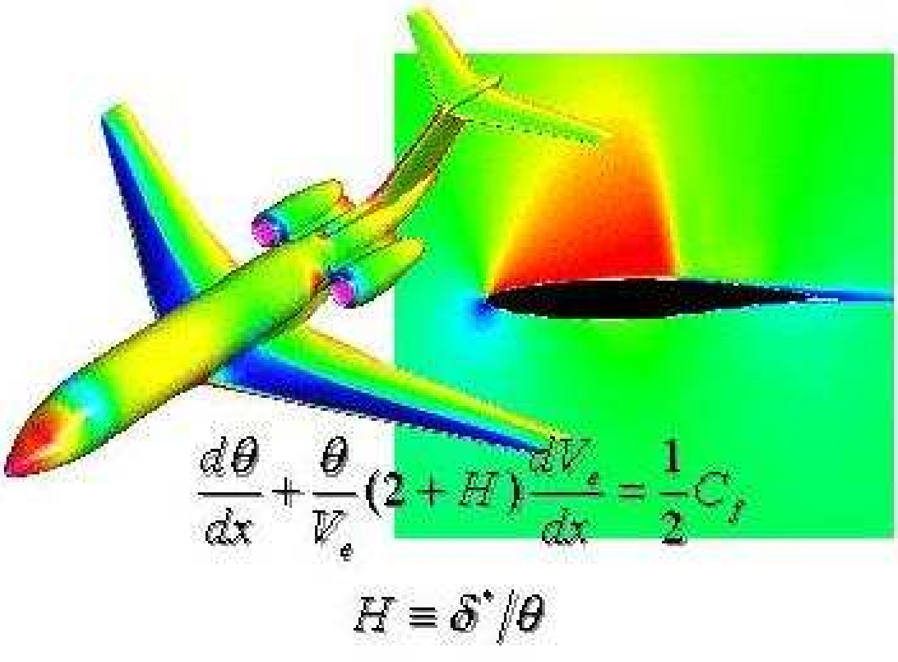
Finite Wing Theory and Details



AA200b

Lecture 11-12

February 17, 2005

Lifting Line Theory

We are now concerned with the application / extension of our two dimensional theories to flows that are truly three-dimensional in nature. The results from two-dimensional theory can be considered to apply to three-dimensional wings of infinite span. However, when the spanwise extent of the wing is abruptly truncated, additional physical phenomena (*trailing vortices*) appear that modify the two-dimensional flow substantially. Accounting for this new modification to the flow is absolutely necessary to obtain data on the aerodynamic performance of finite wings. Otherwise, the results obtained using two-dimensional theories, even for wings of moderately high *aspect ratio* are bound to be incorrect.

Key Elements of Lifting Line Theory

We have already seen that in two dimensions, the governing equations of motion can be used to determine the incompressible, inviscid, flow solution about an arbitrarily shaped body up to a constant. The value of this constant is directly related to the circulation around the body, and therefore, to the total lift produced.

For inviscid flows, the Kutta condition was used to remove this arbitrariness and to yield accurate results in the computation of total lift.

With the Kutta-Joukowski theorem one can see that lift and circulation are intimately related by the equation

$$\mathbf{L} = \rho \mathbf{V}_\infty \times \mathbf{\Gamma} \tag{1}$$

In two dimensions, we had assumed that the circulation was created by an infinite vortex filament, perpendicular to the plane of the airfoil, located at the quarter chord. The vortex filament was allowed to proceed all the way to $+\infty$ and $-\infty$ without questioning what happened to it.

In the case where wings are finite (the wing ends at some well-defined tips), where do the vortex filaments go? Do they continue on to $\pm\infty$ or do they abruptly end in the fluid?

Arguments that can lead to possible solutions to this vortex filament path question can be found in Helmholz's vortex laws (consequences of Kelvin's theorem of circulation), which state that:

1. A vortex filament cannot end in the fluid. Vortex filaments must either propagate to infinity, or must arrange themselves to form closed loops.
2. The strength of a vortex filament must be constant along its length.

This can be shown by using Gauss' theorem on a cylindrical surface that encloses the filament for a certain distance.

3. The path of a vortex filament is such that it *follows* fluid particles. An alternative formulation to this law is the well-known Kelvin theorem.

The vortex filaments that run along the surface of the wing (producing a given circulation at any given spanwise station and therefore creating a given amount of sectional lift) cannot continue along the spanwise direction beyond the wing tips (according to the third theorem). The physical picture of what is occurring in the flow can then be *mathematically approximated* by a superposition of *horseshoe vortices* of varying spanwise extent that trail towards $-\infty$. The key elements of this theory are that

1. The lift distribution along the span of the wing is allowed to vary almost arbitrarily since we can superimpose an infinite number of vortex

filaments there, whose strengths add up to the desired circulation at any given spanwise station.

2. The trailing vortex system convects downstream mainly in the direction of the free stream, although they mutually interact with each other to distort or *roll-up* the shape of the wake.

3. The trailing vortex system generates a *downwash* at the wing that alters the local angle of attack of every wing cross section, therefore changing the amount of lift that it produces.

4. The local strength of the trailing vortex system is a function of the change in circulation in the spanwise direction at the wing

5. The lift distribution on the wing and the strength of the trailing vortex system are intimately related and cannot be chosen arbitrarily: they must

agree with each other as we will see later.

All of these observations lead to the mathematical modeling of finite wings according to the diagram in Figure 1.

