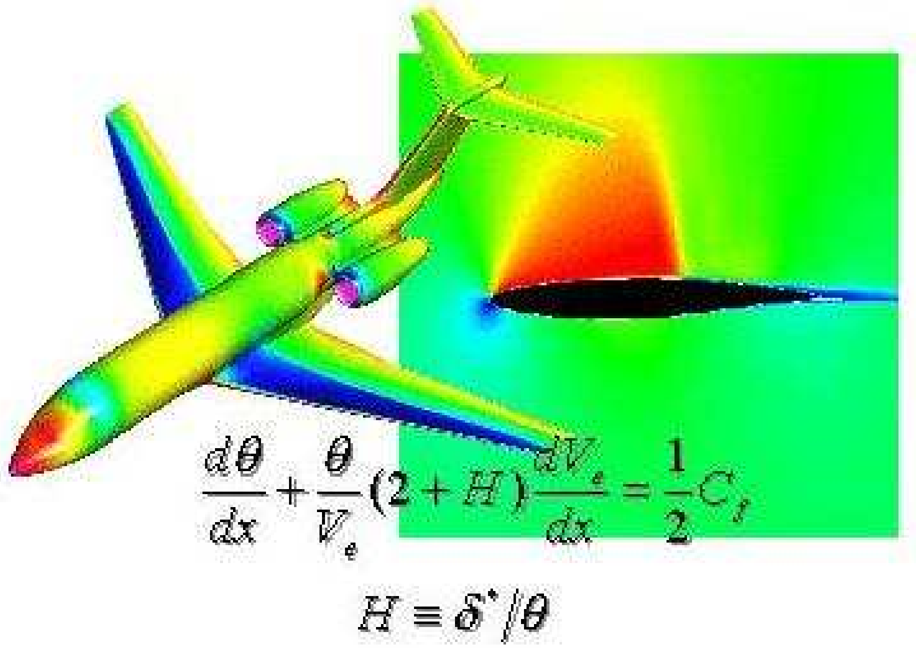
Lifting Airfoils in Incompressible Irrotational Flow



AA200b

Lecture 2

January 6, 2005

Governing Equations

For an incompressible fluid, the continuity equation takes the form

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{1}$$

If the flow is irrotational ($\nabla \times \mathbf{u} \equiv 0$), the velocity field can be written as the gradient of a scalar function called the potential function (typically represented by the symbol ϕ). This is expressed in the following vector identity

$$\mathbf{u} = \nabla \cdot \phi \tag{2}$$

When substituting the definition of the potential, ϕ, into Eq. 1, we obtain the governing equation for ϕ, Laplace's equation:

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \tag{3}$$

which the potential function must satisfy everywhere, including the appropriate boundary conditions. In this way we have reduced the problem of finding the velocity field (2 components, (u, v)) to a single equation. In addition, one can show that the momentum equations can be combined with the statement of *irrotationality* to obtain a relation between velocity magnitude and pressure called the Bernoulli equation

$$p + \frac{1}{2}(u^2 + v^2) = p_0. \tag{4}$$

One way to obtain solutions to Laplace's equation (subject to the appropriate boundary conditions) is to exploit its linear nature and the principle of superposition. Because Laplace's equation is linear, if we have two solutions ϕ_1 and ϕ_2, the linear combination $\alpha\phi_1 + \beta\phi_2$ is also a solution, with α and β arbitrary scalars. Obviously, the principle of superposition can be used to add up an arbitrary number of *elementary solutions* to Laplace's equation.

This method is rather powerful and will be used in the remainder of this lecture to construct the solution of the flow about an arbitrarily shaped lifting airfoil.

In this lecture we will use three types of elementary solutions: *free stream*, *source/sink*, *vortex*. With a *large* number of each of these three kinds of solutions we will be able to construct the flow about rather general airfoils at arbitrary angles of attack.

Free Stream Potential

The potential function for a *free stream* of magnitude V_∞, aligned with the $x-$axis is given by

$$\phi_\infty = V_\infty x. \tag{5}$$

Taking the gradient of this potential we see that the resulting velocity field is given by

$$\begin{aligned} u(x,y) &= V_\infty \\ v(x,y) &= 0. \end{aligned} \tag{6}$$

That is, the velocity is uniform everywhere in the domain. The potential function can be *rotated* at an arbitrary angle α so that $\phi_\infty = V_\infty(\cos\alpha x + \sin\alpha y)$, which for small angles of attack can be approximated as $\phi_\infty \approx V_\infty(x + \alpha y)$.

