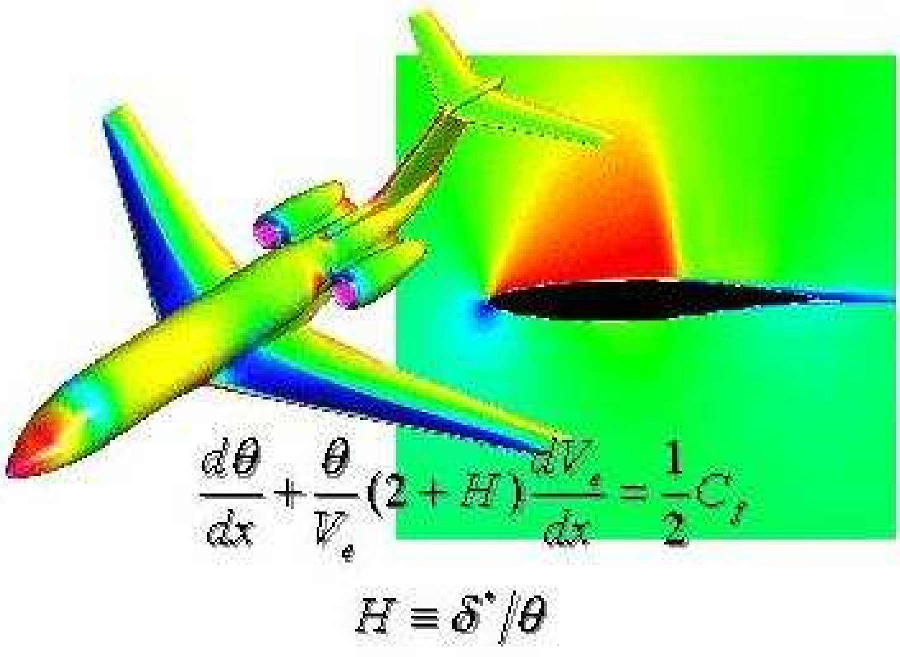
Hess-Smith Panel Method



AA200b

Lecture 3

January 13-18, 2005

Shortcomings of Thin Airfoil Theory

Although thin airfoil theory provides invaluable insights into the generation of lift, the Kutta-condition, the effect of the camber distribution on the coefficients of lift and moment, and the location of the center of pressure and the aerodynamic center, it has several limitations that prevent it from being used in practical applications. Among these we can mention:

- It ignores the effects of the thickness distribution on c_l and c_{mac}.
- Pressure distributions tend to be inaccurate near stagnation points.
- Airfoils with high camber or large thickness violate the assumptions of airfoil theory, and, therefore, the prediction accuracy degrades in these situations even away from stagnation points.

Alternatives

We could consider the following alternatives in order to overcome the limitations of thin airfoil theory

- In addition to sources and vortices, we could use higher order solutions to Laplace's equation that can enhance the accuracy of the approximation (doublet, quadrupoles, octupoles, etc.). This approach falls under the denomination of *multipole expansions*.

- We can use the same solutions to Laplace's equation (sources/sinks and vortices) but place them on the surface of the body of interest, and use the exact flow tangency boundary conditions without the approximations used in thin airfoil theory.

This latter method can be shown to treat a wide range of reasonable problems for the applied aerodynamicist, including multi-element airfoils. It also has the advantage that it can be naturally extended to three-dimensional flows (unlike streamfunction or complex variable methods).

The distribution of the sources/sinks and vortices on the surface of the body can be either *continuous* or *discrete*.

A continuous distribution leads to integral equations similar to those we saw in thin airfoil theory which cannot be treated analytically.

If we discretize the surface of the body into a series of segments or panels, the integral equations are transformed into an easily solvable set of simultaneous linear equations.

These methods come in many flavors and are typically called

PANEL METHODS

Hess-Smith Panel Method

There are many choices as to how to formulate a panel method (singularity solutions, variation within a panel, singularity strength and distribution, etc.) The simplest and first truly practical method was due to Hess and Smith, Douglas Aircraft, 1966. It is based on a distribution of sources and vortices on the surface of the geometry. In their method

$$\phi = \phi_\infty + \phi_S + \phi_V \tag{1}$$

where, ϕ is the total potential function and its three components are the potentials corresponding to the free stream, the source distribution, and the vortex distribution. These last two distributions have potentially locally varying strengths $q(s)$ and $\gamma(s)$, where s is an arc-length coordinate which spans the complete surface of the airfoil in any way you want.

