j=1}^{m} \frac{\partial w_i}{\partial x_j} \delta x_j \right) ds_i$$

Again for an example with three design variables:

$$C_{D_{induced}} = \frac{2(sym)}{V_\infty^2 S}\sum_{i=1}^{n}\left(\mu_{i(o)} + \frac{\partial\mu_i}{\partial x_1}\delta x_1 + \frac{\partial\mu_i}{\partial x_2}\delta x_2 + \frac{\partial\mu_i}{\partial x_3}\delta x_3\right)\left(w_{i(o)} + \frac{\partial w_i}{\partial x_1}\delta x_1 + \frac{\partial w_i}{\partial x_2}\delta x_2 + \frac{\partial w_i}{\partial x_3}\delta x_3\right)ds_i$$

$$= \frac{2(sym)}{V_\infty^2 S}\sum_{i=1}^{n}\left(\mu_{i(o)}\right)\left(w_{i(o)} + \frac{\partial w_i}{\partial x_1}\delta x_1 + \frac{\partial w_i}{\partial x_2}\delta x_2 + \frac{\partial w_i}{\partial x_3}\delta x_3\right)ds_i$$

$$+ \frac{2(sym)}{V_\infty^2 S}\sum_{i=1}^{n}\left(\frac{\partial\mu_i}{\partial x_1}\delta x_1\right)\left(w_{i(o)} + \frac{\partial w_i}{\partial x_1}\delta x_1 + \frac{\partial w_i}{\partial x_2}\delta x_2 + \frac{\partial w_i}{\partial x_3}\delta x_3\right)ds_i$$

$$+ \frac{2(sym)}{V_\infty^2 S}\sum_{i=1}^{n}\left(\frac{\partial\mu_i}{\partial x_2}\delta x_2\right)\left(w_{i(o)} + \frac{\partial w_i}{\partial x_1}\delta x_1 + \frac{\partial w_i}{\partial x_2}\delta x_2 + \frac{\partial w_i}{\partial x_3}\delta x_3\right)ds_i$$

$$+ \frac{2(sym)}{V_\infty^2 S}\sum_{i=1}^{n}\left(\frac{\partial\mu_i}{\partial x_3}\delta x_3\right)\left(w_{i(o)} + \frac{\partial w_i}{\partial x_1}\delta x_1 + \frac{\partial w_i}{\partial x_2}\delta x_2 + \frac{\partial w_i}{\partial x_3}\delta x_3\right)ds_i$$

* * * *   Rearranging and collecting terms --

$$C_{D_{induced}} = \frac{2(sym)}{V_\infty^2 S}\sum_{i=1}^{n}\left(\mu_{i(o)}\right)\left(w_{i(o)} + \frac{\partial w_i}{\partial x_1}\delta x_1 + \frac{\partial w_i}{\partial x_2}\delta x_2 + \frac{\partial w_i}{\partial x_3}\delta x_3\right)ds_i$$

$$+ \frac{2(sym)}{V_\infty^2 S}\sum_{i=1}^{n}\left(w_{i(o)}\right)\left(\frac{\partial\mu_i}{\partial x_1}\delta x_1 + \frac{\partial\mu_i}{\partial x_2}\delta x_2 + \frac{\partial\mu_i}{\partial x_3}\delta x_3\right)ds_i$$

$$+ \frac{2(sym)}{V_\infty^2 S}\sum_{i=1}^{n}\left(\frac{\partial\mu_i}{\partial x_1}\frac{\partial w_i}{\partial x_1}\delta x_1\delta x_1 + \frac{\partial\mu_i}{\partial x_1}\frac{\partial w_i}{\partial x_2}\delta x_1\delta x_2 + \frac{\partial\mu_i}{\partial x_1}\frac{\partial w_i}{\partial x_3}\delta x_1\delta x_3\right)ds_i$$

$$+ \frac{2(sym)}{V_\infty^2 S}\sum_{i=1}^{n}\left(\frac{\partial\mu_i}{\partial x_2}\frac{\partial w_i}{\partial x_1}\delta x_1\delta x_2 + \frac{\partial\mu_i}{\partial x_2}\frac{\partial w_i}{\partial x_2}\delta x_2\delta x_2 + \frac{\partial\mu_i}{\partial x_2}\frac{\partial w_i}{\partial x_3}\delta x_2\delta x_3\right)ds_i$$