Source/Sink Potential

A source/sink that expels/absorbs an amount of fluid volume/unit time $= \pm Q$ can be constructed from the following potential

$$\phi_S = \frac{\pm Q}{2\pi} \ln r. \tag{7}$$

The resulting velocity components are

$$\begin{aligned} u(x,y) &= \frac{\pm Q}{2\pi} \frac{x}{x^2+y^2} \\ v(x,y) &= \frac{\pm Q}{2\pi} \frac{y}{x^2+y^2}. \end{aligned} \tag{8}$$

The streamlines in these flows can be shown to be straight lines emanating radially outwards from the location of the source/sink.

Point Vortex Potential

The potential of a point vortex with circulation Γ can be shown to be

$$\phi_V = -\frac{\Gamma}{2\pi}\theta, \tag{9}$$

where Γ is defined positive if the induced circular flow is clockwise. θ is the angle measured in polar coordinates from some arbitrary origin radial line. Taking the gradient of this function, we see that the velocity field of a vortex is given by

$$\begin{aligned} u(x,y) &= \frac{\pm\Gamma}{2\pi}\frac{y}{x^2+y^2} \\ v(x,y) &= \frac{\pm\Gamma}{2\pi}\frac{-x}{x^2+y^2}, \end{aligned} \tag{10}$$

and has streamlines that are concentric circles centered about the location of the point vortex.

Notice that the circulation around *any* countour that encloses the point vortex is constant and equal to Γ. Furthermore, the flow *outside* of the point vortex is fully irrotational. All of the vorticity in this flow is contained at the singular location of the point vortex. More on this later.

Notice also that the form of the potential for a point vortex is rather similar to that of a source/sink, with the substitutions $x \rightarrow y$, and $y \rightarrow -x$ in the appropriate formulae for the velocity field: the velocity fields are perpendicular to each other (and so are the equipotential lines).

Thickness and Camber Problems

Consider uniform flow past an airfoil at an angle of attack α.

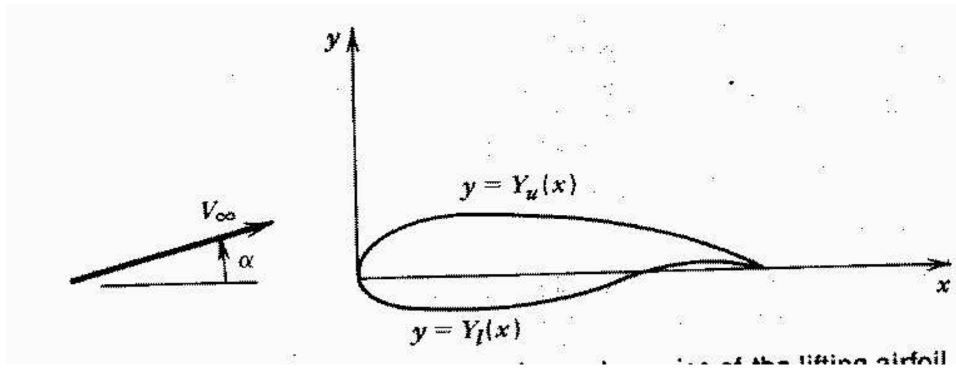


Figure 1: Airfoil Geometry Nomenclature

The airfoil surface is defined by its upper and lower surfaces located at $y = Y_u(x)$ and $y = Y_l(x)$.

The line that joins the *leading* and *trailing* edges is aligned with the x-axis and spans the interval $[0, c]$.

The *thickness* and *camber* distributions are given, by definition, by

$$\begin{aligned} T(x) &= Y_u(x) - Y_l(x) \\ \bar{Y}(x) &= \frac{1}{2}[Y_u(x) + Y_l(x)], \end{aligned} \tag{11}$$

which simply yields

$$\begin{aligned} Y_u(x) &= \bar{Y} + \frac{1}{2}T \\ Y_l(x) &= \bar{Y} - \frac{1}{2}T. \end{aligned} \tag{12}$$

If the airfoil is considered to be "thin", we can safely assume that $u \approx V_\infty$, and furthermore, we can apply the airfoil surface boundary conditions along the x-axis instead of at the exact surface location.