The potentials created by the distribution of sources/sinks and vortices are given by:

$$\phi_S = \int \frac{q(s)}{2\pi} \ln r ds \quad (2)$$
$$\phi_V = -\int \frac{\gamma(s)}{2\pi} \theta ds$$

where the various quantities are defined in the Figure below

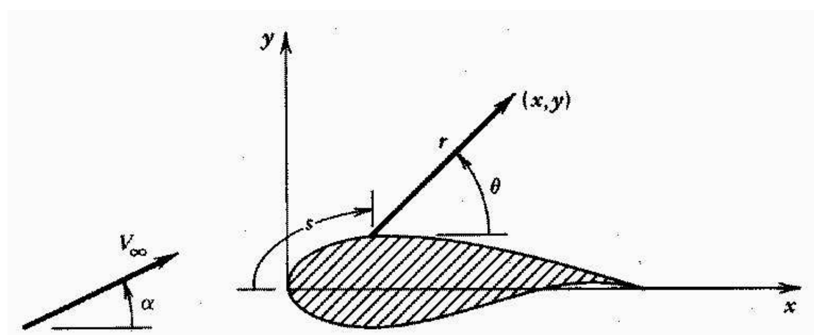


Figure 1: Airfoil Analysis Nomenclature for Panel Methods

Notice that in these formulae, the integration is to be carried out along the complete surface of the airfoil. Using the superposition principle, any such distribution of sources/sinks and vortices satisfies Laplace's equation, but we will need to find conditions for $q(s)$ and $\gamma(s)$ such that the flow tangency boundary condition and the Kutta condition are satisfied.

Notice that we have multiple options. In theory, we could:

- Use the source strength distribution to satisfy flow tangency and the vortex distribution to satisfy the Kutta condition.
- Use arbitrary combinations of both sources/sinks and vortices to satisfy both boundary conditions simultaneously.

Hess and Smith made the following *valid* simplification

Take the vortex strength to be constant over the whole airfoil and use the Kutta condition to fix its value, while allowing the source strength to vary from panel to panel so that, together with the constant vortex distribution, the flow tangency boundary condition is satisfied everywhere.

Alternatives to this choice are possible and result in different types of panel methods. Ask if you want to know more about them. Using the *panel* decomposition from the figure below,

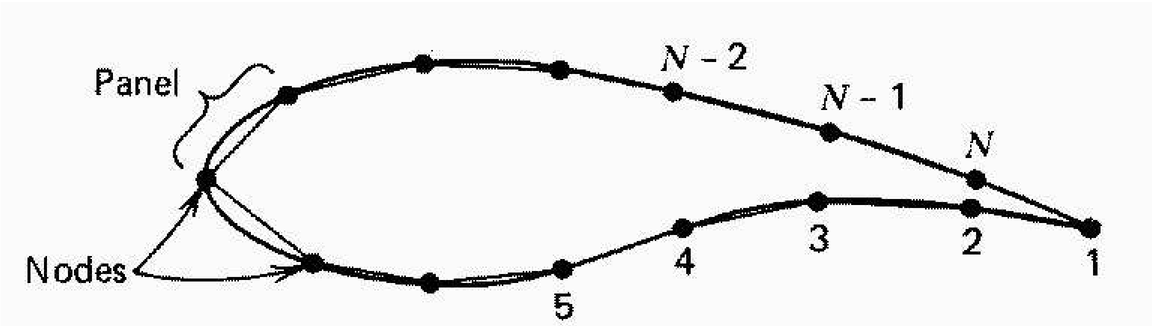


Figure 2: Definition of Nodes and Panels

we can "discretize" Equation 1 in the following way:

$$\phi = V_\infty(x\cos\alpha + y\sin\alpha) + \sum_{j=1}^{N}\int_{panel_j}\left[\frac{q(s)}{2\pi}\ln r - \frac{\gamma}{2\pi}\theta\right]ds \qquad (3)$$

Since Equation 3 involves integrations over each discrete *panel* on the surface of the airfoil, we must somehow parameterize the variation of source and vortex strength within each of the panels. Since the vortex strength was considered to be a constant, we only need worry about the source strength distribution within each panel.

This is the major approximation of the panel method. However, you can see how the importance of this approximation should decrease as the number of panels, $N \to \infty$ (of course this will increase the cost of the computation considerably, so there are more efficient alternatives.)

Hess and Smith decided to take the simplest possible approximation, that is, to take the source strength to be constant on each of the panels

$$q(s) = q_i \text{ on panel } i, \qquad i = 1, \ldots, N$$

Therefore, we have $N + 1$ unknowns to solve for in our problem: the N panel source strengths q_i and the constant vortex strength γ. Consequently, we will need $N + 1$ independent equations which can be obtained by formulating the flow tangency boundary condition at each of the N panels, and by enforcing the Kutta condition discussed previously. The solution of the problem will require the inversion of a matrix of size $(N+1) \times (N+1)$.