$$+ \frac{2(sym)}{V_\infty^2 S}\sum_{i=1}^{n}\left(\frac{\partial\mu_i}{\partial x_3}\frac{\partial w_i}{\partial x_1}\delta x_1\delta x_3 + \frac{\partial\mu_i}{\partial x_3}\frac{\partial w_i}{\partial x_2}\delta x_2\delta x_3 + \frac{\partial\mu_i}{\partial x_3}\frac{\partial w_i}{\partial x_3}\delta x_3\delta x_3\right)ds_i$$

* * * *   Rearranging and collecting terms one more time --

$$C_{D_{induced}} = \frac{2(sym)}{V_\infty^2 S}\sum_{i=1}^{n}\left(\mu_{i(o)} w_{i(o)}\right) ds_i$$

$$+\frac{2(sym)}{V_\infty^2 S}\sum_{i=1}^{n}\left(\left(\mu_{i(o)}\frac{\partial w_i}{\partial x_1}+w_{i(o)}\frac{\partial \mu_i}{\partial x_1}\right)\delta x_1+\left(\mu_{i(o)}\frac{\partial w_i}{\partial x_2}+w_{i(o)}\frac{\partial \mu_i}{\partial x_2}\right)\delta x_2+\left(\mu_{i(o)}\frac{\partial w_i}{\partial x_3}+w_{i(o)}\frac{\partial \mu_i}{\partial x_3}\right)\delta x_3\right) ds_i$$

$$+\frac{2(sym)}{V_\infty^2 S}\sum_{i=1}^{n}\left(\frac{\partial \mu_i}{\partial x_1}\frac{\partial w_i}{\partial x_1}\delta x_1\delta x_1+\frac{\partial \mu_i}{\partial x_2}\frac{\partial w_i}{\partial x_2}\delta x_2\delta x_2+\frac{\partial \mu_i}{\partial x_3}\frac{\partial w_i}{\partial x_3}\delta x_3\delta x_3\right) ds_i$$

$$+\frac{2(sym)}{V_\infty^2 S}\sum_{i=1}^{n}\left(\left(\frac{\partial \mu_i}{\partial x_1}\frac{\partial w_i}{\partial x_2}+\frac{\partial \mu_i}{\partial x_2}\frac{\partial w_i}{\partial x_1}\right)\delta x_1\delta x_2\right) ds_i$$

$$+\frac{2(sym)}{V_\infty^2 S}\sum_{i=1}^{n}\left(\left(\frac{\partial \mu_i}{\partial x_1}\frac{\partial w_i}{\partial x_3}+\frac{\partial \mu_i}{\partial x_3}\frac{\partial w_i}{\partial x_1}\right)\delta x_1\delta x_3\right) ds_i$$

$$+\frac{2(sym)}{V_\infty^2 S}\sum_{i=1}^{n}\left(\left(\frac{\partial \mu_i}{\partial x_2}\frac{\partial w_i}{\partial x_3}+\frac{\partial \mu_i}{\partial x_3}\frac{\partial w_i}{\partial x_2}\right)\delta x_2\delta x_3\right) ds_i$$

Now breaking into components (without δ_x terms, first-order in δ_x , & second-order in δ_x)

$$C_{D0_{induced}} = \frac{2(sym)}{V_\infty^2 S}\sum_{i=1}^{n}\left(\mu_{i(o)} w_{i(o)}\right) ds_i$$

$$\left\{DIC_1\right\} = \frac{2(sym)}{V_\infty^2 S}\begin{Bmatrix}\sum_{i=1}^{n}\left(\mu_{i(o)}\frac{\partial w_i}{\partial x_1}+w_{i(o)}\frac{\partial \mu_i}{\partial x_1}\right) ds_i\\ \sum_{i=1}^{n}\left(\mu_{i(o)}\frac{\partial w_i}{\partial x_2}+w_{i(o)}\frac{\partial \mu_i}{\partial x_2}\right) ds_i\\ \sum_{i=1}^{n}\left(\mu_{i(o)}\frac{\partial w_i}{\partial x_3}+w_{i(o)}\frac{\partial \mu_i}{\partial x_3}\right) ds_i\end{Bmatrix}$$