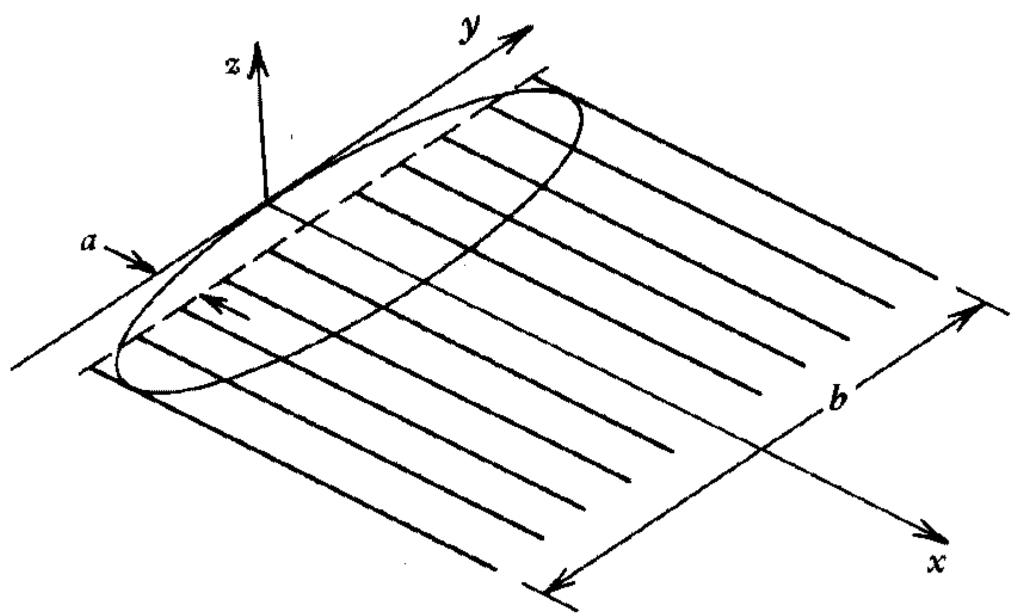


Figure 1: The Lifting Line Approximation

Using small angle approximations, the lift per unit width L' and the induced drag per unit width, D' can be calculated from the Kutta-Joukowski formula taking into account the fact that the incoming free stream is now deflected downwards by an amount that is determined by the strength of the *downwash velocity* $w(y)$, or by the corresponding *induced angle* $\alpha_i(y)$ according to:

$$\begin{aligned} L'(y) &\approx \rho V_\infty \Gamma(y) \\ D'_i(y) &\approx \rho w(y)\Gamma(y) \\ &\approx \rho V_\infty \Gamma(y)\alpha_i(y) \end{aligned} \tag{2}$$

The downwash velocity and the induced angle are related by the following equation

$$\alpha_i(y) \approx \frac{w(y)}{v_\infty} \tag{3}$$

Lifting line theory makes no particular assumption about the geometry of each of the cross sections that make up the wing. Instead, it assumes that the *sectional lift coefficient* at any given spanwise station has been computed (using thin airfoil theory or any other means) and can be expressed as follows:

$$c_l(y) = c_{l_\alpha}(\alpha(y) - \alpha_{L0}(y) - \alpha_i(y)) \tag{4}$$

where c_{l_α} is the lift curve slope (which could also be a function of y), $\alpha(y)$ is the geometric angle of attack, and α_{L0} is the zero-lift angle of attack (due to camber) of that particular cross-section. Since

$$c_l = \frac{L'}{\frac{1}{2}\rho_\infty V_\infty^2 c(y)}$$

where $c(y)$ is the local chord, then

$$\Gamma(y) = \frac{1}{2} c_{l_\alpha} V_\infty c(\alpha - \alpha_{L0} - \alpha_i) \tag{5}$$

This assumes that the flow is locally two-dimensional, and therefore, its validity improves at the aspect ratio, $AR = b^2/S$, increases. In this definition, b is the wing span, and S is its planform area.

Before we are able to compute the induced angle of attack $\alpha_i(y)$ from the strength of the trailing vortex system, we have to understand how the strength of each vortex filament in the wake relates to the strength of the circulation on the wing itself, $\Gamma(y)$. Notice from Figure 2 that the amount of vorticity shed into the wake is proportional to the decrease in circulation along the span of the wing.

