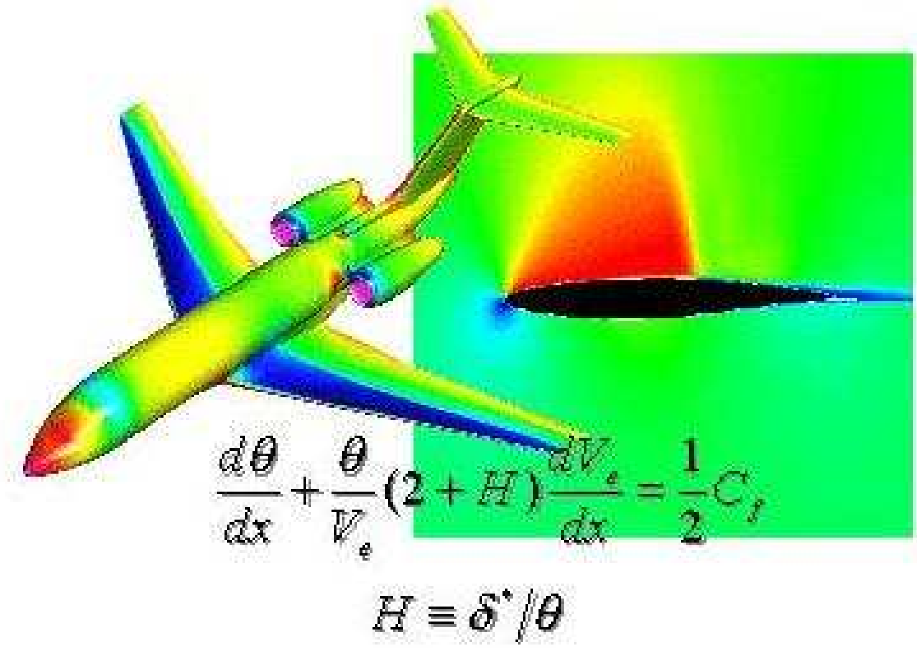
Induced Drag Minimization on Arbitrtrary Configurations



AA200b

Lecture 17

March 1, 2005

Induced Drag Minimization

As you can see from the paper handed out in class (by S. Smith and I. Kroo), it turns out that by using Trefftz-plane methods, the *induced drag*, D_i, of a given aircraft configuration can be represented as a quadratic form of either the vortex element strengths or the geometric design variables such as camber and twist of the various sections along the span of the wing/winglets:

$$D_i = \frac{\rho}{2}\int_{-\frac{b}{2}}^{\frac{b}{2}} \mu\omega ds = \rho\int_{0}^{\frac{b}{2}} \mu\omega ds \qquad (1)$$

where μ is the jump in potential across the wake and $\omega = \frac{\partial\phi}{\partial n}$ is the *normalwash*, or the component of the induced velocity normal to the wake surface.

If the wing is discretized in such a way that the wake is composed of a number of segments or panels, N, this equation can be discretized as follows:

$$D_i = \rho \sum_{i=1}^{N} \mu_i \omega_i \Delta s_i, \tag{2}$$

where the values of the potential jump and the normalwash, μ_i and ω_i are computed at the midpoint of each panel, and Δs_i is simply the length of the panel. More sophisticated integration procedures can also be used; the rectangular rule of integration will be used in these notes for simplicity.

In order to compute the induced drag of a configuration all that needs to be done is to carry out the summation in Equation 2, but for that we have to devise a procedure to calculate μ_i, ω_i, and Δs_i, $i = 1, \ldots, N$. The same procedure that we will outline to compute these quantities, will be used to calculate the necessary information for the calculation of the matrices in the drag minimization problem.

Figure 1 shows the details of the configuration we will discuss in this handout. Although the wing in this particular situation is drawn as being planar there is no need for such requirement. In fact, the whole derivation is valid for non-planar wings (such as wing/winglet combinations) such as the one you are computing in PS4.