$$\left\{DIC_2\right\} = \frac{2(sym)}{V_\infty^2 S}\begin{bmatrix}\sum_{i=1}^{n}\left(\frac{\partial \mu_i}{\partial x_1}\frac{\partial w_i}{\partial x_1}\right) ds_i & \sum_{i=1}^{n}\left(\frac{\partial \mu_i}{\partial x_1}\frac{\partial w_i}{\partial x_2}\right) ds_i & \sum_{i=1}^{n}\left(\frac{\partial \mu_i}{\partial x_1}\frac{\partial w_i}{\partial x_3}\right) ds_i\\ \sum_{i=1}^{n}\left(\frac{\partial \mu_i}{\partial x_2}\frac{\partial w_i}{\partial x_1}\right) ds_i & \sum_{i=1}^{n}\left(\frac{\partial \mu_i}{\partial x_2}\frac{\partial w_i}{\partial x_2}\right) ds_i & \sum_{i=1}^{n}\left(\frac{\partial \mu_i}{\partial x_2}\frac{\partial w_i}{\partial x_3}\right) ds_i\\ \sum_{i=1}^{n}\left(\frac{\partial \mu_i}{\partial x_3}\frac{\partial w_i}{\partial x_1}\right) ds_i & \sum_{i=1}^{n}\left(\frac{\partial \mu_i}{\partial x_3}\frac{\partial w_i}{\partial x_2}\right) ds_i & \sum_{i=1}^{n}\left(\frac{\partial \mu_i}{\partial x_3}\frac{\partial w_i}{\partial x_3}\right) ds_i\end{bmatrix}$$

This allows the induced-drag equation to be written in vector form.

$$C_{D_{induced}} = C_{D0_{induced}} + \left\{DIC_1\right\}^T\left\{x\right\} + \left\{x\right\}^T\left[DIC_2\right]\left\{x\right\}$$

Now calculate the derivatives of the above equation with respect to the design variables.

$$\frac{d}{d_x}\left(\{DIC_1\}^T\{x\}\right) = \{DIC_1\}$$ (note: three derivatives of a scalar → a 3-element vector)

$$\frac{d}{d_{x_1}}\left(\{x\}^T[DIC_2]\{x\}\right) = \frac{2(sym)}{V_\infty^2 S}\left[2\sum_{i=1}^{n}\left(\frac{\partial\mu_i}{\partial x_1}\frac{\partial w_i}{\partial x_1}\right)ds_i\delta x_1 + \sum_{i=1}^{n}\left(\frac{\partial\mu_i}{\partial x_1}\frac{\partial w_i}{\partial x_2}\right)ds_i\delta x_2 + \sum_{i=1}^{n}\left(\frac{\partial\mu_i}{\partial x_1}\frac{\partial w_i}{\partial x_3}\right)ds_i\delta x_3\right]$$
$$+\frac{2(sym)}{V_\infty^2 S}\left[\sum_{i=1}^{n}\left(\frac{\partial\mu_i}{\partial x_2}\frac{\partial w_i}{\partial x_1}\right)ds_i\delta x_2 + 0 + 0\right]$$
$$+\frac{2(sym)}{V_\infty^2 S}\left[\sum_{i=1}^{n}\left(\frac{\partial\mu_i}{\partial x_3}\frac{\partial w_i}{\partial x_1}\right)ds_i\delta x_3 + 0 + 0\right]$$

$$\frac{d}{d_{x_2}}\left(\{x\}^T[DIC_2]\{x\}\right) = \frac{2(sym)}{V_\infty^2 S}\left[0 + \sum_{i=1}^{n}\left(\frac{\partial\mu_i}{\partial x_1}\frac{\partial w_i}{\partial x_2}\right)ds_i\delta x_1 + 0\right]$$
$$+\frac{2(sym)}{V_\infty^2 S}\left[\sum_{i=1}^{n}\left(\frac{\partial\mu_i}{\partial x_2}\frac{\partial w_i}{\partial x_1}\right)ds_i\delta x_1 + 2\sum_{i=1}^{n}\left(\frac{\partial\mu_i}{\partial x_2}\frac{\partial w_i}{\partial x_2}\right)ds_i\delta x_2 + \sum_{i=1}^{n}\left(\frac{\partial\mu_i}{\partial x_2}\frac{\partial w_i}{\partial x_3}\right)ds_i\delta x_3\right]$$
$$+\frac{2(sym)}{V_\infty^2 S}\left[0 + \sum_{i=1}^{n}\left(\frac{\partial\mu_i}{\partial x_3}\frac{\partial w_i}{\partial x_2}\right)ds_i\delta x_3 + 0\right]$$