A fundamental necessity to create lifting flows is that our solution must admit discontinuities across the x-axis. For that purpose, we consider the values of all the relevant variables at $0^{\pm}$, depending on whether we are looking at the upper or lower surface of the airfoil.

With these assumptions, the flow tangency boundary conditions at a solid wall reduce to:

$$v(x,0^+) \approx V_\infty\left(\frac{d\bar{Y}}{dx} + \frac{1}{2}\frac{dT}{dx}\right) \tag{13}$$
$$v(x,0^-) \approx V_\infty\left(\frac{d\bar{Y}}{dx} - \frac{1}{2}\frac{dT}{dx}\right),$$

which can be satisfied in a variety of ways. A physically meaningful approach is to construct a potential function for the flow around the complete airfoil with 3 main components:

$$\phi = \phi_\infty + \phi_T + \phi_C, \tag{14}$$

where ϕ_∞ is the potential of a free stream inclined at an angle of attack α to the x-axis, and ϕ_T and ϕ_C are the potentials created by line distributions of sources/sinks and vortices placed along the x-axis. They will be later related to the *thickness* and *camber* portions of the problem.

Note that ϕ_∞ alone satisfies the boundary conditions at the far field. Therefore, ϕ_T and ϕ_C must vanish at the far field so as not to violate the boundary condition there. ϕ_T is chosen to satisfy the portion of the flow tangency boundary condition associated with the thickness distribution, while ϕ_C will be chosen so that, together with ϕ_∞ the *camber* portion of the boundary condition is satisfied.

In other words, for small α,

$$\phi_\infty \approx V_\infty(x + \alpha y), \tag{15}$$

$$v_T = \frac{\partial \phi_T}{\partial y} = \pm\frac{1}{2}V_\infty\frac{dT}{dx} \text{ at } y = 0^\pm \text{ for } 0 < x < c \tag{16}$$

$$v_C = \frac{\partial \phi_C}{\partial y} = V_\infty\left(\frac{d\bar{Y}}{dx} - \alpha\right) \text{ at } y = 0^\pm \text{ for } 0 < x < c. \tag{17}$$

It is possible to show that, the thickness problem can be solved with a line distribution of sources along the x-axis, while the camber problem can be solved with a distribution of vortices along the same axis. Both distributions will be assumed to be continuously varying with strength $q(x)$ and $\gamma(x)$ respectively.

Thickness problem

The velocity field created by a distribution of sources is given by

$$\begin{aligned} v_s &= \int_0^c \frac{q(t)}{2\pi} \frac{y}{(x-t)^2 + y^2} dt \\ u_s &= \int_0^c \frac{q(t)}{2\pi} \frac{x-t}{(x-t)^2 + y^2} dt. \end{aligned} \tag{18}$$

Taking the limit of these expressions as $y \to 0$ allows us to extend the limits of integration from $-\infty$ to ∞, since the integrand effectively behaves as a delta function as $y \to 0$. With this approximation, we find that

$$v_s(x, 0^{\pm}) = \pm \frac{1}{2} q(x), \tag{19}$$

that is, half of the volume flux per unit depth out of a differential element of source is sent directly upwards, while the other half is directed downwards. This relationship between source strength and flow speed directly produces the strength of a source disctribution that is necessary to generate the shape of a body of interest (i.e. satisfy the flow tangency boundary condition). In particular, for the thickness problem, Equation 16 together with Equation 19 yield

$$q(x) = V_\infty T'(x). \tag{20}$$

The contribution to the x-component of the velocity vector from this distribution of sources is given by

$$u_T(x, 0^\pm) = \frac{V_\infty}{2\pi} \mathrm{P}\!\!\int_0^c T'(t) \frac{dt}{x-t} \text{ for } 0 < x < c \tag{21}$$

Note that u_T is continuous across the x-axis, while v_T is discontinuous

$$\begin{aligned} u_T(x,0^+) &= u_T(x,0^-) \\ v_T(x,0^+) &= -v_T(x,0^-) \end{aligned} \tag{22}$$

Camber Problem

Notice from Equation 17 that the vertical velocity component due to the vortex distribution v_C is continuous across the surface of the airfoil. Therefore, we can infer that the horizontal velocity component due to this vortex distribution must be allowed to be discontinuous across the surface of the airfoil, since, otherwise, the total horizontal velocity would be continuous across the surface of the airfoil, which we know not to be true from a physical standpoint, and which would lead to problems when trying to obtain non-zero lift from this mathematical setup.