As we saw in a previous lecture, Equation 1 is valid under the assumption of a *force-free* wake, or a wake the is parallel to the free stream direction. Since we are not in a position to carry out a detailed calculation of the shape of the force-free wake, we will assume that the wake preserves the shape it had at the trailing edge of the configuration, and that it travels downstream in the direction of the free stream until it reaches the Trefftz-plane.

When using LinAir, the physical location of the wake can be found from the fact that every wing element used to define the configuration is discretized with the specified number of *equal-sized* panels. Therefore, if you know the exact location of the trailing edge and the number of panels in each element, you can work out the $(y_i, z_i),\ i = 1, \ldots, N+1$ locations

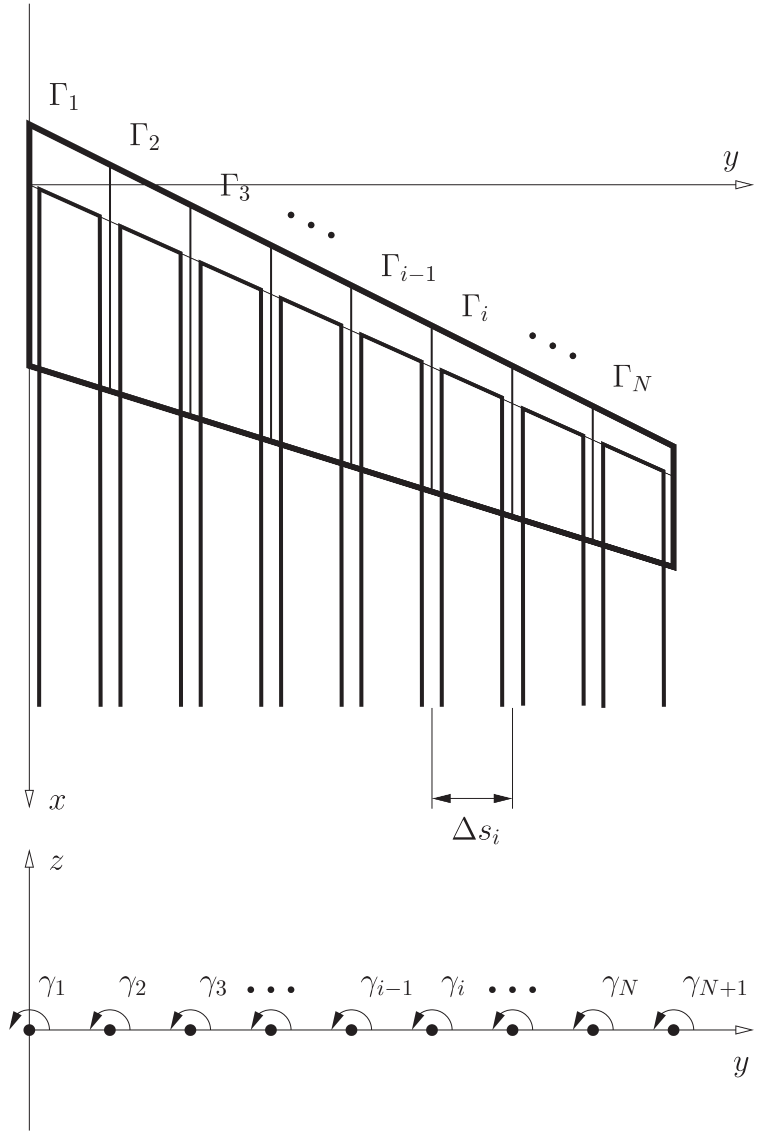


Figure 1: Sample Sweptback Wing and Its Wake

of the endpoints of each panel, starting at the symmetry plane and going all the way to the tip of the configuration.

Once the coordinates of the endpoints of each panel are known, you can work out the panel lengths by simply plugging in their values into the following formula

$$\Delta s_i = \sqrt{(y_{i+1} - y_i)^2 + (z_{i+1} - z_i)^2}$$

Computation of the Normalwash, ω_i

Under the assumption that the Trefftz-plane is *far enough* downstream of the trailing edge of the wing, and, assuming that the trailing vortices propagate a *long* distance behind the Trefftz-plane, the induced velocities in the Trefftz-plane can be considered to be created by two-dimensional point vortices located at the intersection of the trailing vortices and the Trefftz-plane. The situation is depicted in the lower portion of Figure 1.