$$\frac{d}{d_{x_3}}\left(\{x\}^T[DIC_2]\{x\}\right) = \frac{2(sym)}{V_\infty^2 S}\left[0 + 0 + \sum_{i=1}^{n}\left(\frac{\partial\mu_i}{\partial x_1}\frac{\partial w_i}{\partial x_3}\right)ds_i\delta x_1\right]$$
$$+\frac{2(sym)}{V_\infty^2 S}\left[0 + 0 + \sum_{i=1}^{n}\left(\frac{\partial\mu_i}{\partial x_2}\frac{\partial w_i}{\partial x_3}\right)ds_i\delta x_2\right]$$
$$+\frac{2(sym)}{V_\infty^2 S}\left[\sum_{i=1}^{n}\left(\frac{\partial\mu_i}{\partial x_3}\frac{\partial w_i}{\partial x_1}\right)ds_i\delta x_1 + \sum_{i=1}^{n}\left(\frac{\partial\mu_i}{\partial x_3}\frac{\partial w_i}{\partial x_2}\right)ds_i\delta x_2 + 2\sum_{i=1}^{n}\left(\frac{\partial\mu_i}{\partial x_3}\frac{\partial w_i}{\partial x_3}\right)ds_i\delta x_3\right]$$

Or in generic form for any number of design variables:

$$\frac{d}{d_x}\left(\{x\}^T[DIC_2]\{x\}\right) = \left[DIC_2 + DIC_2^T\right]\{x\}$$

Steve Smith from NASA Ames Research Center presented the above method in SAE paper # 971478. It starts with the local circulation and downwash values from a potential-flow analysis, at many points along the wake well downstream of the wing and winglet. It then integrates the influence of the wake columns on each other to predict the lift and drag. The influence coefficients for drag are then integrated with respect to three design variables, the root incidence of the winglet, the tip incidence of the winglet, and the angle of attack of the wing-winglet combination. The variation in angle of attack is required so that the optimization can be done at constant aircraft lift.

Smith developed a quadratic expression for induced drag in terms of the design variables.

$$D_{induced} = \{x\}^T [dic] \{x\} + \{dic_o\}^T \{x\} + D_o$$

where the induced-drag, influence coefficients are defined by

$$[dic]_{jk} = \rho \sum_i \frac{\partial \mu_i}{\partial x_j} \frac{\partial w_i}{\partial x_k} ds_i$$

and

$$\{dic_o\}_j = \rho \sum_i \left(\mu_{oi} \frac{\partial w_i}{\partial x_j} + w_{oi} \frac{\partial \mu_i}{\partial x_j} \right) ds_i$$

The summations integrate the spanwise influences across i and the j and k indices are both for the design variables of x.

The derivative of the drag equation was then used to find the optimum values for the design variables. This equation for minimum induced drag can be written:

$$[dic + dic^T]\{x\} = -\{dic_o\}$$

The following lift equation was used as the basis for a constraint on wing lift.

$$Lift = \{lic\}^T \{x\} + L_o$$

where the lift influence coefficients are defined by:

$$[lic]_j = 2\rho \sum_i \frac{\partial \mu_i}{\partial x_j} dy_i$$

By using the lift equation and a Lagrange-augmented cost function, the optimization problem for induced drag was reduced to the following matrix equation.

$$\begin{bmatrix} dic + dic^T & | & lic \\ ---- & & - \\ lic & | & 0 \end{bmatrix} \begin{Bmatrix} x \\ - \\ \lambda \end{Bmatrix} = \begin{Bmatrix} -dic_o \\ --- \\ Lcon. - L_o \end{Bmatrix}$$