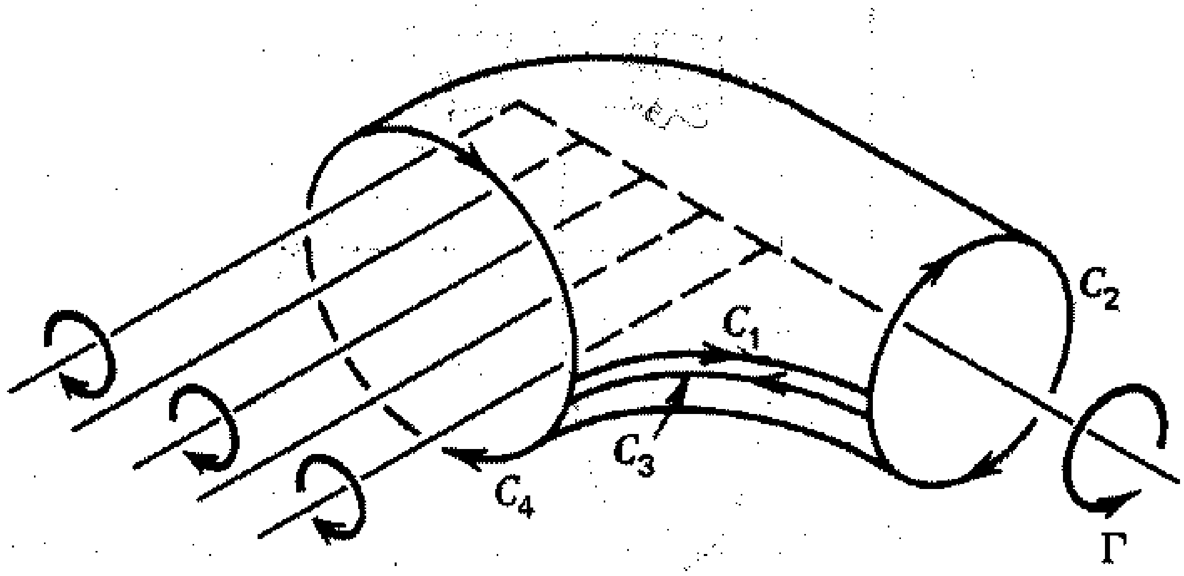


Figure 2: Downwash due to the trailing vortex system of a lifting line

In other words, it can be easily shown that

$$\gamma_x(y) = -\frac{d\Gamma(y)}{dy} \tag{6}$$

where $\gamma_x(y)$ is the strength of the vortex filament located at a spanwise station y. The x subscript in its definition simply indicates that the trailing vortices contain vorticity that is aligned with the free stream direction.

The computation of the induced angle of attack is greatly simplified in lifting line theory by the fact that the trailing vortex system is assumed to be aligned with the free stream (in a planar wake). By symmetry, the downwash at the bound vortex location is exactly *half* of what it would be should the trailing vortex system continue upstream of the wing to $-\infty$. If this were the situation, the downwash can be computed using effectively two-dimensional theory. See Figure 3 below for more details.

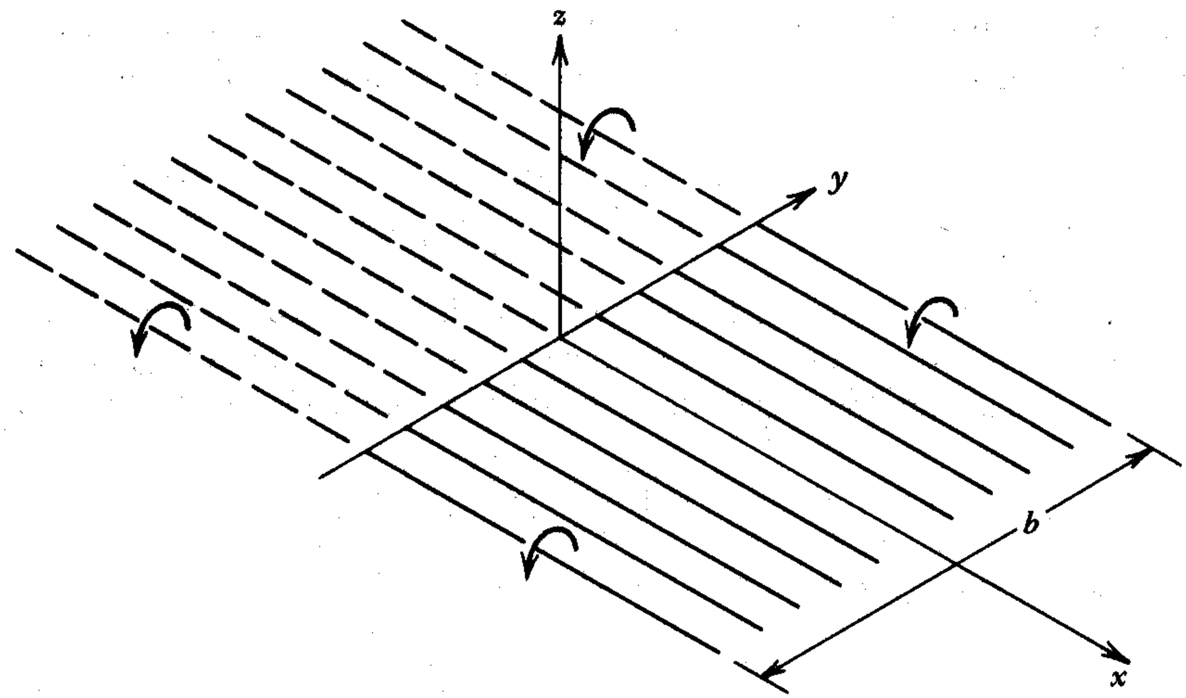


Figure 3: Quasi-Infinite System of Trailing Vortices

A view of this *infinite* trailing vortex system from downstream can be seen in Figure 4. The total downwash velocity generated at any given spanwise station can be calculated as