The tangential velocity created by a point vortex of strength γ_j a distance r_{ij} away from the location of the vortex is given by

$$V_{ij} = \frac{\gamma_j}{2\pi r_{ij}}$$

Instead of a single vortex, we have $N + 1$ discrete vortices, all of which contribute in different measures to the total downwash at any given point along the length of the wake.

From the setup in Figure 2 we can see that the normalwash created by the $j-th$ vortex $(j = 1, \ldots, N+1)$ at the center of the $i-th$ panel $(i = 1, \ldots, N)$ depends on

1. The distance r_{ij} between the two points in question.

2. The angle β_{ij} between the line that joins the point vortex and the center of the panel in question and the panel itself.

3. The strength of the $j-th$ point vortex, γ_j.

The normalwash created by this single vortex at the center of panel i, ω_{ij}, is given by:

$$\omega_{ij} = \frac{\gamma_j}{2\pi r_{ij}} \cos \beta_{ij}$$

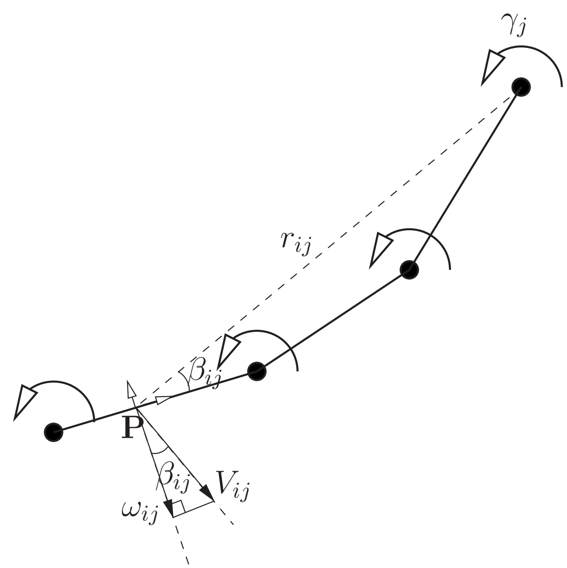


Figure 2: Normalwash Induced by a Point Vortex of Strength γ_j at the Center of Panel i.

Notice that everything is geometry dependent, except for γ_j and therefore, you can precompute the angles and distances if you wish. Finally, the normalwash at the center of panel i (point P) contributed by the left half of the wing is calculated from the contributions of all of the point vortices in the Trefftz-plane as follows

$$\omega_i = \sum_{j=1}^{N+1} \omega_{ij} = \sum_{j=1}^{N+1} \frac{\gamma_j}{2\pi r_{ij}} \cos \beta_{ij} \tag{3}$$

Notice that the sum must be extended over all of the point vortices in the wake (including both sides of the wing) and therefore, the mirror image of the wake has to be constructed, and the angles and distances from all of the point vortices need to be constructed to yield

$$\omega_i = \sum_{j=-(N+1)}^{N+1} \frac{\gamma_j}{2\pi r_{ij}} \cos\beta_{ij}$$

where, in this notation, the vortex represented by $j = 0$ does not exist, and the vortices at $j = \pm 1$ should be counted only once. In fact, $j = \pm 1$ does not present a problem for symmetrically loaded wings, since the vortex strength $\gamma_{\pm 1} = 0$.