ADDING IN THE PROFILE DRAG

Up to this point, the only drag that is has been calculated is that due to the influence of the wake sheet of the finite wing. The profile drag of the wing, which consists of the skin friction on the wing surface and the form drag of the airfoils along the wing, has not been included. Skin friction will be nearly constant for the typical design variables of winglet incidence and twist, but would be important if winglet planform was varied. The form drag, however, is always important. The variation of form drag for the winglet and outer wing can be easily of the same magnitude as the variation of induced drag for the wing and winglet combination with changes in loading due to incidence and twist. For transonic applications, the variation in form drag with lift is doubly important since increased loading can easily cause unwanted compressibility drag. Thus it is important to have a means to calculate the profile drag along the wing and winglet and then combine it with the induced drag component to have a full optimization.

The local section lift of an airfoil along the wing will be viewed as relative to the orientation of the given section and not just in the vertical direction. The section lift coefficient per unit span can be defined as:

$$C_l = \frac{Lift}{(1/2)\rho V_\infty^2} = \frac{\rho \Gamma V_\infty}{(1/2)\rho V_\infty^2} = \frac{2\Gamma}{V_\infty} = \frac{2\mu}{V_\infty}$$

The local section profile drag is composed of two parts; a constant plus a part that is dependent on the square of a local lift term. So for strip 'i' along the wing, the profile drag coefficient is:

$$C_{dp(i)} = C_{dpo(i)} + k_{(i)} (C_{l(i)} - C_{loff(i)})^2$$

where the C_{loff} term is the section lift coefficient for minimum profile drag. Now inserting the above relationship for section drag gives:

$$C_{dp(i)} = C_{dpo(i)} + k_{(i)} \left(\frac{2\mu_{(i)}}{Chord_{(i)} V_\infty} - C_{loff(i)} \right)^2$$

$$= C_{dpo(i)} + k_{(i)} \left(\frac{4\mu_{(i)}^2}{Chord_{(i)}^2 V_\infty^2} - \frac{4\mu_{(i)}}{Chord_{(i)} V_\infty} C_{loff(i)} + C_{loff(i)}^2 \right)$$

Now incorporate the design variables into the μ term using the first term of a Taylor-series expansion.

$$C_{dp(i)} = C_{dpo(i)} + k_{(i)} \left(\frac{4}{Chord_{(i)}^2} \left(\frac{\mu_{i(o)}}{V_\infty} + \sum_{j=1}^{m} \frac{\partial \mu_i / V_\infty}{\partial x_j} \delta x_j \right)^2 - \frac{4}{Chord_{(i)}} \left(\frac{\mu_{i(o)}}{V_\infty} + \sum_{j=1}^{m} \frac{\partial \mu_i / V_\infty}{\partial x_j} \delta x_j \right) C_{loff(i)} + C_{loff(i)}^2 \right)$$

Now sum across all of the spanwise columns to find the total contribution from profile drag.

$$CDp = \frac{(sym)}{S} \sum_{i=1}^{n} C_{dp(i)} Chord_{(i)} ds_{(i)}$$

$$\begin{aligned} CDp = & \frac{(sym)}{S} \sum_{i=1}^{n} C_{dpo(i)} Chord_{(i)} ds_{(i)} \\ & + \frac{4(sym)}{S} \sum_{i=1}^{n} k_{(i)} \left(\frac{\mu_{i(o)}}{V_\infty} + \sum_{j=1}^{m} \frac{\partial \mu_i / V_\infty}{\partial x_j} \delta x_j \right)^2 \frac{Chord_{(i)}}{Chord_{(i)}^2} ds_{(i)} \\ & - \frac{4(sym)}{S} \sum_{i=1}^{n} k_{(i)} \left(\frac{\mu_{i(o)}}{V_\infty} + \sum_{j=1}^{m} \frac{\partial \mu_i / V_\infty}{\partial x_j} \delta x_j \right) C_{loff(i)} \frac{Chord_{(i)}}{Chord_{(i)}} ds_{(i)} \\ & + \frac{(sym)}{S} \sum_{i=1}^{n} k_{(i)} C_{loff(i)}^2 Chord_{(i)} ds_{(i)} \end{aligned}$$