Let's represent the solution to the camber problem via a distribution of vortices along the x-axis, with strength per unit length $\gamma(x)$:

$$\phi_C = -\int_0^c \frac{\gamma(t)}{2\pi} \tan^{-1}\left(\frac{y}{x-t}\right) dt. \tag{23}$$

Following a derivation similar to the one used for the source distribution, we find the velocity components due to ϕ_C to be:

$$
\begin{aligned}
u_C(x, 0^{\pm}) &= \pm\frac{1}{2}\gamma(x) \text{ for } 0 < x < c \qquad (24)\\
v_C(x, 0^{\pm}) &= -\,\mathrm{P}\!\!\int_0^c \frac{\gamma(t)}{2\pi}\frac{dt}{x-t} \text{ for } 0 < x < c
\end{aligned}
$$

In order to satisfy the flow tangency boundary condition in Equation 17, $\gamma(x)$ must satisfy the following integral equation:

$$
\frac{1}{2\pi}\,\mathrm{P}\!\!\int_0^c \gamma(t)\frac{dt}{x-t} = V_\infty\left(\alpha - \frac{d\bar{Y}}{dx}\right) \text{ for } 0 < x < c \qquad (25)
$$

This integral equation is difficult to solve, although there are well-known solutions such as that for a flat plate airfoil at an angle of attack, α (which is shown in the following slides).

Assuming the solution $\gamma(x)$ which satisfies the portion of the boundary conditions for the *camber* problem can be found, the velocity field can be derived from appropriate substitution into Equations 19 and 25, and using the Bernoulli equation, one can found the pressure distribution on the upper and lower surafces of the airfoil, from which forces and moments can be computed.

Note that Bernoulli's equation

$$p + \frac{1}{2}\rho(u^2 + v^2) = p_\infty + \frac{1}{2}\rho V_\infty^2 \tag{26}$$

can be written in a simpler form under the thin airfoil approximation, $u_T + u_C \ll V_\infty$ (by neglecting terms that are higher order) as:

$$p \approx p_\infty - \rho V_\infty(u_T + u_C). \tag{27}$$

Kutta Joukowski Theorem

We can compute the total force that the fluid exerts on the airfoil by simply integrating the pressure around the airfoil surface. The total force can be decomposed into components along the x and y directions as follows:

$$F'_x = \int_0^c \left(p_u \frac{dY_u}{dx} - p_l \frac{dY_l}{dx} \right) dx \tag{28}$$

$$F'_y = \int_0^c (p_l - p_u)\, dx.$$

From our expressions for u_T and u_C on the upper and lower surfaces of the airfoil we see that

$$p_u - p_l \approx -\rho V_\infty \gamma,$$

and therefore, the total force per unit depth in the vertical direction is given by

$$F_y' \approx \rho V_\infty \int_0^c \gamma(x)dx = \rho V_\infty \Gamma, \tag{29}$$

where

$$\Gamma = \int_0^c \gamma(x)dx$$

is the total circulation around the airfoil. With a certain amount of algebra, the x component of the force can be shown to be

$$F_x' \approx -\rho V_\infty \alpha \int_0^c \gamma(x)dx = -\rho V_\infty \alpha \Gamma. \tag{30}$$

The vector addition of these two components of the total force can be shown to be perpendicular to the free stream (remember that for small angles of attack, $\alpha, \tan\alpha \approx \alpha$). This result is often known as D'Alembert's paradox.