Furthermore, note that the direction of the circulation in the trailing vortex wake of the left half of the wing is exactly the opposite to that of the original portion, and therefore the signs have to be reversed so that

$$\gamma_{-i} = -\gamma_i$$

Since LinAir is providing you with quantities that are related to the local coefficient of lift on the wing, you can easily obtain the value of the circulation of each horseshoe vortex in Figure 1. However, in order to calculate the normalwash, you need the value of the circulation in the *trailing vortices*. Since the trailing portions of adjoining horseshoe vortices lie directly on top of each other, their circulations (with the proper sign) can be added to yield (for symmetrically loaded wings)

$$\begin{aligned} \gamma_1 &= 0 \\ \gamma_i &= \Gamma_{i-1} - \Gamma_i \quad i = 2, \ldots, N \\ \gamma_{N+1} &= \Gamma_N \end{aligned} \tag{4}$$

In other words, the circulation strength in the wake is directly related to the change in circulation along the span of the lifting configuration.

Computation of the Potential Jump, μ_i

We have now outlined a procedure to compute the normalwash at the center of each panel in the wake. In order to complete the necessary information to calculate the total induced drag, D_i, in Equation 2 we still need to obtain the values of the potential jump at the center of each wake panel, $\mu_i, \ \ i = 1, \ldots, N$.

From the theory of potential flow methods, one can show that if you were to place a contour around a given spanwise section of our lifting surface the change in potential between two points P_1 and P_2 on the contour was given by

$$\phi(P_2) - \phi(P_1) = \int_{P_1}^{P_2} \nabla\phi \cdot \mathbf{dl} = \int_{P_1}^{P_2} \mathbf{V} \cdot \mathbf{dl}$$

If we allow P_2 to go completely around the closed contour so that it lies directly on top of P_1, we would have that

$$\phi(P_1) - \phi(P_1) = \oint_{\mathcal{C}} \mathbf{V} \cdot \mathbf{dl} = \Gamma$$

which leads to a double valued potential function. This presents a problem that can be resolved by allowing the potential to be discontinuous across a *branch cut* in the domain that we can place along the wake surface. If the points P_2 and P_1 were placed directly above and below the wake at any given spanwise station of a lifting configuration, the *jump* in potential would be equal to the circulation of that particular spanwise station

$$\Delta\phi = \phi(P_2) - \phi(P_1) = \oint_{\mathcal{C}} \mathbf{V} \cdot \mathbf{dl} = \Gamma$$

This formula is a slight approximation since it neglects the contribution of adjoining pieces of the bound vortex to the circulation at a given station.

Therefore, for our optimization problem, the jump in the potential at the center of a given panel i, is simply given by the circulation on the bound vortex of that particular panel, Γ_i. That is

$$\mu_i = \Gamma_i$$

Optimization Problem Setup

Now that you can compute the induced drag of the configuration, and the components of the induced drag (both normalwash and potential jump at every span station) you are in a position to use this information to carry out a winglet (or any other planar/nonplanar wing) shape optimization.

The condition for an unconstrained minimization problem is that the derivative of the cost function to be minimized, D_i, with respect to all of the design variables in the problem be identically zero. Since our cost function is a quadratic form, its first derivative is a linear function of the design variables that can be easily expressed with a matrix-vector multiplication. Therefore, under the conditions stated above, the optimization problem requires only a simple matrix inversion. This is what you are asked to do in PS4.

Notice that the optimum solution (if the changes in the design variables allowed for it) is one where the configuration generates zero lift and therefore zero drag. However optimum this may be, this is not a satisfactory solution. We must minimize D_i subject to the constraint that the lift generated by the aircraft be constant. In order to enforce this constraint, a new *augmented Lagrangian* cost function, J, is constructed such that the constrained problem can be treated as unconstrained if an additional design variable, the Lagrange multiplier λ, is introduced into the formulation.

The rest of the derivation in this handout follows closely the paper by S. Smith that has been mentioned before. Please refer to it for more details.

Notice that the problem definition states that the two design variables of the problem are the root and tip incidence angles of the winglet portion of the configuration. However, in order to make it possible to maintain the lift of the configuration, we must also allow for the angle of attack of the wing/winglet to vary. This yields a total of three design variables.