Expanding the squared term and pulling V_∞ terms out front:

$$\left(\frac{\mu_{i(o)}}{V_\infty} + \sum_{j=1}^{m} \frac{\partial \mu_i / V_\infty}{\partial x_j} \delta x_j \right)^2 = \frac{1}{V_\infty^2} \left(\mu_{i(o)}^2 + 2\mu_{i(o)} \sum_{j=1}^{m} \frac{\partial \mu_i}{\partial x_j} \delta x_j + \left(\sum_{j=1}^{m} \frac{\partial \mu_i}{\partial x_j} \delta x_j \right)^2 \right)$$

Substituting this expression gives:

$$
\begin{aligned}
CDp \; = \; & \frac{(sym)}{S}\sum_{i=1}^{n} C_{dpo(i)}\, Chord_{(i)}\, ds_{(i)} \\
& + \frac{4(sym)}{V_\infty^2 S}\sum_{i=1}^{n} k_{(i)}\left(\mu_{i(o)}^2 + 2\mu_{i(o)}\sum_{j=1}^{m}\frac{\partial\mu_i}{\partial x_j}\delta x_j + \left(\sum_{j=1}^{m}\frac{\partial\mu_i}{\partial x_j}\delta x_j\right)^2 \right)\frac{1}{Chord_{(i)}}\, ds_{(i)} \\
& - \frac{4(sym)}{V_\infty S}\sum_{i=1}^{n} k_{(i)}\left(\mu_{i(o)} + \sum_{j=1}^{m}\frac{\partial\mu_i}{\partial x_j}\delta x_j \right) C_{loff(i)}\; ds_{(i)} \\
& + \frac{(sym)}{S}\sum_{i=1}^{n} k_{(i)}\, C_{loff(i)}^2\, Chord_{(i)}\, ds_{(i)}
\end{aligned}
$$

Collecting terms with the same power of the design variables gives:

$$
\begin{aligned}
CDp \; = \; & \frac{(sym)}{S}\sum_{i=1}^{n}\left(C_{dpo(i)} + k_{(i)} C_{loff(i)}^2 \right) Chord_{(i)}\, ds_{(i)} \\
& + \frac{4(sym)}{V_\infty^2 S}\sum_{i=1}^{n} k_{(i)}\left(\frac{\mu_{i(o)}^2}{Chord_{(i)}} - V_\infty \mu_{i(0)}\, C_{loff(i)} \right) ds_{(i)} \\
& + \frac{4(sym)}{V_\infty^2 S}\sum_{i=1}^{n} k_{(i)}\left(\left(\frac{2\mu_{i(o)}}{Chord_{(i)}} - V_\infty\, C_{loff(i)} \right) \sum_{j=1}^{m}\frac{\partial\mu_i}{\partial x_j}\delta x_j \right) ds_{(i)} \\
& + \frac{4(sym)}{V_\infty^2 S}\sum_{i=1}^{n} \frac{k_{(i)}}{Chord_{(i)}}\left(\sum_{j=1}^{m}\frac{\partial\mu_i}{\partial x_j}\delta x_j \right)^2 ds_{(i)}
\end{aligned}
$$

This can then be written in a matrix form:

$$
CDp \; = \; CDpo + \{DICp1\}^T \{x\} + \{x\}^T [DICp2] \{x\}
$$

The first two lines are independent of the design variables and become

$$
\begin{aligned}
CDpo \; = \; & \frac{(sym)}{S}\sum_{i=1}^{n}\left(C_{dpo(i)} + k_{(i)} C_{loff(i)}^2 \right) chord_{(i)}\, ds_{(i)} \\
& + \frac{4(sym)}{V_\infty^2 S}\sum_{i=1}^{n} k_{(i)}\left(\frac{\mu_{i(o)}^2}{Chord_{(i)}} - V_\infty \mu_{i(0)}\, C_{loff(i)} \right) ds_{(i)}
\end{aligned}
$$

The third line is a function of the first power of the design variables and given three design variables, can be written in the following vector form.