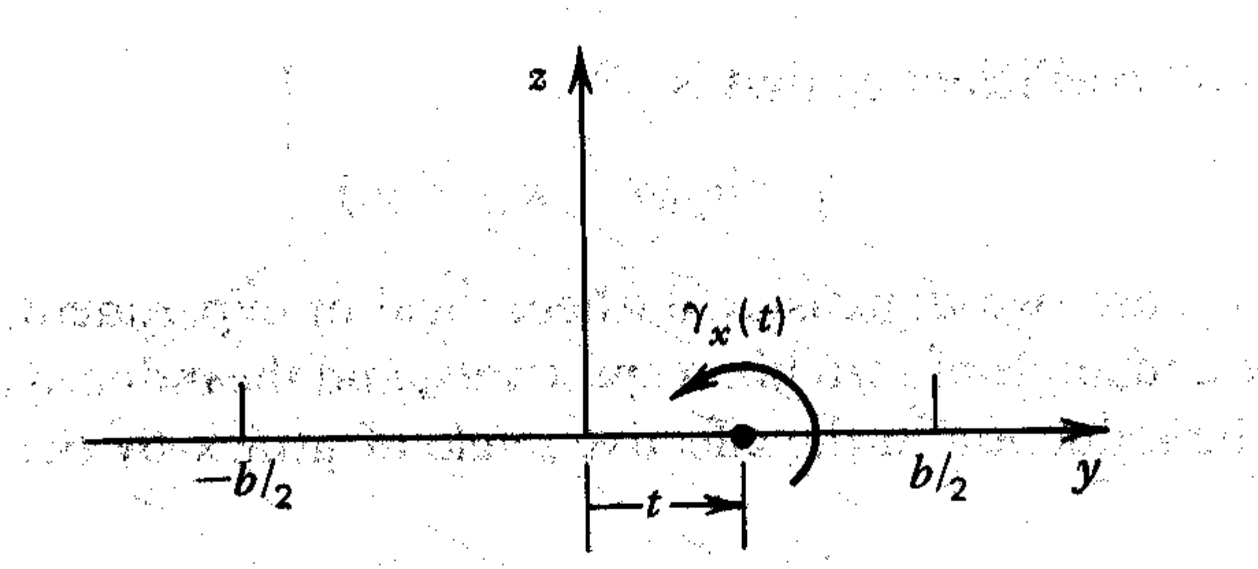


Figure 4: View of Downwash from Downstream

$$2w(y) = \frac{1}{2\pi} \fint_{-\frac{b}{2}}^{\frac{b}{2}} \frac{\gamma_x(t)dt}{t-y} \tag{7}$$

where the integral is a principal value integral in the same sense as discussed during our lectures on thin airfoil theory. Using the definitions of both $\gamma_x(y)$ and $\alpha_i(y)$ in Equations 6 and 3 we can see that

$$\alpha_i(y) = \frac{1}{4\pi V_\infty} \fint_{-\frac{b}{2}}^{\frac{b}{2}} \frac{\Gamma'(t)dt}{y-t} \tag{8}$$

and plugging into Equation 5 we have an *integro-differential* equation for $\Gamma(y)$ assuming that α_{L0}, c, and c_{l_α} are given as functions of y

$$\Gamma(y) = \frac{1}{2} V_\infty c\, c_{l_\alpha} \left[\alpha - \alpha_{L0} - \frac{1}{4\pi V_\infty} \fint_{-\frac{b}{2}}^{\frac{b}{2}} \frac{\Gamma'(t)dt}{y-t} \right] \tag{9}$$

The only unknown in this equation is $\Gamma(y)$. If we were able to solve Equation 9 for $\Gamma(y)$, we could compute every other quantity of interest as we shall show later.

Lifting Line Theory

We are now familiar with the basic concepts of *finite wing* theory, also called *lifting line* theory. Mathematical modelling of a lifting wing via the superposition of a number of *horseshoe* vortices that obey Helmholz's laws yields a good approximation to the flow around a wing of high apect ratio. The key elements of this theory are can be described as follows:

> "The main problem of finite-wing theory is the determination of the distribution of airloads on a wing of given geometry flying at a given speed and orientation in space. The analysis is based on the assumption that at every point along the span the flow is essentially two-dimensional; thus the resultant force per unit span at any point is that calculated for the airfoil using two-dimensional thin airfoil theory, *but* at an angle of attack corrected for the influence of the trailing vortex configuration that itself depends on the spanwise variation of

the lift distribution on the wing."

Figure 5 below presents a good summary of the various elements involved in this theory that we have discussed so far. The key thing to understand is that there is strong feedback between a lift distribution and its wake: the spanwise derivative of the wing lift distribution determines the local strength of the wake, which, in turn, determines the distribution of downwash at the wing, which, again, determines the lift distribution along the span of the wing. The key is to determine a distribution of lift/circulation along the span of the wing which is in *harmony* with the downwash created by its wake. Once this is accomplished all aerodynamic performance parameters can be derived, including the total lift and drag of the wing, the downwash distribution and the deviation of the produced lift distribution from the optimum one. More importantly, lifting line theory can tell us about the effect of variations in the wing planform and twist distributions on the performance of the actual wing.