The normalwash and potential jump at the center of each wake panel can be expanded *exactly* into a first order Taylor series (since the functions are known to be linear) of the variations in the design variables as

$$\mu_i = \mu_{i0} + \sum_j \frac{\partial \mu_i}{\partial x_j} \delta x_j \tag{5}$$

$$\omega_i = \omega_{i0} + \sum_j \frac{\partial \omega_i}{\partial x_j} \delta x_j$$

Notice that the design variables in this derivation are chosen to be the *perturbations* of the three angles with respect to the baseline values. Plugging our expansions for μ_i and ω_i into Equation 2 we obtain the following matrix equation

$$D_i = \{x\}^T [DIC] \{x\} + \{DIC_0\}^T \{x\} + D_0 \tag{6}$$

where the elements of the matrix of drag influence coefficients are given by

$$[DIC]_{jk} = \rho \sum_{i=1}^{N} \frac{\partial \mu_i}{\partial x_j} \frac{\partial \omega_i}{\partial x_k} \Delta s_i$$

and

$$\{DIC_0\}_j = \rho \sum_{i=1}^{N} \left(\mu_{i0} \frac{\partial \omega_i}{\partial x_j} + \omega_{i0} \frac{\partial \mu_i}{\partial x_j} \right) \Delta s_i$$

Lift Constraint

To avoid the trivial solution to the optimization problem, we must enforce a lift constraint. A Trefftz-plane expression for the lift of the configuration is given by

$$L = 2\rho \int_0^{\frac{b}{2}} \mu dy \tag{7}$$

which can be discretized as

$$L = \{LIC\}^T \{x\} + L_0 \tag{8}$$

where the *lift influence coefficients* are given by

$$\{LIC\}_j = 2\rho \sum_{i=1}^{N} \frac{\partial \mu_i}{\partial x_j} \Delta y_i$$

With this definition of lift, an augmented Lagrangian cost function is created

$$J = D_i + \lambda(L - L_{con}) \tag{9}$$

where L_{con} is the value of the lift that the configuration is supposed to achieve and λ is the lagrange multiplier that is simply there to assist us in the optimization by effectively making it unconstrained. If we now differentiate the cost function in Equation 9 with respect to the three design variables in the problem and the Lagrange multiplier we obtain

$$\begin{bmatrix} DIC + DIC^T & LIC \\ LIC & 0 \end{bmatrix} \left\{ \begin{array}{c} x \\ \lambda \end{array} \right\} = \left\{ \begin{array}{c} -DIC_0 \\ L_{con} - L \end{array} \right\} \tag{10}$$

which can be solved for the values of the *increments* in the design variables and, if you care, the value of the Lagrange multiplier, λ.

Sensitivity Computation

The only thing that remains to be explained in order to be able to setup the optimization problem is the procedure that needs to be followed in order to compute the values of the derivatives $\frac{\partial \mu_i}{\partial x_j}$ and $\frac{\partial \omega_i}{\partial x_j}$. They effectively represent the change in either the jump in potential or the normalwash at a given section when a change in one of the design variables is made. In theory, these sensitivities could be calculated analytically if you had the source code of the panel method that is (directly or indirectly) producing them. For simplicity (and since we only have a small number of design variables), we will use the finite-difference method.

The finite-difference method makes small steps in each of the deisgn variables in turn and recomputes the function of interest to determine the change in its value. For example, for the jump in potential we can say that

$$\frac{\partial \mu_i}{\partial x_j} \approx \frac{\mu_i(x_j + \delta x_j) - \mu_i(x_j)}{\delta x_j} + \mathcal{O}(\delta x_j)$$

Since this approximation is first order accurate, $\mathcal{O}(\delta x_j)$, we would typically want a very small step to minimize the truncation error in the approximation. However, very small steps lead to subtractive cancellation so they cannot be tolerated. In addition, the output you are getting from LinAir does not have many significant digits, so for very small steps you would not be able to even see the difference in the function!

Furthermore, we are dealing with a simple dependence of μ_i and ω_i on the design variables: their variations are supposed to be *linearly dependent* on the variation of the design variables. Therefore, the step size can be made quite large and, in this PS, a step of around $1 - 2^\circ$ is suggested.