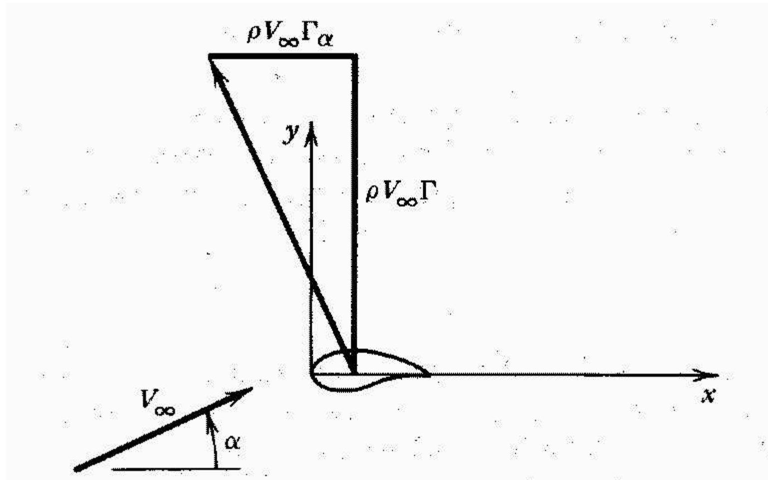


Figure 2:

Theorem 1. [Kutta-Joukowski Theorem] *An isolated two-dimensional airfoil in an incompressible inviscid flow feels a force per unit depth of*

$$\mathbf{F}' = \rho_\infty \mathbf{V}_\infty \times \mathbf{\Gamma}.$$

Since the lift and drag forces are defined as the forces in the directions normal and parallel to the free stream, a direct consequence is that

$$
\begin{aligned}
L' &= \rho V_\infty \Gamma \\
D' &= 0,
\end{aligned}
$$

and thus, the airfoil produces lift while creating no drag. This result is clearly unphysical since it would amount to a perpetual motion machine, which would violate the second law of thermodynamics. The result for the lift force is very close to experiment (for high Reynolds numbers), while the result for the drag force is clearly off and will be taken care of during our lectures on viscous flow and boundary layer theory.

Symmetric Airfoil at an Angle of Attack

In order to show an example of the solution to the integral equation 25 lets assume we have a symmetric airfoil ($Y_u(x) = -Y_l(x)$), such that the camberline is a flat plate aligned with the x-axis ($\bar{Y}(x) = 0$). The integral equation for $\gamma(x)$ then reduces to

$$\mathrm{P}\!\!\int_0^c \frac{\gamma(t)dt}{(x-t)} = 2\pi V_\infty \alpha, \text{ for } 0 < x < c. \tag{31}$$

While the integrand of this expression depends on x, the right hand side is a constant. Because of this fact, the integral equation can be solved using the following coordinate transformation

$$x = \frac{c}{2}(1 - \cos\theta_0)$$

$$\begin{aligned} t &= \frac{c}{2}(1-\cos\theta) \\ \gamma(t) &= g(\theta). \end{aligned}$$

The integral equation is transformed into

$$\int_0^\pi \frac{g(\theta)\sin\theta d\theta}{\cos\theta - \cos\theta_0} = 2\pi V_\infty \alpha.$$

A useful result in applied aerodynamics (which will come back when we discuss lifting line theory) is that (look in your CRC)

$$\int_0^\pi \frac{\cos n\theta}{\cos\theta - \cos\theta_0} d\theta = \pi \frac{\sin n\theta_0}{\sin\theta_0}. \tag{32}$$

Since the right hand side of this integral reduces to π when $n = 1$, a solution to the integral equation is certainly

$$g(\theta)\sin\theta = 2V_\infty\alpha\cos\theta.$$

However, for $n = 0$, the integral vanishes, so that we may add an arbitrary constant to the term $\sin\theta\ g(\theta)$ while still retaining a valid solution to the integral equation 31!!! Hmmmm . . . the implications of this constant are quite relevant. In general,

$$g(\theta) = \frac{k}{\sin\theta} + 2V_\infty\alpha\frac{\cos\theta}{\sin\theta},$$

where k is the arbitrary constant. It can be shown that the total circulation around the airfoil is given by

$$\Gamma = \int_0^\pi g(\theta_0)\frac{c}{2}\sin\theta_0 d\theta_0 = \frac{kc\pi}{2}.$$

Since the constant k can change the value of the total circulation, Γ, it will change the total value of the force exerted by the fluid on the airfoil . . . you may have already guessed that something is wrong here.