$$\{DICp1\}^T\{x\} = \frac{4(sym)}{V_\infty^2 S}\begin{Bmatrix} \sum_{i=1}^{n} k_{(i)}\left(\left(\frac{2\mu_{i(o)}}{Chord_{(i)}} - V_\infty\, C_{loff(i)}\right)\frac{\partial \mu_i}{\partial x_1}\right) ds_{(i)} \\ \sum_{i=1}^{n} k_{(i)}\left(\left(\frac{2\mu_{i(o)}}{Chord_{(i)}} - V_\infty\, C_{loff(i)}\right)\frac{\partial \mu_i}{\partial x_2}\right) ds_{(i)} \\ \sum_{i=1}^{n} k_{(i)}\left(\left(\frac{2\mu_{i(o)}}{Chord_{(i)}} - V_\infty\, C_{loff(i)}\right)\frac{\partial \mu_i}{\partial x_3}\right) ds_{(i)} \end{Bmatrix}^T \begin{Bmatrix} \delta_{x1} \\ \delta_{x2} \\ \delta_{x3} \end{Bmatrix}$$

Similarly the fourth line with the squared terms can be written as:

$$\{x\}^T[DICp2]\{x\} =$$

$$\frac{4(sym)}{V_\infty^2 S}\begin{Bmatrix} \delta_{x1} \\ \delta_{x2} \\ \delta_{x3} \end{Bmatrix}^T \begin{bmatrix} \sum_{i=1}^{n}\frac{k_{(i)}}{Chord_{(i)}}\left(\frac{\partial \mu_i}{\partial x_1}\right)^2 ds_{(i)} & \sum_{i=1}^{n}\frac{k_{(i)}}{Chord_{(i)}}\left(\frac{\partial \mu_i}{\partial x_1}\frac{\partial \mu_i}{\partial x_2}\right) ds_{(i)} & \sum_{i=1}^{n}\frac{k_{(i)}}{Chord_{(i)}}\left(\frac{\partial \mu_i}{\partial x_1}\frac{\partial \mu_i}{\partial x_3}\right) ds_{(i)} \\ \sum_{i=1}^{n}\frac{k_{(i)}}{Chord_{(i)}}\left(\frac{\partial \mu_i}{\partial x_1}\frac{\partial \mu_i}{\partial x_2}\right) ds_{(i)} & \sum_{i=1}^{n}\frac{k_{(i)}}{Chord_{(i)}}\left(\frac{\partial \mu_i}{\partial x_2}\right)^2 ds_{(i)} & \sum_{i=1}^{n}\frac{k_{(i)}}{Chord_{(i)}}\left(\frac{\partial \mu_i}{\partial x_2}\frac{\partial \mu_i}{\partial x_3}\right) ds_{(i)} \\ \sum_{i=1}^{n}\frac{k_{(i)}}{Chord_{(i)}}\left(\frac{\partial \mu_i}{\partial x_1}\frac{\partial \mu_i}{\partial x_3}\right) ds_{(i)} & \sum_{i=1}^{n}\frac{k_{(i)}}{Chord_{(i)}}\left(\frac{\partial \mu_i}{\partial x_2}\frac{\partial \mu_i}{\partial x_3}\right) ds_{(i)} & \sum_{i=1}^{n}\frac{k_{(i)}}{Chord_{(i)}}\left(\frac{\partial \mu_i}{\partial x_3}\right)^2 ds_{(i)} \end{bmatrix}\begin{Bmatrix} \delta_{x1} \\ \delta_{x2} \\ \delta_{x3} \end{Bmatrix}$$

Differentiating with respect to these design variables gives the following two results.

$$\frac{d}{dx}\left(\{DICp1\}^T\{x\}\right) = \frac{4(sym)}{V_\infty^2 S}\begin{Bmatrix} \sum_{i=1}^{n} k_{(i)}\left(\frac{2\mu_{i(o)}}{Chord_{(i)}} - V_\infty\, C_{loff(i)}\right)\left(\frac{\partial \mu_i}{\partial x_1}\right) ds_{(i)} \\ \sum_{i=1}^{n} k_{(i)}\left(\frac{2\mu_{i(o)}}{Chord_{(i)}} - V_\infty\, C_{loff(i)}\right)\left(\frac{\partial \mu_i}{\partial x_2}\right) ds_{(i)} \\ \sum_{i=1}^{n} k_{(i)}\left(\frac{2\mu_{i(o)}}{Chord_{(i)}} - V_\infty\, C_{loff(i)}\right)\left(\frac{\partial \mu_i}{\partial x_3}\right) ds_{(i)} \end{Bmatrix}$$

(Which gets added to the right-hand-side drag terms of the matrix equation below.)