The final question that remains is: where should we impose the flow tangency boundary condition? The following options are available

- The nodes of the surface panelization.

- The points on the surface of the actual airfoil, halfway between each adjacent pair of nodes.

- The points located at the midpoint of each of the panels.

We will see in a moment that the velocities are infinite at the nodes of our panelization which makes them a poor choice for boundary condition imposition.

The second option is reasonable, but rather difficult to implement in practice.

The last option is the one Hess and Smith chose. Although it suffers from a slight alteration of the surface geometry, it is easy to implement and yields fairly accurate results for a reasonable number of panels. This location is also used for the imposition of the Kutta condition (on the last panels on upper and lower surfaces of the airfoil, assuming that their midpoints remain at equal distances from the trailing edge as the number of panels is increased).

Implementation

Consider the ith panel to be located between the ith and $(i+1)$th nodes, with its orientation to the x-axis given by

$$\begin{aligned}\sin\theta_i &= \frac{y_{i+1}-y_i}{l_i}\\ \cos\theta_i &= \frac{x_{i+1}-x_i}{l_i}\end{aligned}$$

where l_i is the length of the panel under consideration. The normal and tangential vectors to this panel, are then given by

$$\begin{aligned}\hat{\mathbf{n}}_i &= -\sin\theta_i\hat{\mathbf{i}}+\cos\theta_i\hat{\mathbf{j}}\\ \hat{\mathbf{t}}_i &= \cos\theta_i\hat{\mathbf{i}}+\sin\theta_i\hat{\mathbf{j}}\end{aligned}$$

The tangential vector is oriented in the direction from node i to node $i+1$, while the normal vector, if the airfoil is traversed clockwise, points into the fluid.

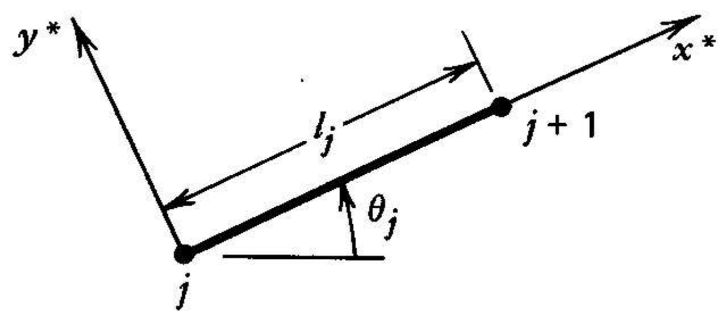


Figure 3: Local Panel Coordinate System

Furthermore, the coordinates of the midpoint of the panel are given by

$$\bar{x}_i = \frac{x_i + x_{i+1}}{2}$$

$$\bar{y}_i = \frac{y_i + y_{i+1}}{2}$$

and the velocity components at these midpoints are given by

$$u_i = u(\bar{x}_i, \bar{y}_i)$$
$$v_i = v(\bar{x}_i, \bar{y}_i)$$

The flow tangency boundary condition can then be simply written as $(\vec{\mathbf{u}} \cdot \vec{\mathbf{n}}) = 0$, or, for each panel

$$-u_i \sin\theta_i + v_i \cos\theta_i = 0 \quad \text{for} \;\; i = 1, \ldots, N$$

while the Kuttta condition is simply given by

$$u_1 \cos\theta_1 + v_1 \sin\theta_1 = -u_N \cos\theta_N - v_N \sin\theta_N \tag{4}$$

where the negative signs are due to the fact that the tangential vectors at the first and last panels have nearly opposite directions.

Now, the velocity at the midpoint of each panel can be computed by superposition of the contributions of all sources and vortices located at the midpoint of *every* panel (including itself). Since the velocity induced by the source or vortex on a panel is proportional to the source or vortex strength in that panel, q_i and γ can be pulled out of the integral in Equation 3 to yield

$$
\begin{aligned}
u_i &= V_\infty \cos\alpha + \sum_{j=1}^{N} q_j u_{sij} + \gamma \sum_{j=1}^{N} u_{vij} \\
v_i &= V_\infty \sin\alpha + \sum_{j=1}^{N} q_j v_{sij} + \gamma \sum_{j=1}^{N} v_{vij}
\end{aligned}
\tag{5}
$$

where u_{sij}, v_{sij} are the velocity components at the midpoint of panel i induced by a source of unit strength at the midpoint of panel j. A similar interpretation can be found for u_{vij}, v_{vij}. In a coordinate system tangential and normal to the panel, we can perform the integrals in Equation 3 by noticing that the local velocity components can be expanded into absolute ones according to the following transformation:

$$\begin{aligned} u &= u^* \cos\theta_j - v^* \sin\theta_j \\ v &= u^* \sin\theta_j + v^* \cos\theta_j \end{aligned} \tag{6}$$