Notice that the reverse transformation is given by

$$\begin{aligned}\cos\theta &= 1 - \frac{2t}{c}\\ \sin\theta &= 2\sqrt{\frac{t}{c}\left(1-\frac{t}{c}\right)},\end{aligned}$$

and therefore, the vortex strength distribution is given by

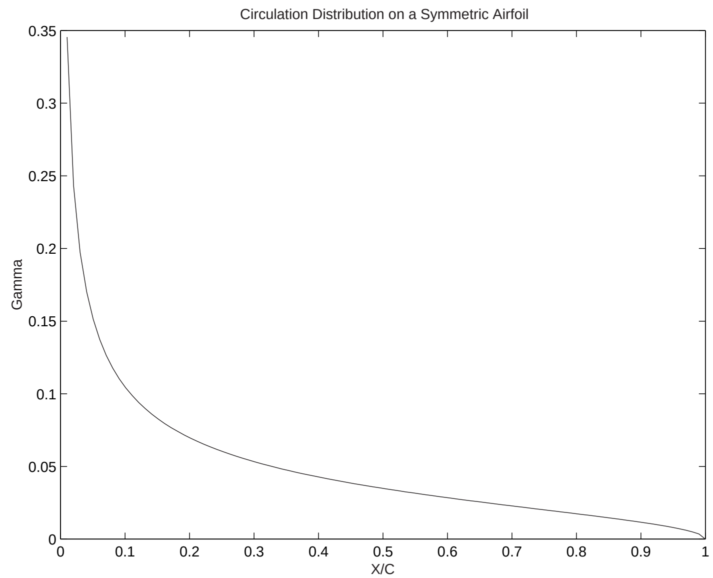
$$\gamma(t) = \frac{k}{2\sqrt{\frac{t}{c}\left(1-\frac{t}{c}\right)}} + 2V_\infty\alpha\frac{1-\frac{2t}{c}}{\sqrt{\frac{t}{c}\left(1-\frac{t}{c}\right)}}.$$

If k is set to $2V_\infty\alpha$, one can show by applying L'Hopital's rule that the circulation vanishes at the trailing edge point $x = c$, leaving the following expression for the vortex strength distribution

$$\gamma(x) = 2V_\infty\alpha\frac{\sqrt{1-(x/c)}}{\sqrt{x/c}}.$$

This choice of k is the one that satisfies the Kutta condition which we will discuss next.

The figure below shows the resulting circulation distribution on a symmetric airfoil at a 3° angle of attack. Note that there is still a singularity at the leading edge of the airfoil, but all ambiguity has been removed by fixing the value of the constant k.



Finally, the section lift and moment can be found to be

$$L' = \pi \rho V_\infty^2 \alpha c \tag{33}$$

$$M'_{l.e.} = -\frac{\pi}{4} \rho V_\infty^2 \alpha c^2, \tag{34}$$

so that the center of pressure (point on the airfoil at which the lift vector appears to be acting) is located at the quarter chord point

$$x_{c.p.} = -\frac{M'_{l.e.}}{L'} = \frac{c}{4}.$$

Furthermore, one can see that since the lift vector appears to be acting at the quarter chord location (independently of angle of attack for this symmetric airfoil...is that true for non-symmetric airfoils?) we can compute the moment about any point (other than the leading edge) for this airfoil. If we do this, we would find that for $\frac{x}{c} = \frac{1}{4}$ the moment is independent of the angle of attack of the airfoil. This point is rather useful and is called the *aerodynamic center*. As we will see later, thin airfoil theory predicts this location of the aerodynamic center for airfoils with arbitrary camber as well.

Kutta Condition

As we saw in the previous example, the circulation distribution on an arbitrary airfoil is fixed by both the far field and solid wall boundary conditions up to a constant. In order to obtain a unique solution to this problem, we must find some additional information that helps us determine the proper value of the constant, k.

Observation of the flow near the trailing edge shows that we must have one of the following alternatives:

1. The velocity at the trailing edge is infinite.

2. The flow leaves the trailing edge *smoothly* along the direction of the bisector of the trailing edge angle.

In a real flow, the velocity cannot be infinite, since it would lead to very low pressure (high adverse pressure gradient) and the flow would separate. The second option is the one that occurs in reality.

Theorem 2. [Kutta Condition] *The flow leaves the trailing edge of a sharp-tailed airfoil smoothly; that is, the velocity is finite there.*

The following corollaries for the Kutta-condition can also be derived:

- The streamline that leaves a sharp trailing edge is an extension of the bisector of the trailing edge angle.
- Near the trailing edge, the flow speeds on the upper and lower surfaces of the airfoil are equal at equal distances from the trailing edge.
- Unless the trailing edge is cusped, the flow has a stagnation point at the trailing edge.