And:

$$\frac{d}{dx1}\left(\{x\}^T[DICp2]\{x\}\right) = \frac{8(sym)}{V_\infty^2 S}\left[\sum_{i=1}^{n}\frac{k_{(i)}}{Chord_{(i)}}\left(\frac{\partial \mu_i}{\partial x_1}\right)^2 ds_{(i)}\right]\delta_{x1}$$
$$+ \frac{8(sym)}{V_\infty^2 S}\left[\sum_{i=1}^{n}\frac{k_{(i)}}{Chord_{(i)}}\left(\frac{\partial \mu_i}{\partial x_1}\frac{\partial \mu_i}{\partial x_2}\right) ds_{(i)}\right]\delta_{x2}$$
$$+ \frac{8(sym)}{V_\infty^2 S}\left[\sum_{i=1}^{n}\frac{k_{(i)}}{Chord_{(i)}}\left(\frac{\partial \mu_i}{\partial x_1}\frac{\partial \mu_i}{\partial x_3}\right) ds_{(i)}\right]\delta_{x3}$$

$$\frac{d}{dx2}\left(\{x\}^T [DICp2]\{x\}\right) = \frac{8(sym)}{V_\infty^2 S}\left[\sum_{i=1}^{n} \frac{k_{(i)}}{Chord_{(i)}}\left(\frac{\partial\mu_i}{\partial x_1}\frac{\partial\mu_i}{\partial x_2}\right) ds_{(i)}\right]\delta_{x1}$$
$$+ \frac{8(sym)}{V_\infty^2 S}\left[\sum_{i=1}^{n} \frac{k_{(i)}}{Chord_{(i)}}\left(\frac{\partial\mu_i}{\partial x_2}\right)^2 ds_{(i)}\right]\delta_{x2}$$
$$+ \frac{8(sym)}{V_\infty^2 S}\left[\sum_{i=1}^{n} \frac{k_{(i)}}{Chord_{(i)}}\left(\frac{\partial\mu_i}{\partial x_2}\frac{\partial\mu_i}{\partial x_3}\right) ds_{(i)}\right]\delta_{x3}$$

$$\frac{d}{dx3}\left(\{x\}^T [DICp2]\{x\}\right) = \frac{8(sym)}{V_\infty^2 S}\left[\sum_{i=1}^{n} \frac{k_{(i)}}{Chord_{(i)}}\left(\frac{\partial\mu_i}{\partial x_1}\frac{\partial\mu_i}{\partial x_3}\right) ds_{(i)}\right]\delta_{x1}$$
$$+ \frac{8(sym)}{V_\infty^2 S}\left[\sum_{i=1}^{n} \frac{k_{(i)}}{Chord_{(i)}}\left(\frac{\partial\mu_i}{\partial x_2}\frac{\partial\mu_i}{\partial x_3}\right) ds_{(i)}\right]\delta_{x2}$$
$$+ \frac{8(sym)}{V_\infty^2 S}\left[\sum_{i=1}^{n} \frac{k_{(i)}}{Chord_{(i)}}\left(\frac{\partial\mu_i}{\partial x_3}\right)^2 ds_{(i)}\right]\delta_{x3}$$

$$\frac{d}{dx}\left(\{x\}^T [DICp2]\{x\}\right) = \left[DICp2 + DICp2^T\right]\{x\}$$

(Which is added to the left-side, influence coefficients for drag in the matrix equation.)

The induced and profile drag terms are combined in the following matrix equation to optimize for the total drag at a constrained lift.

$$\begin{bmatrix} dic + dic^T + dic_{p2} + dic_{p2}^T & | & lic \\ ------------- & & - \\ lic & | & 0 \end{bmatrix} \begin{Bmatrix} x \\ - \\ \lambda \end{Bmatrix} = \begin{Bmatrix} -dic_o - dic_{p1} \\ ----- \\ Lcon. - L_o \end{Bmatrix}$$

It is important to insure that the terms for induced and profile drag have the same dimensional or non-dimensional level so their relative weights are preserved.