Now, the local velocity components at the midpoint of the ith panel due to a unit-strength source distribution on this jth panel can be written as

$$u^*_{sij} = \frac{1}{2\pi}\int_0^{l_j} \frac{x^* - t}{(x^* - t)^2 + {y^*}^2} dt \tag{7}$$

$$v^*_{sij} = \frac{1}{2\pi}\int_0^{l_j} \frac{y^*}{(x^*-t)^2 + {y^*}^2} dt$$

where (x^*, y^*) are the coordinates of the midpoint of panel i in the local coordinate system of panel j. Carrying out the integrals in Equation 7 we find that

$$u^*_{sij} = \frac{-1}{2\pi}\ln\left[(x^*-t)^2 + {y^*}^2\right]^{\frac{1}{2}}\Bigg|_{t=0}^{t=l_j} \tag{8}$$

$$v^*_{sij} = \frac{1}{2\pi}\tan^{-1}\frac{y^*}{x^*-t}\Bigg|_{t=0}^{t=l_j}$$

These results have a simple geometric interpretation that can be discerned by looking at the figure below. One can say that

$$u^*_{sij} = \frac{-1}{2\pi} \ln \frac{r_{ij+1}}{r_{ij}}$$

$$v^*_{sij} = \frac{\nu_l - \nu_0}{2\pi} = \frac{\beta_{ij}}{2\pi}$$

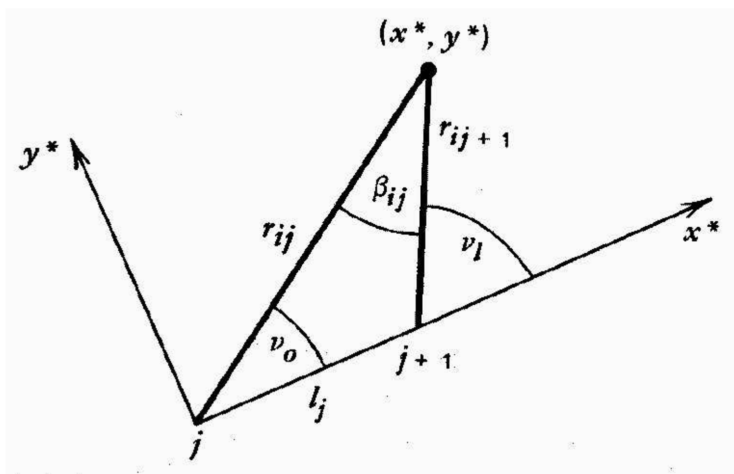


Figure 4: Geometric Interpretation of Source and Vortex Induced Velocities

r_{ij} is the distance from the midpoint of panel i to the jth node, while β_{ij} is the angle subtended by the jth panel at the midpoint of panel i. Notice that $u^*_{sii} = 0$, but the value of v^*_{sii} is not so clear. When the point of interest approaches the midpoint of the panel from the *outside* of the airfoil, this angle, $\beta_{ii} \rightarrow \pi$. However, when the midpoint of the panel is approached from the *inside* of the airfoil, $\beta_{ii} \rightarrow -\pi$. Since we are interested in the flow *outside* of the airfoil only, we will always take $\beta_{ii} = \pi$.

Similarly, for the velocity field induced by the vortex on panel j at the midpoint of panel i we can simply see that

$$u^*_{vij} = -\frac{1}{2\pi}\int_0^{l_j} \frac{y^*}{(x^*-t)^2 + {y^*}^2} dt = \frac{\beta_{ij}}{2\pi} \quad (9)$$

$$v^*_{vij} = -\frac{1}{2\pi}\int_0^{l_j} \frac{x^*-t}{(x^*-t)^2 + {y^*}^2} dt = \frac{1}{2\pi}\ln\frac{r_{ij+1}}{r_{ij}} \quad (10)$$

and finally, the flow tangency boundary condition, using Equation 5, and undoing the local coordinate transformation of Equation 6 can be written as

$$\sum_{j=1}^{N} A_{ij} q_j + A_{iN+1}\gamma = b_i$$

where

$$\begin{aligned} A_{ij} &= -u_{sij}\sin\theta_i + v_{sij}\cos\theta_i \\ &= -u^*_{sij}(\cos\theta_j\sin\theta_i - \sin\theta_j\cos\theta_i) + v^*_{sij}(\sin\theta_j\sin\theta_i + \cos\theta_j\cos\theta_i) \end{aligned}$$

which yields

$$2\pi A_{ij} = \sin(\theta_i - \theta_j)\ln\frac{r_{ij+1}}{r_{ij}} + \cos(\theta_i - \theta_j)\beta_{ij}$$