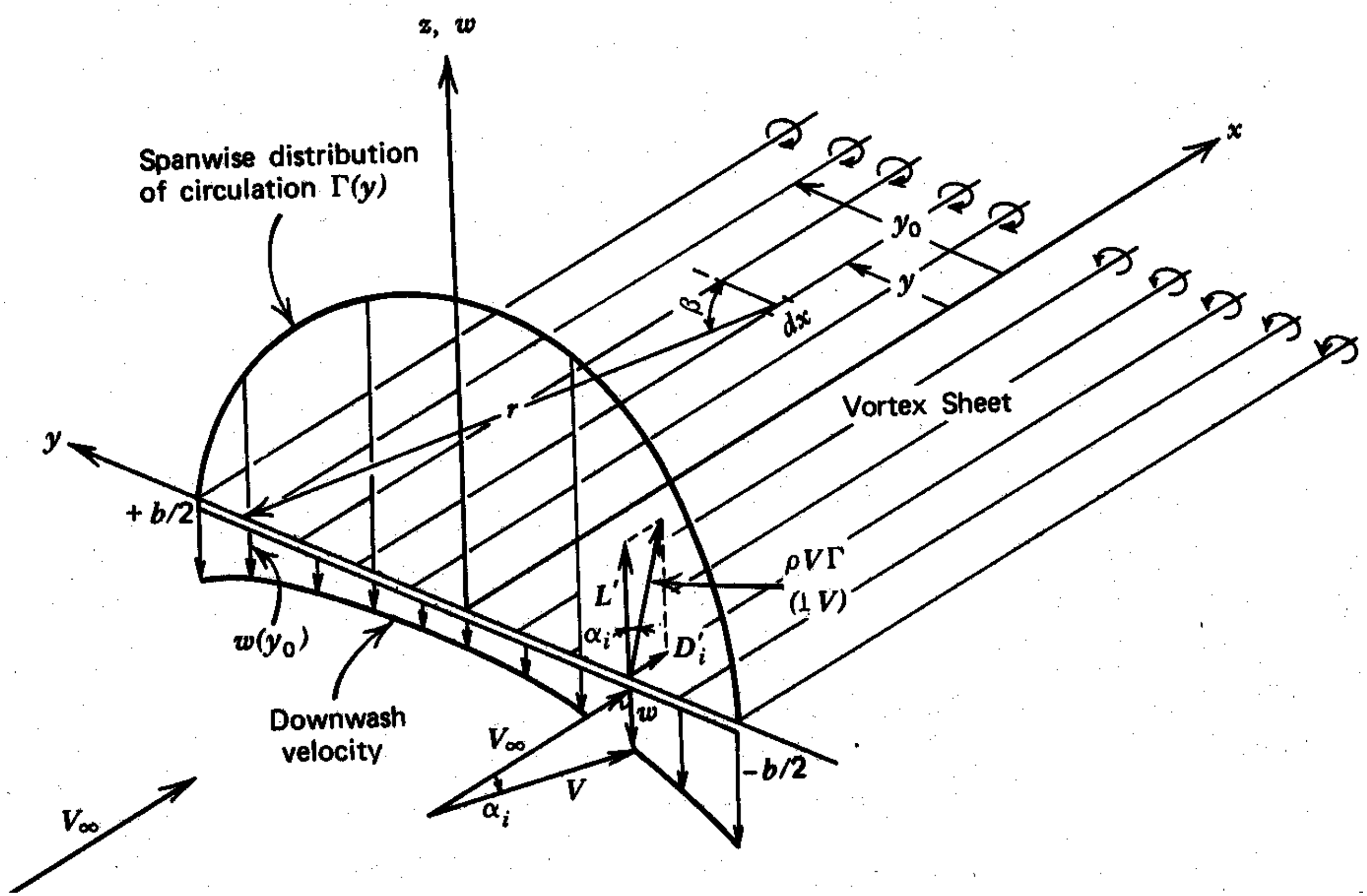


Figure 5: Summary of Key Elements of Lifting Line Theory

Consider the unknown spanwise distribution of circulation to be given by $\Gamma(y)$. The strength of the circulation at any point of the wake is then given by

$$\gamma_x(y) = \frac{d\Gamma(y)}{dy}.$$

The net effect of this trailing vortex system on any point on the *bound vortex* is a downwash, $w(y)$, whose magnitude at at each point is given by the integrated effect of the continuous distribution of semi-infinite vorticity over the range $-b/2 < y < b/2$. Notice from the Figure that resultant velocity at the wing has two components V_∞ and w, at each point, which define the *induced angle of attack*

$$\alpha_i(y) = \tan^{-1} \frac{w(y)}{V_\infty} \tag{10}$$

Using the Kutta-Joukwoski law, the force per unit span on the bound vortex has magnitude $\rho V \Gamma$ and it is normal to V, that is, it is inclined with respect to the z-axis at an angle α_i. Using small angle approximations, these various quantities can be shown to be

$$\begin{aligned} \alpha_i(y) &= \frac{w}{V_\infty} \\ L'(y) &= \rho V_\infty \Gamma \\ D_i'(y) &= -L'\alpha_i = -\rho w \Gamma \end{aligned}$$

We have shown that the net effect of the wake on the wing is the downwash velocity which can be calculated as

$$\alpha_i(y) = \frac{w(y)}{V_\infty} = -\frac{1}{4\pi V_\infty} \text{P}\!\!\int_{-\frac{b}{2}}^{\frac{b}{2}} \frac{\frac{d\Gamma(t)}{dt}}{y-t} dt \tag{11}$$

We had also seen that

$$c_l(y) = c_{l_\alpha}(\alpha(y) - \alpha_{L0}(y) - \alpha_i(y)) \tag{12}$$

where c_{l_α} is the lift curve slope (which could also be a function of y), $\alpha(y)$ is the geometric angle of attack, and α_{L0} is the zero-lift angle of attack (due to camber) of that particular cross-section. Since

$$c_l = \frac{L'}{\frac{1}{2}\rho_\infty V_\infty^2 c(y)}$$

where $c(y)$ is the local chord, then

$$\Gamma(y) = \frac{1}{2} c_{l_\alpha} V_\infty c(\alpha - \alpha_{L0} - \alpha_i) \tag{13}$$

and plugging in Equation 13 into Equation 11 we obtain the governing equation for $\Gamma(y)$ as

$$\Gamma(y) = \frac{1}{2} V_\infty c \, c_{l_\alpha} \left[\alpha - \alpha_{L0} - \frac{1}{4\pi V_\infty} \mathrm{P}\!\!\int_{-\frac{b}{2}}^{\frac{b}{2}} \frac{\Gamma'(t) dt}{y - t} \right] \tag{14}$$

Elliptical Lift Distribution

Equation 14 is easily solved for $c(y)$ if we were given $\Gamma(y)$. This procedure involves the solution of an algebraic equation, rather than an integral equation. The most important case of this type of solution procedure is that of the elliptical lift distribution. Suppose that Γ_s represents the value of the circulation at the plane of symmetry. Then, an elliptical lift distribution can be written as

$$\Gamma = \Gamma_s \sqrt{1 - \left(\frac{y}{b/2}\right)^2}. \tag{15}$$

The induced angle of attack created by this elliptical load distribution can be computed by subtituting Equation 15 into Equation 11. One can then show that

$$\alpha_i(y) = -\frac{\Gamma_s}{4\pi V_\infty} \mathcal{P}\!\!\int_{-\frac{b}{2}}^{\frac{b}{2}} \frac{\frac{d}{dt}\sqrt{1 - \left(\frac{t}{b/2}\right)^2}}{y - t} dt$$