The most useful of these corollaries is the second one. This leads to the conclusion that there cannot be a jump in u_C velocity at the trailing edge. Since, according to Equation 25 the jump in u_C is equal to $\gamma(x)$, we must have $\gamma(c) = 0$. This is the condition that we imposed in the previous example leading to the elimnation of the constant k.

Cambered Thin Airfoil

Let's now assume that the camberline of the airfoil is no longer flat, $\bar{Y} \neq 0$. As in the previous example, we need to find the circulation distribution $\gamma(x)$ that satisfies Equation 25, and that satisfies the Kutta condition $\gamma(c) = 0$.

Using the same coordinate transformation as before we have

$$\frac{1}{2\pi V_\infty} P\!\!\!\!\int_0^\pi \frac{g(\theta)\sin\theta d\theta}{\cos\theta - \cos\theta_0} = \alpha - s(\theta_0), \text{ for } 0 < \theta_0 < \pi, \tag{35}$$

where

$$\bar{Y}'(x) = s(\theta_0),$$

and the Kutta condition is given by

$$g(\pi) = 0.$$

One option to solve this equation is to expand $g(\theta)\sin\theta$ into a Fourier cosine series, but we would run into trouble evaluating the Kutta condition. Alternatively, we can start directly with a form of the solution that always satisfies the Kutta condition. One such form is given by

$$g(\theta)\sin\theta = 2V_\infty\left[A_0(1+\cos\theta) + \sum_{n=1} A_n \sin n\theta \sin\theta\right]. \tag{36}$$

Using Equation 32, some trigonometric identities, and the fact that it can be shown that one can expand

$$\alpha - s(\theta_0) = A_0 - \sum_{n=1} A_n \cos n\theta_0$$

it can be shown that the solution to the thin cambered airfoil is given by

$$A_0 = \alpha - \frac{1}{\pi}\int_0^\pi s(\theta_0)d\theta_0$$

and

$$A_m = \frac{2}{\pi}\int_0^\pi s(\theta_0)cosm\theta_0 d\theta_0 \text{ for } m = 1, 2, \ldots$$

Note that in the Fourier series above we are expanding the camber slope distribution $\bar{Y}'(x)$ into a series that converges even if the function being expanded contains jump discontinuities (i.e. the camberline slope is not continuous, which would occur for airfoils with flaps and slats).

Formulae for the lift and moment coefficient, as well as the location of the center of pressure can be worked out in a manner similar to the one followed for the symmetric airfoil at an angle of attack. In summary

$$
\begin{aligned}
L' &= \pi\rho V_\infty^2 c(A_0 + \frac{1}{2}A_1) \\
M'_{l.e.} &= \frac{-\pi}{4}\rho V_\infty^2 c^2(A_0 + A_1 - \frac{1}{2}A_2) \\
x_{c.p.} &= \frac{c}{4}\frac{A_0 + A_1 - \frac{1}{2}A_2}{A_0 + \frac{1}{2}A_1}
\end{aligned}
$$

One can see that for this airfoil, the location of the center of pressure changes with angle of attack (since the A_0 coefficient will change with α). However, one can show there is a point on the airfoil about which the aerodynamic moment does not change with angle of attack. This point is called the *aerodynamic* center, and, according to thin airfoil theory it is located at the quarter chord, that is

$$x_{a.c} = \frac{c}{4}$$

Finally, the coefficients of lift and moment can be shown to be

$$\begin{aligned} c_l &= 2\pi(A_0 + \frac{1}{2}A_1) = 2\pi\left[\alpha - \frac{1}{\pi}\int_0^\pi s(\theta_0)(1-\cos\theta_0)d\theta_0\right] \\ c_{mac} &= -\frac{\pi}{4}(A_1 - A_2) = -\frac{1}{2}\int_0^\pi s(\theta_0)(\cos\theta_0 - \cos 2\theta_0)d\theta_0 \end{aligned} \quad (37)$$

This airfoil theory then predicts that

$$c_l = m_0(\alpha - \alpha_{L0})$$

where m_0 is the lift-curve slope

$$m_0 = 2\pi$$

and α_{L0} is the zero-lift angle of attack

$$\alpha_{L0} = \frac{1}{\pi}\int_0^\pi s(\theta_0)(1 - \cos\theta_0)d\theta_0$$