Similarly for the vortex strength coefficient

$$2\pi A_{iN+1} = \sum_{j=1}^{N} \cos(\theta_i - \theta_j) \ln \frac{r_{ij+1}}{r_{ij}} - \sin(\theta_i - \theta_j)\beta_{ij}$$

The right hand side of this matrix equation is given by

$$b_i = V_\infty \sin(\theta_i - \alpha)$$

The flow tangency boundary condition gives us N equations. We need an additional one provided by the Kutta condition in order to obtain a system that can be solved. According to Equation 4

$$\sum_{j=1}^{N} A_{N+1,j} q_j + A_{N+1,N+1}\gamma = b_{N+1}$$

After similar manipulations we find that

$$
\begin{aligned}
2\pi A_{N+1,j} &= \sum_{k=1,N} \sin(\theta_k - \theta_j)\beta_{kj} - \cos(\theta_k - \theta_j)\ln\frac{r_{kj+1}}{r_{kj}} && (11)\\
2\pi A_{N+1,N+1} &= \sum_{k=1,N}\sum_{j=1}^{N} \sin(\theta_k - \theta_j)\ln\frac{r_{kj+1}}{r_{kj}} + \cos(\theta_k - \theta_j)\beta_{kj}\\
b_{N+1} &= -V_\infty\cos(\theta_1 - \alpha) - V_\infty\cos(\theta_N - \alpha)
\end{aligned}
$$

where the sums $\sum_{k=1,N}$ are carried out only over the first and last panels, and not the range $[1, N]$. These various expressions set up a matrix problem of the kind

$$Ax = b$$

where the matrix A is of size $(N+1)\times(N+1)$. This system can be sketched as follows:

$$\begin{bmatrix} A_{11} & \dots & A_{1i} & \dots & A_{1N} & A_{1,N+1} \\ \vdots & & \vdots & & \vdots & \vdots \\ A_{i1} & \dots & A_{ii} & \dots & A_{iN} & A_{i,N+1} \\ \vdots & & \vdots & & \vdots & \vdots \\ A_{N1} & \dots & A_{Ni} & \dots & A_{NN} & A_{N,N+1} \\ A_{N+1,1} & \dots & A_{N+1,i} & \dots & A_{N+1,N} & A_{N+1,N+1} \end{bmatrix} \begin{bmatrix} q_1 \\ \vdots \\ q_i \\ \vdots \\ q_N \\ \gamma \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_i \\ \vdots \\ b_N \\ b_{N+1} \end{bmatrix}$$

Notice that the cost of inversion of a full matrix such as this one is $\mathcal{O}(N+1)^3$, so that, as the number of panels increases without bounds, the cost of solving the panel problem increases rapidly. This is usually not a problem for *two-dimensional* flows, but becomes a serious problem in *three-dimensional* flows where the number of panels, instead of being in the neighborhood of 100, is usually closer to $10,000$. Iterative solution methods and panel method implementations using fast multipole methods can help alleviate this problem. More on this later.

Finally, once you have solved the system for the unknowns of the problem, it is easy to construct the tangential velocity at the midpoint of each panel according to the following formula

$$\begin{aligned} V_{ti} &= V_\infty \cos(\theta_i - \alpha) + \sum_{j=1}^{N} \frac{q_j}{2\pi} \left[\sin(\theta_i - \theta_j)\beta_{ij} - \cos(\theta_i - \theta_j) \ln \frac{r_{ij+1}}{r_{ij}} \right] \\ &+ \frac{\gamma}{2\pi} \sum_{j=1}^{N} \left[\sin(\theta_i - \theta_j) \ln \frac{r_{ij+1}}{r_{ij}} + \cos(\theta_i - \theta_j)\beta_{ij} \right] \end{aligned}$$

And knowing the tangential velocity component, we can compute the pressure coefficient (no approximation since $V_{ni} = 0$) at the midpoint of each panel according to the following formula

$$C_p(\bar{x}_i, \bar{y}_i) = 1 - \frac{V_{ti}^2}{V_\infty^2}$$

from which the force and moment coefficients can be computed assuming that this value of C_p is constant over each panel and by performing the discrete sum. How close is the c_d to zero? You will find out in your next homework.