Using the Glauert trigonometric substitution that we have seen several times by now, we can show that

$$\alpha_i(\theta_0) = \frac{\Gamma_s}{2\pi b V_\infty} \mathrm{P}\!\!\int_0^\pi \frac{\cos\theta d\theta}{\cos\theta_0 - \cos\theta}$$

which can be shown to be equal to a constant

$$\alpha_i = -\frac{\Gamma_s}{2bV_\infty} \tag{16}$$

Therefore, the angle of attack of every airfoil section on the wing is constant (assuming no geometric twist). If we further assume that the sectional lift-curve slope is independent of y, both the sectional lift and drag coefficient will also be independent of y. Therefore

$$c_l = c_{l_\alpha}\alpha_0$$

$$c_{d_i} = \frac{D_i'}{q_\infty c} = -c_l \alpha_i$$

Using these conditions, the product of $c_{l_\alpha} \alpha_0 c$ must vary elliptically, since

$$L' = \rho V_\infty \Gamma_s \sqrt{1 - \left(\frac{y}{b/2}\right)^2} = c_{l_\alpha} \alpha_0 \frac{1}{2} \rho V_\infty^2 c$$

$$c_{l_\alpha} \alpha_0 c = \frac{2\Gamma_s}{V_\infty} \sqrt{1 - \left(\frac{y}{b/2}\right)^2}$$

Assuming that $c_{l_\alpha} \alpha_0$ is a constant (untwisted wing of identical cross-sectional properties) this indicates that in order to obtain an elliptically loaded wing, we must have an elliptical planform. If the planform were not elliptical, an elliptical load distribution can still be achieved by making sure that the wing be twisted in such a way that the product of $c_{l_\alpha} \alpha_0 c$ still varied elliptically. Notice that although this is fine in theory, extreme taper ratios may lead to excessively high sectional lift coefficients in the outboard wing sections that may exceed the local $c_{l_{max}}$ and lead to stall.

The wing properties of this elliptical lift distribution can be computed by integrating the spanload variations as follows

$$C_L = \frac{1}{\frac{1}{2}\rho V_\infty^2 S} \int_{-\frac{b}{2}}^{\frac{b}{2}} L' dy = \frac{\Gamma_s \pi b}{2 V_\infty S}$$

The wing lift coefficient and the sectional lift coefficient are equal when the sectional lift coefficients are constant along the span, and therefore, $C_L = c_l$.

Solving for Γ_s in the expression above, and substituting into the expression for the induced angle of attack we find that

$$\alpha_i = -\frac{C_L}{\pi AR} = -\frac{c_l}{\pi AR}$$

and then, the wing induced drag coefficient is given by

$$C_{D_i} = c_{d_i} = -C_L \alpha_i = \frac{C_L^2}{\pi AR}$$

The expression for the lift curve slope is then given by

$$C_{L_\alpha} = \frac{c_{l_\alpha}}{1 + c_{l_\alpha}/\pi AR}$$

$\Gamma(y)$ Solution for Arbitrary Lift Distributions

Assume that you are given the geometry of the wing and the two-dimensional lift curve slope of each of its cross sections. That is, the quantities α_{L0}, $c(y)$, and $c_{l_\alpha}(y)$ are known. What is the best approach to solve for $\Gamma(y)$? The solution procedure should bring about a sense of *déja-vu* from thin airfoil theory.

Let us transform the spanwise coordinate y into a polar variable θ according to

$$
\begin{aligned}
t &= \frac{b}{2}\cos\theta \\
y &= \frac{b}{2}\cos\theta_0
\end{aligned}
$$

Then,

$$\Gamma(y) = g(\theta_0) = \frac{1}{2} c_{l_{\alpha_s}} c_s V_\infty \sum_{n=1} A_n \sin n\theta_0 \tag{17}$$

where the constant outside the summation is included for convenience and the s subscripts indicate values at the *symmetry plane*. Note that the A_n coefficients are all non-dimensional, since the circulation Γ has units of $L\ L/T$. In addition

$$\begin{aligned} \Gamma'(t)dt &= \frac{d\Gamma}{d\theta}\frac{d\theta}{dt}dt = g'(\theta)d\theta \\ &= \frac{1}{2} c_{l_{\alpha_s}} c_s V_\infty \sum_{n=1} nA_n \cos n\theta d\theta \end{aligned}$$

and

$$⨍_{-\frac{b}{2}}^{\frac{b}{2}} \frac{\Gamma'(t)dt}{y-t} = \frac{1}{2} c_{l_{\alpha_s}} c_s V_\infty \sum_{n=1} nA_n ⨍_{\pi}^{0} \frac{\cos n\theta d\theta}{\frac{b}{2}(\cos\theta_0 - \cos\theta)}$$

You may remember the value of the following principal value integral from thin airfoil theory

$$P\int_0^{\pi} \frac{\cos n\theta}{\cos\theta - \cos\theta_0} = \frac{\pi \sin n\theta_0}{\sin\theta_0}$$

which can be used to evaluate the integral and, substituting the result and Equation 17 into Equation 14 we get

$$\sum_{n=1} A_n \sin n\theta_0 = \frac{c\ c_{l_\alpha}}{c_s c_{l_{\alpha s}}}(\alpha - \alpha_{L0}) - \frac{c\ c_{l_\alpha}}{4b} \sum_{n=1} n A_n \frac{\sin n\theta_o}{\sin\theta_0} \qquad (18)$$

which must be solved for the values of A_n. This is typically done with the *collocation method* by truncating the series after a certain value of n and evaluating the expression at n different values of θ_0 to create a linear system of n equations in n unknowns A_n. Unfortunately, the only way to check if your truncation was too crude is to repeat the procedure with a higher value of n and compute the difference between the results.

Assuming that we are able to solve the Equation 18 for the unknown coefficients of the circulation distribution A_n, we can now use these values to compute the lift and drag coefficients of the wing. Firstly, the sectional lift and drag coefficients can be found to be

$$
\begin{aligned}
c_l &= \frac{\rho V_\infty \Gamma}{q_\infty c} = \frac{c_{l_{\alpha_s}} c_s}{c} \sum_{m=1}^{\infty} A_n \sin n\theta \qquad (19) \\
c_{d_i} &= -c_l \alpha_i = \frac{c_{l_{\alpha_s}}^2 c_s^2}{4bc} \left(\sum_{n=1}^{\infty} A_n \sin n\theta \right) \left(\sum_{k=1}^{\infty} k A_k \frac{\sin k\theta}{\sin\theta} \right)
\end{aligned}
$$

which can be integrated to yield

$$
C_L = \int_{-\frac{b}{2}}^{\frac{b}{2}} \frac{c_l q_\infty c dy}{q_\infty S} = \frac{c_{l_{\alpha_s}} c_s}{S} \int_0^\pi \sum_{n=1}^{\infty} A_n \sin n\theta \frac{b}{2} \sin\theta d\theta
$$

The integration is carried out easily by switching the orders of integration and summation and noting that

$$\int_0^{\pi} \sin n\theta \sin k\theta d\theta = \begin{cases} 0 & \text{for } n \neq k \\ \frac{\pi}{2} & \text{for } n = k \end{cases}$$

Since $k = 1$ in all of the terms of the interal above, all terms vanish except for the $n = 1$ term and therefore we have

$$C_L = \frac{c_{l_{\alpha_s}} c_s \pi b}{4S} A_1 \tag{20}$$

and therefore, the wing lift coefficient for an arbitrary symmetrical circulation distribution is proportional to A_1 and independent of all other Fourier coefficients. For an elliptical lift distribution C_L was proportional to Γ_s which is the only surviving term in the Fourier expansion.

Similarly, the coefficient of drag of the wing can be computed as follows

$$C_{D_i} = \int_{-\frac{b}{2}}^{\frac{b}{2}} \frac{c_{d_i} q_\infty c dy}{q_\infty S} = \frac{c_{l_{\alpha_s}}^2 c_s^2}{8S} \int_0^\pi \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} k A_n A_k \sin n\theta \sin k\theta d\theta$$

which can be shown to be

$$C_{D_i} = \frac{c_{l_{\alpha_s}}^2 c_s^2 \pi}{16S} \sum_{n=1}^{\infty} n A_n^2$$

which proves that, for a given lift coefficient ($A_1 \neq 0$), the optimum lift distribution (the one that yields minimum induced drag) is the one that minimizes C_{D_i}. In other words, all A_i except for A_1 must be equal to zero. This corresponds to the elliptic lift distribution that we have been discussing earlier.

Interestingly, using the result for C_L in Equation 20, one can show that the induced drag coefficient for the wing is given by

$$C_{D_i} = \frac{C_L^2}{\pi AR} \sum_{n=1}^{\infty} n \left(\frac{A_n}{A_1} \right)^2$$

which is simply the result for the elliptically loaded wing with a correction factor that is sometimes called the span efficiency factor, e, defined as

$$\frac{1}{e} = \sum_{n=1}^{\infty} n \left(\frac{A_n}{A_1} \right)^2$$

Wings with Symmetric Loading

In the case of symmetric loadings, the situation is simplified considerably. If we can safely assume that

$$\Gamma(y) = -\Gamma(-y)$$

then all of the even coefficients of the series expansions, A_2, A_4, ... are zero.

Comparison with Experiment

Classical results on rectangular planforms reported by Prandtl can be seen in the following Figures. Seven wings of aspect ratios between 1 and 7 were tested in a wind tunnel. Figure 6 shows the variation of C_L vs the geometric angle of attack. In Figure 6 the results have also been corrected by the induced angle for $AR = 5$

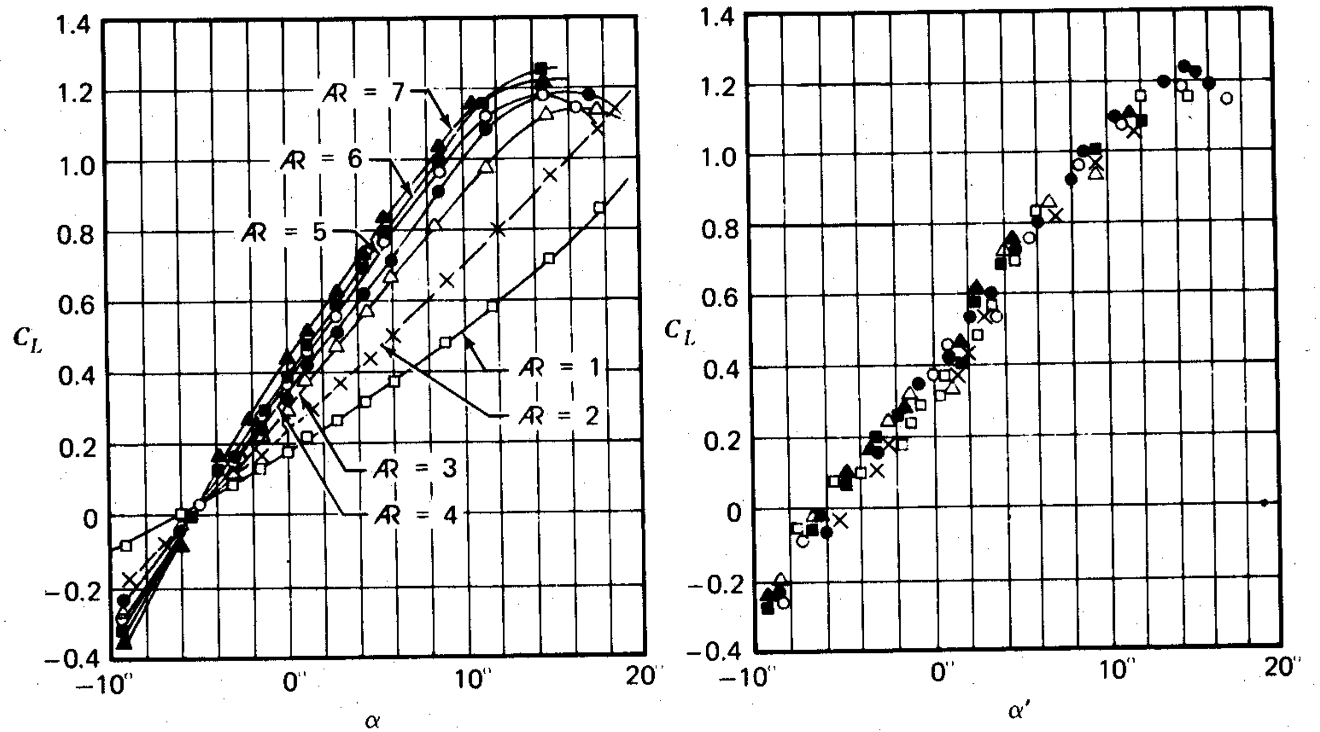


Figure 6: C_L vs. α and α' for $1 \leq AR \leq 7$

$$\alpha' = \alpha + \frac{C_L}{\pi}\left(\frac{1}{5} - \frac{1}{AR}\right)$$

Figure 7 show the drag polars for each of these wings, where

$$C_D = C_{D_0} + C_{D_i}$$

In Figure 7 these graphs have also been corrected to $AR = 5$ so the new drag coefficient, $C_{D'}$ becomes

$$C_{D'} = C_D + \frac{C_L^2}{\pi}\left(\frac{1}{5} - \frac{1}{AR}\right)$$

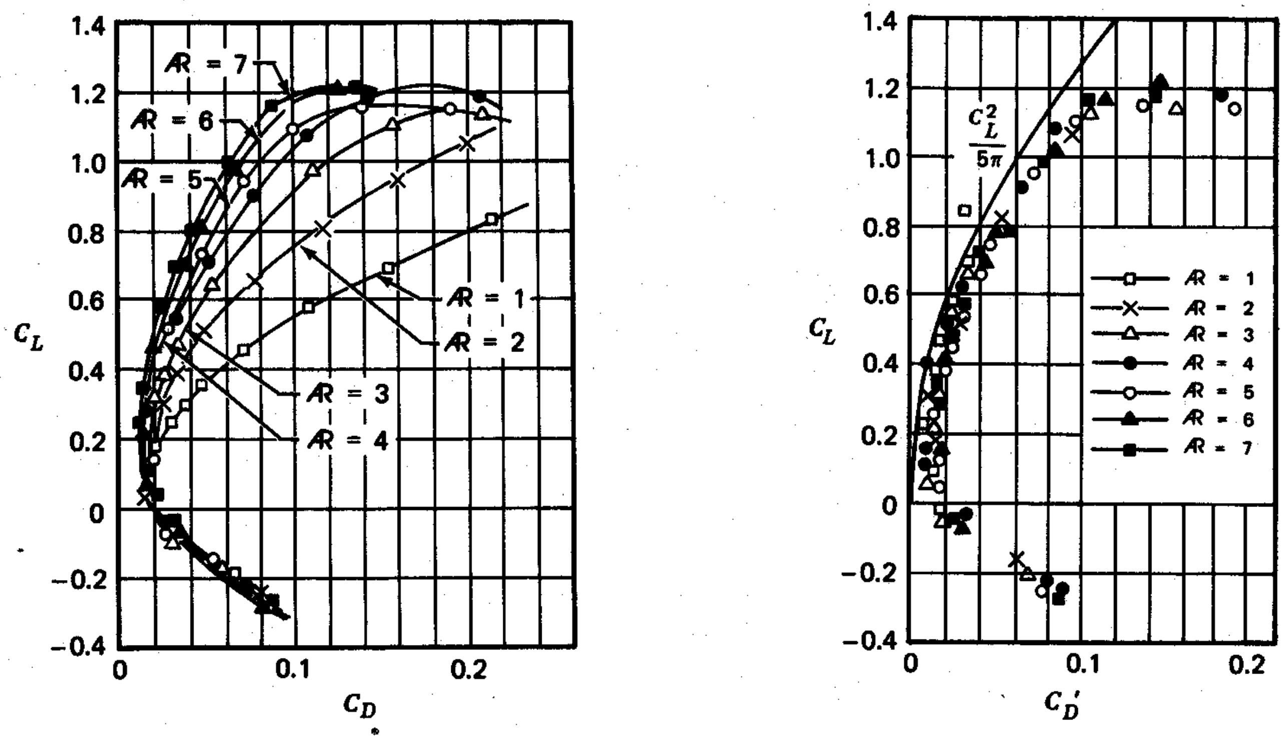


Figure 7: C_L vs. C_D and $C'_D$for $1 \leq AR \leq 7$

The corrected Figures show that there is indeed significant departure from the theory for aspect ratios under $AR = 3$. If AR is larger than this value, significant departures from elliptical wing loading can still be tolerated by this theory. In fact, it is typical to introduce the *span efficiency factor*, s, to account for some of these deviations so that

$$C_{D_i} = \frac{C_L^2}{\pi A R s}$$

where $s = 1.0$ for a perfectly elliptically loaded wing, while it is smaller (typically in the range $0.9 \leq s \leq 1.0$) for wings with lift distributions that deviate from the elliptical load.